Chapter Nine

SEQUENCES AND SERIES

In this chapter, we look at infinite lists of numbers, called sequences, and infinite sums, called series. In Section 9.1, we study sequences. In Section 9.2, we begin with a particular type of series, called a geometric series. In Section 9.3, we consider general series of constants, what it means for such a series to converge, and how series are related to improper integrals. Tests that allow us to determine convergence are in Sections 9.3 and 9.4.

Section 9.5 introduces power series, in which the terms are constants multiplied by powers of x. These series converge for some x-values and not for others; the radius of convergence is introduced to identify the interval on which the series converges.

9.1 SEQUENCES

A sequence[1] is an infinite list of numbers $s_1, s_2, s_3, \ldots, s_n, \ldots$. We call s_1 the first term, s_2 the second term; s_n is the general term. For example, we can denote the sequence of squares, $1, 4, 9, \ldots, n^2, \ldots$ by the general term, $s_n = n^2$. Thus, a sequence is a function whose domain is the positive integers, but it is traditional to denote the terms of a sequence using subscripts, s_n, rather than function notation, $s(n)$. In addition, we may talk about sequences whose general term has no simple formula, such as the sequence 3, 3.1, 3.14, 3.141, 3.1415, . . ., in which s_n gives the first n digits of π.

The Numerical, Algebraic, and Graphical Viewpoint

Just as we can view a function algebraically, numerically, graphically, or verbally, we can view sequences in different ways. We may give an algebraic formula for the general term. We may give the numerical values of the first few terms of the sequence, suggesting a pattern for the later terms.

Example 1 Give the first six terms of the following sequences:

(a) $s_n = \dfrac{n(n+1)}{2}$ (b) $s_n = \dfrac{n+(-1)^n}{n}$

Solution (a) Substituting $n = 1, 2, 3, 4, 5, 6$ into the formula for the general term, we get

$$\frac{1\cdot 2}{2}, \frac{2\cdot 3}{2}, \frac{3\cdot 4}{2}, \frac{4\cdot 5}{2}, \frac{5\cdot 6}{2}, \frac{6\cdot 7}{2} = 1,\ 3,\ 6,\ 10,\ 15,\ 21.$$

(b) Substituting $n = 1, 2, 3, 4, 5, 6$ into the formula for the general term, we get

$$\frac{1-1}{1}, \frac{2+1}{2}, \frac{3-1}{3}, \frac{4+1}{4}, \frac{5-1}{5}, \frac{6+1}{6} = 0,\ \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}.$$

Example 2 Give a general term for the following sequences:

(a) 1, 2, 4, 8, 16, 32, . . . (b) $\dfrac{7}{2}, \dfrac{7}{5}, \dfrac{7}{8}, \dfrac{7}{11}, \dfrac{1}{2}, \dfrac{7}{17}, \ldots$

Solution Although the first six terms do not determine the sequence, we can sometimes use them to guess a possible formula for the general term.

(a) We have powers of 2, so we guess $s_n = 2^n$. When we check by substituting in $n = 1, 2, 3, 4, 5, 6$, we get $2, 4, 8, 16, 32, 64$, instead of $1, 2, 4, 8, 16, 32$. We fix our guess by subtracting 1 from the exponent, so the general term is

$$s_n = 2^{n-1}.$$

Substituting the first six values of n shows that the formula checks.

(b) In this sequence, the fifth term looks different from the others, whose numerators are all 7. We can fix this by rewriting $1/2 = 7/14$. The sequence of denominators is then $2, 5, 8, 11, 14, 17$. This looks like a linear function with slope 3, so we expect the denominator has formula $3n + k$ for some k. When $n = 1$, the denominator is 2, so

$$2 = 3 \cdot 1 + k \quad \text{giving} \quad k = -1$$

and the denominator of s_n is $3n - 1$. Our general term is then

$$s_n = \frac{7}{3n-1}.$$

To check this, evaluate s_n for $n = 1, \ldots, 6$.

[1] In everyday English, the words "sequence" and "series" are used interchangeably. In mathematics, they have different meanings and cannot be interchanged.

There are two ways to visualize a sequence. One is to plot points with n on the horizontal axis and s_n on the vertical axis. The other is to label points on a number line s_1, s_2, s_3, ... See Figure 9.1 for the sequence $s_n = 1 + (-1)^n/n$.

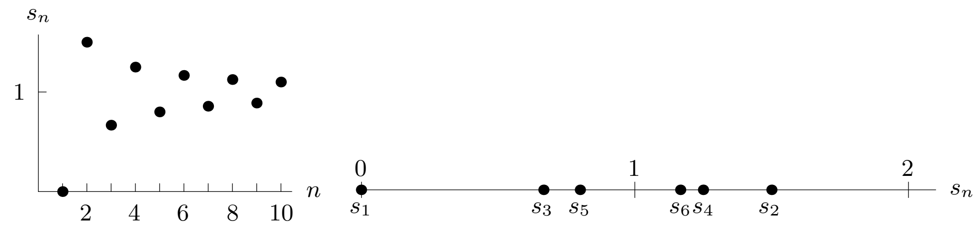


Figure 9.1: The sequence $s_n = 1 + (-1)^n/n$

Defining Sequences Recursively

Sequences can also be defined *recursively*, by giving an equation relating the n^{th} term to the previous terms and as many of the first terms as are needed to get started.

Example 3 Give the first six terms of the recursively defined sequences.

(a) $s_n = s_{n-1} + 3$ for $n > 1$ and $s_1 = 4$
(b) $s_n = -3s_{n-1}$ for $n > 1$ and $s_1 = 2$
(c) $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$ for $n > 2$ and $s_1 = 0, s_2 = 1$
(d) $s_n = ns_{n-1}$ for $n > 1$ and $s_1 = 1$

Solution

(a) When $n = 2$, we obtain $s_2 = s_1 + 3 = 4 + 3 = 7$. When $n = 3$, we obtain $s_3 = s_2 + 3 = 7 + 3 = 10$. In words, we obtain each term by adding 3 to the previous term. The first six terms are

$$4,\ 7,\ 10,\ 13,\ 16,\ 19.$$

(b) Each term is -3 times the previous term, starting with $s_1 = 2$. We have $s_2 = -3s_1 = -3 \cdot 2 = -6$ and $s_3 = -3s_2 = -3(-6) = 18$. Continuing, we get

$$2,\ -6,\ 18,\ -54,\ 162,\ -486.$$

(c) Each term is the average of the previous two terms, starting with $s_1 = 0$ and $s_2 = 1$. We get $s_3 = (s_2 + s_1)/2 = (1 + 0)/2 = 1/2$. Then $s_4 = (s_3 + s_2)/2 = ((1/2) + 1)/2 = 3/4$. Continuing, we get

$$0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}.$$

(d) Here $s_2 = 2s_1 = 2 \cdot 1 = 2$ so $s_3 = 3s_2 = 3 \cdot 2 = 6$ and $s_4 = 4s_3 = 4 \cdot 6 = 24$. Continuing gives

$$1,\ 2,\ 6,\ 24,\ 120,\ 720.$$

The general term of part (d) of the previous example is given by $s_n = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$, which is denoted $s_n = n!$ and is called n factorial .

We can also look at the first few terms of a sequence and try to guess a recursive definition by looking for a pattern.

Example 4 Give a recursive definition of the following sequences.

(a) $1, 3, 7, 15, 31, 63, \ldots$ (b) $1, 4, 9, 16, 25, 36, \ldots$

Solution

(a) Each term is twice the previous term plus one; for example $7 = 2 \cdot 3 + 1$ and $63 = 2 \cdot 31 + 1$. Thus, a recursive definition is

$$s_n = 2s_{n-1} + 1 \text{ for } n > 1 \text{ and } s_1 = 1.$$

There are other ways to define the sequence recursively. We might notice, for example, that the differences of consecutive terms are powers of 2. Thus, we could also use

$$s_n = s_{n-1} + 2^{n-1} \quad \text{for } n > 1 \text{ and } s_1 = 1.$$

(b) We recognize the terms as the squares of the positive integers, but we are looking for a recursive definition which relates consecutive terms. We see that

$$\begin{aligned} s_2 &= s_1 + 3 \\ s_3 &= s_2 + 5 \\ s_4 &= s_3 + 7 \\ s_5 &= s_4 + 9, \end{aligned}$$

so the differences between consecutive terms are consecutive odd integers. The difference between s_n and s_{n-1} is $2n - 1$, so a recursive definition is

$$s_n = s_{n-1} + 2n - 1, \text{ for } n > 1 \text{ and } s_1 = 1.$$

Recursively defined sequences, sometimes called recurrence relations, are powerful tools used frequently in computer science, as well as differential equations. Finding a formula for the general term—which can be surprisingly difficult—uses the principle of mathematical induction.

Convergence of Sequences

The limit of a sequence s_n as $n \to \infty$ is defined the same way as the limit of a function $f(x)$ as $x \to \infty$; see also Problem 44.

> The sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ has a **limit** L, written $\lim_{n\to\infty} s_n = L$, if s_n is as close to L as we please whenever n is sufficiently large. If a limit, L, exists, we say the sequence **converges** to its limit L. If no limit exists, we say the sequence **diverges**.

To calculate the limit of a sequence, we use what we know about the limits of functions, including the properties in Theorem 1.2 and the following facts:

- The sequence $s_n = x^n$ converges to 0 if $|x| < 1$ and diverges if $|x| > 1$
- The sequence $s_n = 1/n^p$ converges to 0 if $p > 0$

Example 5 Do the following sequences converge or diverge? If a sequence converges, find its limit.

(a) $s_n = (0.8)^n$ (b) $s_n = \dfrac{1 - e^{-n}}{1 + e^{-n}}$ (c) $s_n = 1 + (-1)^n$

Solution

(a) Since $0.8 < 1$, the sequence converges by the first fact and the limit is 0.
(b) Since $e^{-1} < 1$, we have $\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} (e^{-1})^n = 0$ by the first fact, so $\lim_{n\to\infty} s_n = 1$.
(c) Since $(-1)^n$ alternates in sign, the sequence alternates between 0 and 2. Thus the sequence s_n diverges, since it does not get close to any fixed value.

Convergence and Bounded Sequences

A sequence s_n is *bounded* if there are numbers K and M such that $K \le s_n \le M$ for all terms. If $\lim_{n\to\infty} s_n = L$, then from some point on, the terms are bounded between $L - 1$ and $L + 1$. Thus we have the following fact:

> A convergent sequence is bounded.

On the other hand, a bounded sequence need not be convergent. In Example 5, we saw that $1+(-1)^n$ diverges, but it is bounded between 0 and 2. To ensure that a bounded sequence converges we need to rule out this sort of oscillation. The following theorem gives a condition that ensures convergence for a bounded sequence. A sequence s_n is called *monotone* if it is either increasing, that is $s_n < s_{n+1}$ for all n, or decreasing, that is $s_n > s_{n+1}$ for all n.

Theorem 9.1: Convergence of a Monotone, Bounded Sequence

If a sequence s_n is bounded and monotone, it converges.

To understand this theorem graphically, see Figure 9.2. The sequence s_n is increasing and bounded above by M, so the values of s_n must "pile up" at some number less than or equal to M. This number is the limit.[2]

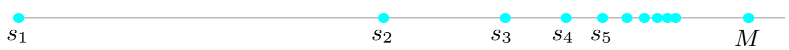


Figure 9.2: Values of s_n for $n = 1, 2, \cdots, 10$

As an example, the sequence $s_n = (1 + 1/n)^n$ can be shown to be increasing and bounded (see Project 1 on page 475). Theorem 9.1 then guarantees that this sequence has a limit, which turns out to be e. (In fact, the sequence can be used to define e.)

Example 6 If $s_n = (1 + 1/n)^n$, find s_{100} and s_{1000}. How many decimal places agree with e?

Solution We have $s_{100} = (1.01)^{100} = 2.7048$ and $s_{1000} = (1.001)^{1000} = 2.7169$. Since $e = 2.7183\ldots$,we see that s_{100} agrees with e to one decimal place and s_{1000} agrees with e to two decimal places.

Exercises and Problems for Section 9.1

Exercises

For Exercises 1–6, find the first five terms of the sequence from the formula for s_n, $n \geq 1$.

1. $2^n + 1$

2. $n + (-1)^n$

3. $\dfrac{2n}{2n+1}$

4. $(-1)^n \left(\dfrac{1}{2}\right)^n$

5. $(-1)^{n+1} \left(\dfrac{1}{2}\right)^{n-1}$

6. $\left(1 - \dfrac{1}{n+1}\right)^{n+1}$

In Exercises 7–12, find a formula for s_n, $n \geq 1$.

7. $4, 8, 16, 32, 64, \ldots$

8. $1, 3, 7, 15, 31, \ldots$

9. $2, 5, 10, 17, 26, \ldots$

10. $1, -3, 5, -7, 9, \ldots$

11. $1/3, 2/5, 3/7, 4/9, 5/11, \ldots$

12. $1/2, -1/4, 1/6, -1/8, 1/10, \ldots$

In Exercises 13–16, find the first six terms of the recursively defined sequence.

13. $s_n = s_{n-1} + n$ for $n > 1$ and $s_1 = 1$

14. $s_n = 2s_{n-1} + 3$ for $n > 1$ and $s_1 = 1$

15. $s_n = s_{n-1} + \left(\frac{1}{2}\right)^{n-1}$ for $n > 1$ and $s_1 = 0$

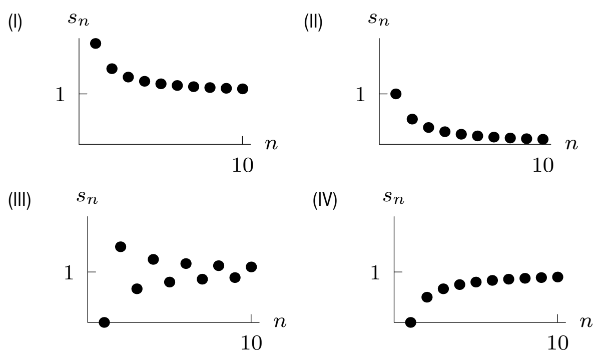
16. $s_n = s_{n-1} + 2s_{n-2}$ for $n > 2$ and $s_1 = 1$, $s_2 = 5$

[2]See the online supplement for a proof.

Problems

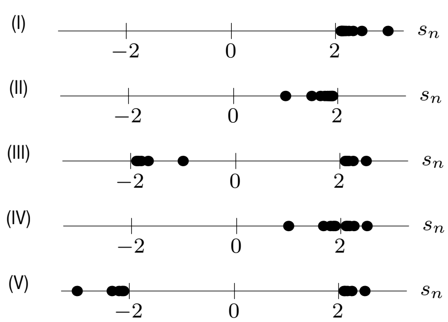
17. Match formulas (a)–(d) with graphs (I)–(IV).

(a) $s_n = 1 - 1/n$ **(b)** $s_n = 1 + (-1)^n/n$
(c) $s_n = 1/n$ **(d)** $s_n = 1 + 1/n$



18. Match formulas (a)–(e) with graphs (I)–(V).

(a) $s_n = 2 - 1/n$
(b) $s_n = (-1)^n 2 + 1/n$
(c) $s_n = 2 + (-1)^n/n$
(d) $s_n = 2 + 1/n$
(e) $s_n = (-1)^n 2 + (-1)^n/n$



19. Match formulas (a)–(e) with descriptions (I)–(V) of the behavior of the sequence as $n \to \infty$.

(a) $s_n = n(n+1) - 1$
(b) $s_n = 1/(n+1)$
(c) $s_n = 1 - n^2$
(d) $s_n = \cos(1/n)$
(e) $s_n = (\sin n)/n$

(I) Diverges to $-\infty$
(II) Diverges to $+\infty$
(III) Converges to 0 through positive numbers
(IV) Converges to 1
(V) Converges to 0 through positive and negative numbers

Do the sequences in Problems 20–31 converge or diverge? If a sequence converges, find its limit.

20. $(0.2)^n$
21. 2^n
22. $(-0.3)^n$
23. $3 + e^{-2n}$
24. $\dfrac{2^n}{3^n}$
25. $\dfrac{n}{10} + \dfrac{10}{n}$
26. $\dfrac{(-1)^n}{n}$
27. $\dfrac{2n+1}{n}$
28. $\cos(\pi n)$
29. $\dfrac{\sin n}{n}$
30. $\dfrac{2n + (-1)^n 5}{4n - (-1)^n 3}$
31. $\dfrac{2^n}{n^3}$

In electrical engineering, a continuous function like $f(t) = \sin t$, where t is time in seconds, is referred to as an analog signal. To digitize the signal, we sample $f(t)$ every Δt seconds to form the sequence $s_n = f(n\Delta t)$. For example, sampling f every 1/10 second produces the sequence $\sin(1/10), \sin(2/10), \sin(3/10), \ldots$. In Problems 32–34, give the first 6 terms of a sampling of the signal every Δt seconds.

32. $f(t) = \cos 5t$, $\Delta t = 0.1$
33. $f(t) = (x-1)^2$, $\Delta t = 0.5$
34. $f(t) = \dfrac{\sin t}{t}$, $\Delta t = 1$

To smooth a sequence, $s_1, s_2, s_3, \ldots$, we replace each term s_n by t_n, the average of s_n with its neighboring terms

$$t_n = \frac{(s_{n-1} + s_n + s_{n+1})}{3} \text{ for } n > 1.$$

We start with $t_1 = (s_1 + s_2)/2$, since s_1 has only one neighbor. For Problems 35–37, smooth the sequence once and then smooth the resulting sequence. What do you notice?

35. 18, −18, 18, −18, 18, −18, 18 . . .
36. 0, 0, 0, 18, 0, 0, 0, 0 . . .
37. 1, 2, 3, 4, 5, 6, 7, 8 . . .
38. Let V_n be the number of new SUVs sold in the US in month n, where $n = 1$ is January 2004. In terms of SUVs, what do the following represent?

(a) V_{10}
(b) $V_n - V_{n-1}$
(c) $\sum_{i=1}^{12} V_i$ and $\sum_{i=1}^{n} V_i$

[3]www.bp.com, accessed May 28, 2004.

39. World oil consumption was 75.747 million barrels per day in 2002 and is increasing by about 0.3% per year.[3] Let c_n be daily world oil consumption n years after 2002.

(a) Find a formula for c_n.
(b) Find and interpret $c_n - c_{n-1}$.
(c) What does the sum $\sum_{n=1}^{18} c_n$ represent? (You do not need to compute this sum.)

40. **(a)** Let s_n be the number of ancestors a person has n generations ago. What is s_1? s_2? Find a formula for s_n.
(b) For which n is s_n greater than 6 billion, the current world population? What does this tell you about your ancestors?

41. For $0 \leq n \leq 10$, find a formula for p_n, the payment in year n on a loan of \$100,000. Interest is 5% per year, compounded annually, and payments are made at the end of each year for ten years. Each payment is \$10,000 plus the interest on the amount of money outstanding.

42. Baby formula can contain bacteria which double in number every half hour at room temperature and every 10 hours in the refrigerator.[4] Suppose there are B_0 bacteria initially.

(a) Write a formula for
(i) R_n, the number of bacteria n hours later if the baby formula is kept at room temperature.
(ii) F_n, the number of bacteria n hours later if the baby formula is kept in the refrigerator.
(iii) Y_n, the ratio of the number of bacteria at room temperature to the number of bacteria in the refrigerator.

(b) How many hours does it take before there are a million times as many bacteria in baby formula kept at room temperature as in baby formula kept in the refrigerator?

43. You are deciding whether to buy a new or a two-year-old car (of the same make) based on which will have cost you less when you resell it at the end of three years. Your cost consists of two parts: the loss in value of the car and the repairs. A new car costs \$20,000 and loses 12% of its value each year. Repairs are \$400 the first year and increase by 18% each subsequent year.

(a) For a new car, find the first three terms of the sequence d_n giving the depreciation (loss of value) in dollars in year n. Give a formula for d_n.
(b) Find the first three terms of the sequence r_n, the repair cost in dollars for a new car in year n. Give a formula for r_n.
(c) Find the total cost of owning a new car for three years.
(d) Find the total cost of owning the two-year-old car for three years. Which should you buy?

44. Write a definition for $\lim_{n\to\infty} s_n = L$ similar to the ϵ, δ definition for $\lim_{x\to a} f(x) = L$ in Section 1.8. Instead of δ, you will need N, a value of n.

45. The sequence s_n is increasing, the sequence t_n converges, and $s_n \leq t_n$ for all n. Show that s_n converges.

In Exercises 46–51, find a recursive definition for the sequence.

46. $1, 3, 5, 7, 9, \ldots$

47. $2, 4, 6, 8, 10, \ldots$

48. $3, 5, 9, 17, 33, \ldots$

49. $1, 5, 14, 30, 55, \ldots$

50. $1, 3, 6, 10, 15, \ldots$

51. $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$

In Problems 52–54, show that the sequence s_n satisfies the recurrence relation.

52. $s_n = 3n - 2$
$s_n = s_{n-1} + 3$ for $n > 1$ and $s_1 = 1$

53. $s_n = n(n+1)/2$
$s_n = s_{n-1} + n$ for $n > 1$ and $s_1 = 1$

54. $s_n = 2n^2 - n$
$s_n = s_{n-1} + 4n - 3$ for $n > 1$ and $s_1 = 1$

55. **(a)** Cans are stacked in a triangle on a shelf. The bottom row contains k cans, the row above contains one can fewer, and so on. How many rows are there? Find a_n, the number of cans in the n^{th} row from the top, $1 \leq n \leq k$.
(b) Let T_n be the total number of cans in the top n rows. Find a recurrence relation for T_n in terms of T_{n-1}.
(c) Show that $T_n = \frac{1}{2}n(n+1)$ satisfies the recurrence relation.

56. The Fibonacci sequence first studied by the thirteenth century Italian mathematician Leonardo di Pisa, also known as Fibonacci, is defined recursively by

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2 \text{ and } F_1 = 1, F_2 = 1.$$

The Fibonacci sequence occurs in many branches of mathematics and can be found in patterns of plant growth (phyllotaxis).

(a) Find the first 12 terms.
(b) Show that the sequence of successive ratios F_{n+1}/F_n appears to converge to a number r satisfying the equation $r^2 = r + 1$. (The number r was known as the golden ratio to the ancient Greeks.)
(c) Let r satisfy $r^2 = r + 1$. Show that the sequence $s_n = Ar^n$, where A is constant, satisfies the Fibonacci equation $s_n = s_{n-1} + s_{n-2}$ for $n > 2$.

[4] Iverson, C. and Forsythe, F., reported in "Baby Food Could Trigger Meningitis," www.newscientist.com, June 3, 2004.

For a function f, define a sequence recursively by $x_n = f(x_{n-1})$ for $n > 1$ and $x_1 = a$. Depending on f and the starting value a, this sequence may converge to a limit L. If L exists, it has the property that $f(L) = L$. For the functions and starting values in Problems 57–58, use a calculator to see if the sequence converges. [To obtain the terms of the sequence, repeatedly push the function button.]

57. $f(x) = \cos x$, $a = 0$

58. $f(x) = e^{-x}$, $a = 0$

9.2 GEOMETRIC SERIES

This section introduces infinite series of constants, which are sums of the form

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
$$0.4 + 0.04 + 0.004 + 0.0004 + \cdots .$$

The individual numbers, $1, \frac{1}{2}, \frac{1}{3}, \cdots$, or $0.4, 0.04, \cdots$, etc., are called *terms* in the series. To talk about the *sum* of the series, we must first explain how to add infinitely many numbers.

Let us look at the repeated administration of a drug. In this example, the terms in the series represent each dose; the sum of the series represents the drug level in the body in the long run.

Repeated Drug Dosage

A person with an ear infection is told to take antibiotic tablets regularly for several days. Since the drug is being excreted by the body between doses, how can we calculate the quantity of the drug remaining in the body at any particular time?

To be specific, let's suppose the drug is ampicillin (a common antibiotic) taken in 250 mg doses four times a day (that is, every six hours). It is known that at the end of six hours, about 4% of the drug is still in the body. What quantity of the drug is in the body right after the tenth tablet? The fortieth?

Let Q_n represent the quantity, in milligrams, of ampicillin in the blood right after the n^{th} tablet. Then

$$Q_1 = 250 \qquad = 250 \text{ mg}$$

$$Q_2 = \underbrace{250(0.04)}_{\text{Remnants of first tablet}} + \underbrace{250}_{\text{New tablet}} \qquad = 260 \text{ mg}$$

$$\begin{aligned} Q_3 &= Q_2(0.04) + 250 = (250(0.04) + 250)\,(0.04) + 250 \\ &= \underbrace{250(0.04)^2 + 250(0.04)}_{\text{Remnants of first and second tablets}} + \underbrace{250}_{\text{New tablet}} \qquad = 260.4 \text{ mg} \end{aligned}$$

$$\begin{aligned} Q_4 &= Q_3(0.04) + 250 = \left(250(0.04)^2 + 250(0.04) + 250\right)(0.04) + 250 \\ &= \underbrace{250(0.04)^3 + 250(0.04)^2 + 250(0.04)}_{\text{Remnants of first, second, and third tablets}} + \underbrace{250}_{\text{New tablet}} \qquad = 260.416 \text{ mg.} \end{aligned}$$

Looking at the pattern that is emerging, we guess that

$$Q_5 = 250(0.04)^4 + 250(0.04)^3 + 250(0.04)^2 + 250(0.04) + 250$$
$$Q_{10} = 250(0.04)^9 + 250(0.04)^8 + \cdots + 250(0.04) + 250.$$

Notice that there are 10 terms in this sum—one for every tablet—but that the highest power of 0.04 is the ninth, because no tablet has been in the body for more than 9 six-hour time periods. (Do you see why?) Now suppose we actually want to find the numerical value of Q_{10}. It seems that we have to add 10 terms—and if we want the value of Q_{40}, we would be faced with adding 40 terms:

$$Q_{40} = 250(0.04)^{39} + 250(0.04)^{38} + \cdots + 250(0.04) + 250.$$

Fortunately, there's a better way. Let's start with Q_{10}.

$$Q_{10} = 250(0.04)^9 + 250(0.04)^8 + 250(0.04)^7 + \cdots + 250(0.04)^2 + 250(0.04) + 250.$$

Notice the remarkable fact that if you subtract $(0.04)Q_{10}$ from Q_{10}, a great many terms (all but two, in fact) drop out. First multiplying by 0.04, we get

$$(0.04)Q_{10} = 250(0.04)^{10} + 250(0.04)^9 + 250(0.04)^8 + \cdots + 250(0.04)^3 + 250(0.04)^2 + 250(0.04).$$

Subtracting gives

$$Q_{10} - (0.04)Q_{10} = 250 - 250(0.04)^{10}.$$

Factoring Q_{10} on the left and solving for Q_{10} gives

$$Q_{10}(1 - 0.04) = 250\left(1 - (0.04)^{10}\right)$$

$$Q_{10} = \frac{250\left(1 - (0.04)^{10}\right)}{1 - 0.04}.$$

This is called the *closed-form* expression for Q_{10}. It is easy to evaluate on a calculator, giving $Q_{10} = 260.42$ (to two decimal places). Similarly, Q_{40} is given in closed-form by

$$Q_{40} = \frac{250\left(1 - (0.04)^{40}\right)}{1 - 0.04}.$$

Evaluating this on a calculator shows $Q_{40} = 260.42$, which is the same (to two decimal places) as Q_{10}. Thus after ten tablets, the value of Q_n appears to have stabilized at just over 260 mg.

Looking at the closed-forms for Q_{10} and Q_{40}, we can see that, in general, Q_n must be given by

$$Q_n = \frac{250\left(1 - (0.04)^n\right)}{1 - 0.04}.$$

What Happens as $n \to \infty$?

What does this closed-form for Q_n predict about the long-run level of ampicillin in the body? As $n \to \infty$, the quantity $(0.04)^n \to 0$. In the long run, assuming that 250 mg continue to be taken every six hours, the level right after a tablet is taken is given by

$$Q_n = \frac{250\left(1 - (0.04)^n\right)}{1 - 0.04} \to \frac{250(1 - 0)}{1 - 0.04} = 260.42.$$

The Geometric Series in General

In the previous example we encountered sums of the form $a + ax + ax^2 + \cdots + ax^8 + ax^9$ (with $a = 250$ and $x = 0.04$). Such a sum is called a finite *geometric series*. A geometric series is one in which each term is a constant multiple of the one before. The first term is a, and the constant multiplier, or *common ratio* of successive terms, is x. (In our example, $a = 250$ and $x = 0.04$.)

A **finite geometric series** has the form

$$a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1}.$$

An **infinite geometric series** has the form

$$a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1} + ax^n + \cdots.$$

The "$\cdots$" at the end of the second series tells us that the series is going on forever—in other words, that it is infinite.

Sum of a Finite Geometric Series

The same procedure that enabled us to find the closed-form for Q_{10} can be used to find the sum of any finite geometric series. Suppose we write S_n for the sum of the first n terms, which means up to the term containing x^{n-1}:

$$S_n = a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1}.$$

Multiply S_n by x:

$$xS_n = ax + ax^2 + ax^3 + \cdots + ax^{n-1} + ax^n.$$

Now subtract xS_n from S_n, which cancels out all terms except for two, giving

$$S_n - xS_n = a - ax^n$$
$$(1-x)S_n = a(1-x^n).$$

Provided $x \neq 1$, we can solve to find a closed form for S_n as follows:

The **sum of a finite geometric series** is given by

$$S_n = a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}, \qquad \text{provided } x \neq 1.$$

Note that the value of n in the formula for S_n is the number of terms in the sum S_n.

Sum of an Infinite Geometric Series

In the ampicillin example, we found the sum Q_n and then let $n \to \infty$. We do the same here. The sum Q_n, which shows the effect of the first n doses, is an example of a *partial sum*. The first three partial sums of the series $a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots$ are

$$S_1 = a$$
$$S_2 = a + ax$$
$$S_3 = a + ax + ax^2.$$

To find the sum, S, of this infinite series, we consider the partial sum, S_n, of the first n terms. The formula for the sum of a finite geometric series gives

$$S_n = a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}.$$

What happens to S_n as $n \to \infty$? It depends on the value of x. If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-x^n)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x}.$$

Thus, provided $|x| < 1$, as $n \to \infty$ the partial sums S_n approach a limit of $a/(1-x)$. When this happens, we define the sum of the infinite geometric series to be that limit and say the series *converges* to $a/(1-x)$.

For $|x| < 1$, the **sum of the infinite geometric series** is given by

$$S = a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots = \frac{a}{1-x}.$$

If, on the other hand, $|x| > 1$, then x^n and the partial sums have no limit as $n \to \infty$ (if $a \neq 0$). In this case, we say the series *diverges*. If $x > 1$, the terms in the series become larger and larger in magnitude, and the partial sums diverge to $+\infty$ (if $a > 0$) or $-\infty$ (if $a < 0$). When $x < -1$, the terms become larger in magnitude, the partial sums oscillate as $n \to \infty$, and the series diverges.

What happens when $x = 1$? The series is

$$a + a + a + a + \cdots,$$

and if $a \neq 0$, the partial sums grow without bound, and the series does not converge. When $x = -1$, the series is

$$a - a + a - a + a - \cdots,$$

and, if $a \neq 0$, the partial sums oscillate between a and 0, and the series does not converge.

Example 1 For each of the following infinite geometric series, find several partial sums and the sum (if it exists).

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ (b) $1+2+4+8+\cdots$ (c) $6-2+\frac{2}{3}-\frac{2}{9}+\frac{2}{27}-\cdots$

Solution (a) This series may be written

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots$$

which we can identify as a geometric series with $a=1$ and $x=\frac{1}{2}$, so $S=\frac{1}{1-(1/2)}=2$. Let's check this by finding the partial sums:

$$\begin{aligned}
S_1 &= 1\\
S_2 &= 1+\frac{1}{2}=\frac{3}{2}=2-\frac{1}{2}\\
S_3 &= 1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}=2-\frac{1}{4}\\
S_4 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}=2-\frac{1}{8}\\
S_5 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{31}{16}=2-\frac{1}{16}.
\end{aligned}$$

The formula for S_n gives

$$S_n=\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=2-\left(\frac{1}{2}\right)^{n-1}.$$

Thus, the partial sums are creeping up to the value $S=2$, so $S_n\to 2$ as $n\to\infty$.

(b) The partial sums of this geometric series (with $a=1$ and $x=2$) grow without bound, so the series has no sum:

$$\begin{aligned}
S_1 &= 1\\
S_2 &= 1+2=3\\
S_3 &= 1+2+4=7\\
S_4 &= 1+2+4+8=15\\
S_5 &= 1+2+4+8+16=31.
\end{aligned}$$

The formula for S_n gives

$$S_n=\frac{1-2^n}{1-2}=2^n-1.$$

(c) This is an infinite geometric series with $a=6$ and $x=-\frac{1}{3}$. The partial sums,

$$S_1=6.00,\quad S_2=4.00,\quad S_3\approx 4.67,\quad S_4\approx 4.44,\quad S_5\approx 4.52,\quad S_6\approx 4.49,$$

appear to be converging to 4.5. This turns out to be correct because the sum is

$$S=\frac{6}{1-(-1/3)}=4.5.$$

Regular Deposits into a Savings Account

People who save money often do so by putting some fixed amount aside regularly. To be specific, suppose \$1000 is deposited every year in a savings account earning 5% a year, compounded annually. What is the balance, B_n, in dollars, in the savings account right after the n^{th} deposit?

As before, let's start by looking at the first few years:

$$B_1 = 1000$$
$$B_2 = B_1(1.05) + 1000 = \underbrace{1000(1.05)}_{\text{Original deposit}} + \underbrace{1000}_{\text{New deposit}}$$
$$B_3 = B_2(1.05) + 1000 = \underbrace{1000(1.05)^2 + 1000(1.05)}_{\text{First two deposits}} + \underbrace{1000}_{\text{New deposit}}$$
$$B_4 = B_3(1.05) + 1000 = \underbrace{1000(1.05)^3 + 1000(1.05)^2 + 1000(1.05)}_{\text{First three deposits}} + \underbrace{1000}_{\text{New deposit}}$$

Observing the pattern, we see

$$B_n = 1000(1.05)^{n-1} + 1000(1.05)^{n-2} + \cdots + 1000(1.05) + 1000.$$

So B_n is a finite geometric series with $a = 1000$ and $x = 1.05$. Thus we have

$$B_n = \frac{1000\left(1 - (1.05)^n\right)}{1 - 1.05}.$$

We can rewrite this so that both the numerator and denominator of the fraction are positive:

$$B_n = \frac{1000\left((1.05)^n - 1\right)}{1.05 - 1}.$$

What Happens as $n \to \infty$?

Common sense tells you that if you keep depositing \$1000 in an account and it keeps earning interest, your balance grows without bound. This is what the formula for B_n shows also: $(1.05)^n \to \infty$ as $n \to \infty$, so B_n has no limit. (Alternatively, observe that the infinite geometric series of which B_n is a partial sum has $x = 1.05$, which is greater than 1, so the series does not converge.)

Exercises and Problems for Section 9.2

Exercises

In Exercises 1–10, decide which of the following are geometric series. For those which are, give the first term and the ratio between successive terms. For those which are not, explain why not.

1. $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$

3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

4. $5 - 10 + 20 - 40 + 80 - \cdots$

5. $1 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$

6. $3 + 3z + 6z^2 + 9z^3 + 12z^4 + \cdots$

7. $1 + 2z + (2z)^2 + (2z)^3 + \cdots$

8. $y^2 + y^3 + y^4 + y^5 + \cdots$

9. $1 - x + x^2 - x^3 + x^4 - \cdots$

10. $1 - y^2 + y^4 - y^6 + \cdots$

11. Find the sum of the series in Exercise 7.

12. Find the sum of the series in Exercise 8.

13. Find the sum of the series in Exercise 9.

14. Find the sum of the series in Exercise 10.

For each finite geometric series in Exercises 15–17, say how many terms are in the series and find its sum.

15. $2 + 2(0.1) + 2(0.1)^2 + \cdots + 2(0.1)^{25}$

16. $2(0.1) + 2(0.1)^2 + \cdots + 2(0.1)^{10}$

17. $2(0.1)^5 + 2(0.1)^6 + \cdots + 2(0.1)^{13}$

Find the sum of the series in Exercises 18–21.

18. $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

19. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^{10}}$

20. $\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$

21. $\sum_{n=4}^{20} \left(\frac{1}{3}\right)^n$

Problems

22. Every term of a geometric series (finite or infinite) is multiplied by the same nonzero constant c. Is the new series also a geometric series? Explain.

23. A new series is obtained from a geometric series (finite or infinite) by taking the reciprocal of each term. Is the new series that results also a geometric series? Explain.

24. This problem shows another way of deriving the long-run ampicillin level. (See page 444.) In the long run the ampicillin levels off to Q mg right after each tablet is taken. Six hours later, right before the next dose, there will be less ampicillin in the body. However, if stability has been reached, the amount of ampicillin that has been excreted is exactly 250 mg because taking one more tablet raises the level back to Q mg. Use this to solve for Q.

25. Figure 9.3 shows the quantity of the drug atenolol in the blood as a function of time, with the first dose at time $t = 0$. Atenolol is taken in 50 mg doses once a day to lower blood pressure.

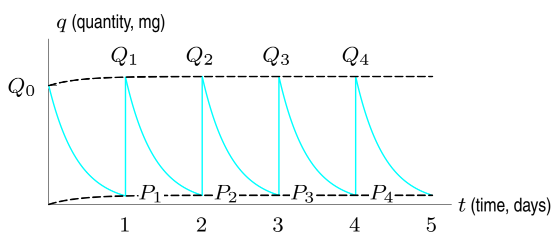


Figure 9.3

(a) If the half-life of atenolol in the blood is 6.3 hours, what percentage of the atenolol present at the start of a 24-hour period is still there at the end?

(b) Find expressions for the quantities $Q_0, Q_1, Q_2, Q_3, \ldots$, and Q_n shown in Figure 9.3. Write the expression for Q_n in closed-form.

(c) Find expressions for the quantities $P_1, P_2, P_3, \ldots$, and P_n shown in Figure 9.3. Write the expression for P_n in closed-form.

26. On page 444, you saw how to compute the quantity Q_n mg of ampicillin in the body right after the n^{th} tablet of 250 mg, taken once every six hours.

(a) Do a similar calculation for P_n, the quantity of ampicillin (in mg) in the body right *before* the n^{th} tablet is taken.

(b) Express P_n in closed form.

(c) What is $\lim_{n\to\infty} P_n$? Is this limit the same as $\lim_{n\to\infty} Q_n$? Explain in practical terms why your answer makes sense.

27. Draw a graph like that in Figure 9.3 for 250 mg of ampicillin taken every 6 hours, starting at time $t = 0$. Put on the graph the values of $Q_1, Q_2, Q_3, \ldots$ introduced in the text on page 444 and the values of $P_1, P_2, P_3, \ldots$ calculated in Problem 26.

28. A ball is dropped from a height of 10 feet and bounces. Each bounce is $\frac{3}{4}$ of the height of the bounce before. Thus, after the ball hits the floor for the first time, the ball rises to a height of $10(\frac{3}{4}) = 7.5$ feet, and after it hits the floor for the second time, it rises to a height of $7.5(\frac{3}{4}) = 10(\frac{3}{4})^2 = 5.625$ feet. (Assume $g = 32$ ft/sec^2 and that there is no air resistance.)

(a) Find an expression for the height to which the ball rises after it hits the floor for the n^{th} time.

(b) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the first, second, third, and fourth times.

(c) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the n^{th} time. Express your answer in closed-form.

29. You might think that the ball in Problem 28 keeps bouncing forever since it takes infinitely many bounces. This is not true!

(a) Show that a ball dropped from a height of h feet reaches the ground in $\frac{1}{4}\sqrt{h}$ seconds.

(b) Show that the ball in Problem 28 stops bouncing after

$$\frac{1}{4}\sqrt{10}+\frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}}\left(\frac{1}{1-\sqrt{3/4}}\right) \approx 11 \text{ seconds.}$$

30. One way of valuing a company is to calculate the present value of all its future earnings. Suppose a farm expects to sell \$1000 worth of Christmas trees once a year forever, with the first sale in the immediate future. What is the present value of this Christmas tree business? Assume that the interest rate is 4% per year, compounded continuously.

31. This problem deals with the question of estimating the cumulative effect of a tax cut on a country's economy. Suppose the government proposes a tax cut totaling \$100 million. We assume that all the people who have extra money to spend would spend 80% of it and save 20%. Thus, of the extra income generated by the tax cut, \$100(0.8) million = \$80 million would be spent and so become extra income to someone else. Assume that these people also spend 80% of their additional income, or \$80(0.8) million, and so on. Calculate the total additional spending created by such a tax cut.

32. Suppose the government proposes a tax cut of \$100 million as in Problem 31, but that economists now predict that people will spend 90% of their extra income and save only 10%. How much additional spending would be generated by the tax cut under these assumptions?

9.3 CONVERGENCE OF SERIES

We now consider general series in which each term a_n is a number. The series can be written compactly using a $\sum$ sign as follows

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

For any particular values of a and x, the geometric series is such a series, with general term $a_n = ax^{n-1}$.

Partial Sums and Convergence of Series

As in Section 9.2, we define the *partial sum*, S_n, of the first n terms of a series as

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

To investigate the convergence of the series, we consider the sequence of partial sums

$$S_1, S_2, S_3, \ldots, S_n, \ldots.$$

If S_n has a limit as $n \to \infty$, then we define the sum of the series to be that limit.

If the sequence S_n of partial sums converges to S, so $\lim_{n\to\infty} S_n = S$, then we say the series $\sum_{n=1}^{\infty} a_n$ **converges** and that its sum is S. We write $\sum_{n=1}^{\infty} a_n = S$. If $\lim_{n\to\infty} S_n$ does not exist, we say that the series **diverges**.

Visualizing Series

We can visualize the terms of a series as in Figure 9.4. In this figure, we assume $a_n \geq 0$ for all n, so each rectangle has area a_n. Then the series converges if the total area of the rectangles is finite and the sum of the series is the total area of the rectangles. This is similar to an improper integral $\int_0^\infty f(x)\,dx$, in which the area under the graph of f can be finite, even on an infinite interval.

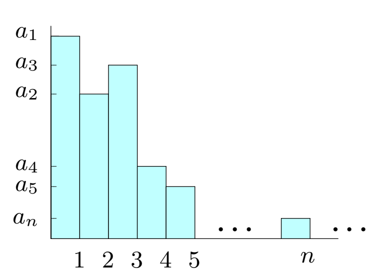


Figure 9.4: Height and area of the n^{th} rectangle is a_n

Here are some properties that are useful in determining whether or not a series converges.

Theorem 9.2: Convergence Properties of Series

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and if k is a constant, then
   - $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
   - $\sum_{n=1}^{\infty} ka_n$ converges to $k\sum_{n=1}^{\infty} a_n$.
2. Changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.
3. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
4. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ diverges if $k \neq 0$.

For proofs of these properties, see Problems 26–29. As for improper integrals, the convergence of a series is determined by its behavior for large n. (See the "behaves like" principle on page 357.) From Property 2 we see that, if N is a positive integer, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=N}^{\infty} a_n$ either both converge or both diverge. Thus, if all we care about is the convergence of a series, we can omit the limits and write $\sum a_n$.

Example 1 Does the series $\sum(1 - e^{-n})$ converge?

Solution Since the terms in the series, $a_n = 1 - e^{-n}$ tend to 1, not 0, as $n \to \infty$, the series diverges by Property 3 of Theorem 9.2.

Comparison of Series and Integrals

We investigate the convergence of some series by comparison with an improper integral. The *harmonic series* is the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

Convergence of this sum would mean that the sequence of partial sums

$$S_1 = 1, \quad S_2 = 1 + \frac{1}{2}, \quad S_3 = 1 + \frac{1}{2} + \frac{1}{3}, \quad \ldots, \quad S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad \cdots$$

tends to a limit as $n \to \infty$. Let's look at some values:

$$S_1 = 1, \quad S_{10} \approx 2.93, \quad S_{100} \approx 5.19, \quad S_{1000} \approx 7.49, \quad S_{10000} \approx 9.79.$$

The growth of these partial sums is slow, but they do in fact grow without bound, so the harmonic series diverges. This is justified in the following example and in Problem 33.

Example 2 Show that the harmonic series $1 + 1/2 + 1/3 + 1/4 + \ldots$ diverges.

Solution The idea is to approximate $\int_1^\infty (1/x)\,dx$ by a left-hand sum, where the terms $1, 1/2, 1/3, \ldots$ are heights of rectangles of base 1. In Figure 9.5, the sum of the areas of the 3 rectangles is larger than the area under the curve between $x = 1$ and $x = 4$, and the same kind of relationship holds for the first n rectangles. Thus, we have

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1).$$

Since $\ln(n+1)$ gets arbitrarily large as $n \to \infty$, so do the partial sums, S_n. Thus, the partial sums have no limit, so the series diverges.

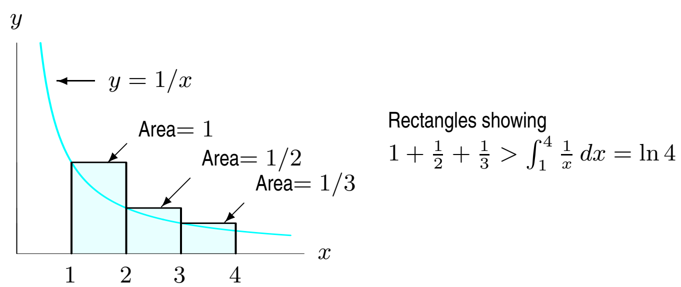


Figure 9.5: Comparing the harmonic series to $\int_1^\infty (1/x)\,dx$

Notice that the harmonic series diverges, even though $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1/n) = 0$.

Example 3 By comparison with the improper integral $\int_1^\infty (1/x^2)\,dx$, show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots.$$

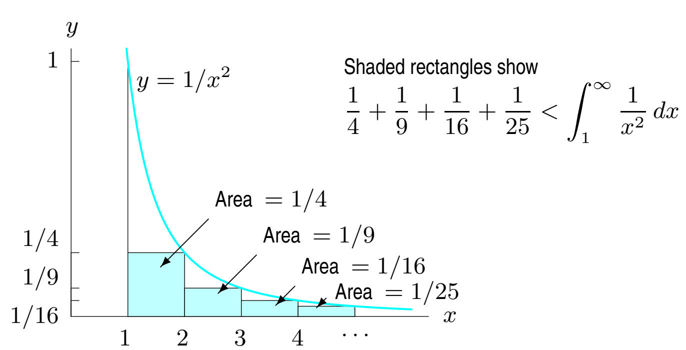


Figure 9.6: Comparing $\sum_{n=1}^{\infty} 1/n^2$ to $\int_1^\infty (1/x^2)\,dx$

Solution Since we want to show that $\sum_{n=1}^{\infty} 1/n^2$ converges, we want to show that the partial sums of this series tend to a limit. We do this by showing that the sequence of partial sums increases and is bounded above, so Theorem 9.1 applies.

Each successive partial sum is obtained from the previous one by adding one more term in the series. Since all the terms are positive, the sequence of partial sums is increasing.

To show that the partial sums of $\sum_{n=1}^{\infty} 1/n^2$ are bounded, we consider the right-hand sum represented by the area of the rectangles in Figure 9.6. We start at $x = 1$, since the area under the curve is infinite for $0 \leq x \leq 1$. The shaded rectangles in Figure 9.6 suggest that:

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx.$$

The area under the graph is finite, since

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1.$$

To get S_n, we add 1 to both sides, giving

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2.$$

Thus, the sequence of partial sums is bounded above by 2. Hence, by Theorem 9.1 the sequence of partial sums converges, so the series converges.

Notice that we have shown that the series in the Example 3 converges, but we have not found its sum. The integral gives us a bound on the partial sums, but it does not give us the limit of the partial sums. Euler proved the remarkable fact that the sum is $\pi^2/6$.

The method of Examples 2 and 3 can be used to prove the following theorem. See Problem 32.

Theorem 9.3: The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive for $x \geq c$.

- If $\int_c^{\infty} f(x)\, dx$ converges, then $\sum a_n$ converges.
- If $\int_c^{\infty} f(x)\, dx$ diverges, then $\sum a_n$ diverges.

The integral test allows us to analyze a family of series, the p-series, and see how convergence depends on the parameter p.

Example 4 For what values of p does the series $\sum_{n=1}^{\infty} 1/n^p$ converge?

Solution If $p \leq 0$, the terms in the series $a_n = 1/n^p$ do not tend to 0 as $n \to \infty$. Thus the series diverges for $p \leq 0$.

If $p > 0$, we compare $\sum_{n=1}^{\infty} 1/n^p$ to the integral $\int_1^{\infty} 1/x^p\, dx$. In Example 3 of Section 7.7 we saw that the integral converges if $p > 1$ and diverges if $p \leq 1$. By the integral test, we conclude that $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

We can summarize Example 4 as follows:

The p**-series** $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

Exercises and Problems for Section 9.3

Exercises

Use the integral test to decide whether the series in Exercises 1–4 converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

5. Use comparison with $\int_1^{\infty} x^{-3}\,dx$ to show that $\sum_{n=2}^{\infty} 1/n^3$ converges to a number less than or equal to $1/2$.

6. Use comparison with $\int_0^{\infty} 1/(x^2+1)\,dx$ to show that $\sum_{n=1}^{\infty} 1/(n^2+1)$ converges to a number less than or equal to $\pi/2$.

Explain why the integral test cannot be used to decide if the series in Exercises 7–9 converge or diverge.

7. $\displaystyle\sum_{n=1}^{\infty} n^2$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

9. $\displaystyle\sum_{n=1}^{\infty} e^{-n}\sin n$

Problems

Do the series in Problems 10–19 converge or diverge?

10. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n+2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{\sqrt{2+n}}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(2n+1)^3}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n^2+4}$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\left(\frac{3}{4}\right)^n + \frac{1}{n}\right)$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n2^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

19. $\displaystyle\sum_{n=3}^{\infty} \frac{n+1}{n^2+2n+2}$

20. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(2^n)}$ diverges.

21. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln(2^n))^2}$ converges.

22. **(a)** Find the partial sum, S_n, of $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

(b) Does the series in part (a) converge or diverge?

23. **(a)** Show $r^{\ln n} = n^{\ln r}$ for positive numbers n and r.

(b) For what values $r > 0$ does $\sum_{n=1}^{\infty} r^{\ln n}$ converge?

24. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots$.

(a) Show that $\displaystyle\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$.

(b) Use part (a) to find the partial sums S_3, S_{10}, and S_n.

(c) Use part (b) to show that the sequence of partial sums S_n, and therefore the series, converges to 1.

25. Consider the series

$$\sum_{k=2}^{\infty} \ln\left(\frac{(k-1)(k+1)}{k^2}\right) = \ln\left(\frac{1\cdot 3}{2\cdot 2}\right) + \ln\left(\frac{2\cdot 4}{3\cdot 3}\right) + \cdots.$$

(a) Show that the partial sum $S_4 = \ln(5/8)$.

(b) Show that the partial sum $S_n = \ln\left(\dfrac{n+1}{2n}\right)$.

(c) Use part (b) to show that the partial sums S_n, and therefore the series, converge to $\ln(1/2)$.

26. Show that if $\sum a_n$ and $\sum b_n$ converge and if k is a constant, then $\sum(a_n+b_n)$, $\sum(a_n-b_n)$, and $\sum ka_n$ converge.

27. Let N be a positive integer. Show that if $a_n = b_n$ for $n \ge N$, then $\sum a_n$ and $\sum b_n$ either both converge, or both diverge.

28. Show that if $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$. [Hint: Consider $\lim_{n\to\infty}(S_n - S_{n-1})$, where S_n is the n^{th} partial sum.]

29. Show that if $\sum a_n$ diverges and $k \ne 0$, then $\sum ka_n$ diverges.

30. The series $\sum a_n$ converges. Explain, by looking at partial sums, why the series $\sum(a_{n+1} - a_n)$ also converges.

31. The series $\sum a_n$ diverges. Give examples that show the series $\sum(a_{n+1} - a_n)$ could converge or diverge.

32. In this problem, you will justify the integral test. Suppose $c \ge 0$ and $f(x)$ is a decreasing positive function, defined for all $x \ge c$, with $f(n) = a_n$ for all n.

(a) Suppose $\int_c^{\infty} f(x)\,dx$ converges. By considering rectangles under the graph of f, show that $\sum a_n$ converges. [Hint: See Example 3 on page 452.]

(b) Suppose that $\int_c^{\infty} f(x)\,dx$ diverges. By considering rectangles above the graph of f, show that $\sum a_n$ diverges. [Hint: See Example 2 on page 452.]

33. Consider the following grouping of terms in the harmonic series:

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots$$

(a) Show that the sum of each group of fractions is more than $1/2$.
(b) Explain why this shows that the harmonic series does not converge.

34. Show that $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges.

(a) Using the integral test.
(b) By considering the grouping of terms

$$\left(\frac{1}{2\ln 2}\right)+\left(\frac{1}{3\ln 3}+\frac{1}{4\ln 4}\right)+\left(\frac{1}{5\ln 5}+\frac{1}{6\ln 6}+\frac{1}{7\ln 7}+\frac{1}{8\ln 8}\right)+\cdots.$$

35. Consider the sequence given by

$$a_n=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)-\ln n.$$

(a) Show that a_n is positive for all n. [Use a left-sum approximation to $\int_1^n (1/x)\,dx$ with $\Delta x=1$.]
(b) Show that $a_{n+1}<a_n$ for all n. [Use one rectangle to give a right approximation to $\int_n^{n+1}(1/x)\,dx$.]
(c) Explain why $\lim_{n\to\infty} a_n$ exists.
(d) The number $\gamma=\lim_{n\to\infty} a_n$ is called *Euler's constant.* Estimate γ to two decimal places by computing a_{200}.

36. On page 453, we gave Euler's result

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

(a) Find the sum of the first 20 terms of this series. Give your answer to three decimal places.
(b) Use your answer to estimate π. Give your answer to two decimal places.
(c) Repeat parts (a) and (b) with 100 terms.
(d) Use a right sum approximation to bound the error in approximating $\pi^2/6$ by $\sum_{n=1}^{20}(1/n^2)$ and by $\sum_{n=1}^{100}(1/n^2)$.

37. This problem approximates e using

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}.$$

(a) Find a lower bound for e by evaluating the first five terms of the series.
(b) Show that $1/n!\le 1/2^{n-1}$ for $n\ge 1$.
(c) Find an upper bound for e using part (b).

38. In this problem we investigate how fast the partial sums $S_N=1^5+2^5+3^5+\cdots+N^5$ of the divergent series $\sum_{n=1}^{\infty} n^5$ grow as N gets larger and larger. Show that

(a) $S_N>N^6/6$ by considering the right-hand Riemann sum for $\int_0^N x^5dx$ with $\Delta x=1$.
(b) $S_N<((N+1)^6-1)/6$ by considering the left-hand Riemann sum for $\int_1^{N+1} x^5dx$ with $\Delta x=1$.
(c) $\lim_{N\to\infty} S_N/(N^6/6) = 1$. We say that S_N is asymptotic to $N^6/6$ as N goes to infinity.

9.4 TESTS FOR CONVERGENCE

Comparison of Series

In Section 7.8, we compared two integrals to decide whether an improper integral converged. In Theorem 9.3 we compared an integral and a series. Now we compare two series.

> **Theorem 9.4: Comparison Test**
>
> Suppose $0\le a_n\le b_n$ for all n.
> - If $\sum b_n$ converges, then $\sum a_n$ converges.
> - If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Since $a_n\le b_n$, the plot of the a_n lies under the plot of the b_n. (See Figure 9.7.) The comparison test says that if the total area for $\sum b_n$ is finite, then the total area for $\sum a_n$ is finite also. If the total area for $\sum a_n$ is not finite, then neither is the total area for $\sum b_n$.

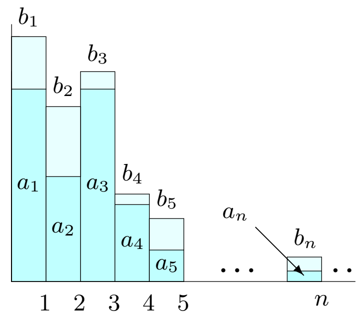


Figure 9.7: Each a_n is represented by the area of a dark rectangle, and each b_n by a dark plus a light rectangle

Example 1 Use the comparison test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges.

Solution For $n \geq 1$, we know that $n^3 \leq n^3 + 1$, so

$$0 \leq \frac{1}{n^3} \leq \frac{1}{n^3+1}.$$

Thus, every term in the series $\sum_{n=1}^{\infty} 1/(n^3+1)$ is less than or equal to the corresponding term in $\sum_{n=1}^{\infty} 1/n^3$. Since we saw that $\sum_{n=1}^{\infty} 1/n^3$ converges as a p-series with $p > 1$, we know that $\sum_{n=1}^{\infty} 1/(n^3+1)$ converges.

Example 2 Decide whether the following series converge: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$.

Solution (a) Since the convergence is determined by the behavior of the terms for large n, we observe that

$$\frac{n-1}{n^3+3} \quad \text{behaves like} \quad \frac{n}{n^3} = \frac{1}{n^2} \quad \text{as} \quad n \to \infty.$$

Since $\sum 1/n^2$ converges, we guess that $\sum (n-1)/(n^3+3)$ converges. To confirm this, we use the comparison test. Since a fraction increases if its numerator is made larger or its denominator is made smaller, we have

$$0 \leq \frac{n-1}{n^3+3} \leq \frac{n}{n^3} = \frac{1}{n^2} \quad \text{for all} \quad n \geq 1.$$

Thus, the series $\sum (n-1)/(n^3+3)$ converges by comparison with $\sum 1/n^2$.

(b) First, we observe that

$$\frac{6n^2+1}{2n^3-1} \quad \text{behaves like} \quad \frac{6n^2}{2n^3} = \frac{3}{n} \quad \text{as} \quad n \to \infty.$$

Since $\sum 1/n$ diverges, so does $\sum 3/n$, and we guess that $\sum (6n^2+1)/(2n^3-1)$ diverges. To confirm this, we use the comparison test. Since a fraction decreases if its numerator is made smaller or its denominator is made larger, we have

$$0 \leq \frac{6n^2}{2n^3} \leq \frac{6n^2+1}{2n^3-1},$$

so

$$0 \leq \frac{3}{n} \leq \frac{6n^2+1}{2n^3-1}.$$

Thus, the series $\sum (6n^2+1)/(2n^3-1)$ diverges by comparison with $\sum 3/n$.

Limit Comparison Test

The convergence or divergence of a series $\sum a_n$ is determined by the values of a_n as $n \to \infty$. This often enables us to predict convergence or divergence by looking at the long-run behavior of a_n.

Example 3 Predict convergence or divergence of

$$\sum \frac{n^2 - 5}{n^3 + n + 2}.$$

Solution As $n \to \infty$, the highest power terms in the numerator and denominator, n^2 and n^3, dominate. Thus the term

$$a_n = \frac{n^2 - 5}{n^3 + n + 2}$$

behaves, as $n \to \infty$, like

$$\frac{n^2}{n^3} = \frac{1}{n}.$$

Since the harmonic series $\sum 1/n$ diverges, we guess that $\sum a_n$ also diverges.

The comparison test is needed to confirm a prediction of convergence or divergence. Because the comparison test requires showing that $a_n \le b_n$, we may prefer to use the following test, which avoids these inequalities.

Theorem 9.5: Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \qquad \text{where } c > 0,$$

then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

The limit $\lim_{n\to\infty} a_n/b_n = c$ captures the idea that a_n "behaves like" cb_n as $n \to \infty$.

Example 4 Use the limit comparison test to determine if the following series converge or diverge.

(a) $\displaystyle\sum \frac{n^2 + 6}{n^4 - 2n + 3}$ (b) $\displaystyle\sum \sin\left(\frac{1}{n}\right)$

Solution

(a) We take $a_n = \dfrac{n^2 + 6}{n^4 - 2n + 3}$. Because a_n behaves like $\dfrac{n^2}{n^4} = \dfrac{1}{n^2}$ as $n \to \infty$ we take $b_n = 1/n^2$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 + 6n^2}{n^4 - 2n + 3} = 1.$$

The limit comparison test applies with $c = 1$. Since $\sum 1/n^2$ converges, the limit comparison test shows that $\displaystyle\sum \frac{n^2 + 6}{n^4 - 2n + 3}$ also converges.

(b) Since $\sin(x) \approx x$ for x near 0, we know that $\sin\left(\frac{1}{n}\right)$ behaves like $1/n$ as $n \to \infty$. We apply the limit comparison test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

Thus $c = 1$ and since $\sum 1/n$ diverges, the series $\sum \sin(1/n)$ also diverges.

Series of Both Positive and Negative Terms

If $\sum a_n$ has both positive and negative terms, then its plot has rectangles lying both above and below the horizontal axis. See Figure 9.8. The total area of the rectangles is no longer equal to $\sum a_n$. However, it is still true that if the total area of the rectangles above and below the axis is finite, then the series converges. The area of the n^{th} rectangle is $|a_n|$, so we have:

Theorem 9.6: Convergence of Absolute Values Implies Convergence

If $\sum |a_n|$ converges, then so does $\sum a_n$.

Problem 67 shows how to prove this result.

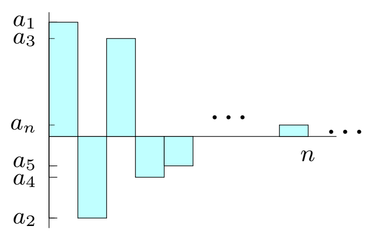


Figure 9.8: Representing a series with positive and negative terms

Example 5 Explain how we know that the following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots.$$

Solution Writing $a_n = (-1)^{n-1}/n^2$, we have

$$|a_n| = \left| \frac{(-1)^{n-1}}{n^2} \right| = \frac{1}{n^2}.$$

The p-series $\sum 1/n^2$ converges, since $p > 1$, so $\sum (-1)^{n-1}/n^2$ converges.

Comparison with a Geometric Series: The Ratio Test

A geometric series $\sum a_n$ has the property that the ratio a_{n+1}/a_n is constant for all n. For many other series, this ratio, although not constant, tends to a constant as n increases. In some ways, such series behave like geometric series. In particular, a geometric series converges if the ratio $|a_{n+1}/a_n| < 1$. A non-geometric series also converges if the ratio $|a_{n+1}/a_n|$ tends to a limit which is less than 1. This idea leads to the following test.

Theorem 9.7: The Ratio Test

For a series $\sum a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

- If $L < 1$, then $\sum a_n$ converges.
- If $L > 1$, or if L is infinite,[5] then $\sum a_n$ diverges.
- If $L = 1$, the test does not tell us anything about the convergence of $\sum a_n$.

[5] That is, the sequence $|a_{n+1}|/|a_n|$ grows without bound.

Proof Here are the main steps in the proof. Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$. Let x be a number between L and 1. Then for all sufficiently large n, say for all $n \geq k$, we have

$$\frac{|a_{n+1}|}{|a_n|} < x.$$

Then,

$$\begin{aligned} |a_{k+1}| &< |a_k|x, \\ |a_{k+2}| &< |a_{k+1}|x < |a_k|x^2, \\ |a_{k+3}| &< |a_{k+2}|x < |a_k|x^3, \end{aligned}$$

and so on. Thus, writing $a = |a_k|$, we have for $i = 1, 2, 3, \ldots,$

$$|a_{k+i}| < ax^i.$$

Now we can use the comparison test: $\sum |a_{k+i}|$ converges by comparison with the geometric series $\sum ax^i$. Since $\sum |a_{k+i}|$ converges, Theorem 9.6 tells us that $\sum a_{k+i}$ converges. So, by property 2 of Theorem 9.2, we see that $\sum a_n$ converges too.

If $L > 1$, then for sufficiently large n, say $n \geq m$,

$$|a_{n+1}| > |a_n|,$$

so the sequence $|a_m|, |a_{m+1}|, |a_{m+2}|, \cdots$, is increasing. Thus, $\lim_{n\to\infty} a_n \neq 0$, so $\sum a_n$ diverges (by Theorem 9.2, property 3). The argument in the case that $|a_{n+1}|/|a_n|$ is unbounded is similar.

Example 6 Show that the following series converges:[6]

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Solution Since $a_n = 1/n!$ and $a_{n+1} = 1/\left(n+1\right)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{(n+1)n(n-1)\cdots 2\cdot 1}.$$

We cancel $n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1$, giving

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Because the limit is 0, which is less than 1, the ratio test tells us that $\sum_{n=1}^{\infty} 1/n!$ converges. In Chapter 10, we show the limit is e.

Example 7 What does the ratio test tell us about the convergence of the following two series?

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Solution Because $|(-1)^n| = 1$, in both cases we have $\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} n/(n+1) = 1$. Thus, the ratio test does not tell us anything about the convergence of either series. In fact, the first series is the harmonic series, which diverges. Example 8 will show that the second series converges.

[6]We define $2!$ to be $2\cdot 1 = 2$. Similarly, $3! = 3\cdot 2\cdot 1 = 6$ and $n! = n\cdot(n-1)\cdots 2\cdot 1$. We also define $0! = 1$.

Alternating Series

A series is called an *alternating series* if the terms alternate in sign. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots.$$

The convergence of an alternating series can often be determined using the following test:

Theorem 9.8: Alternating Series Test

A series of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$$

converges if

$$0 < a_{n+1} < a_n \quad \text{for all } n \qquad \text{and} \qquad \lim_{n\to\infty} a_n = 0.$$

Although we do not prove this result, we can see why it is reasonable. The first partial sum, $S_1 = a_1$, is positive. The second, $S_2 = a_1 - a_2$, is still positive, since $a_2 < a_1$, but S_2 is smaller than S_1. (See Figure 9.9.) The next sum, $S_3 = a_1 - a_2 + a_3$, is greater than S_2 but smaller than S_1. The partial sums oscillate back and forth, and since the distance between them tends to 0, they eventually converge.

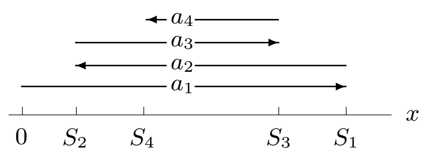


Figure 9.9: Partial sums, S_1, S_2, S_3, S_4 of an alternating series

Example 8 Show that the following alternating harmonic series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Solution We have $a_n = 1/n$ and $a_{n+1} = 1/(n+1)$. Thus,

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n \quad \text{for all } n, \quad \text{and} \quad \lim_{n\to\infty} 1/n = 0.$$

Thus, the hypothesis of Theorem 9.8 is satisfied, so the alternating harmonic series converges.

Suppose S is the sum of an alternating series, so $S = \lim_{n\to\infty} S_n$. Then S is trapped between any two consecutive partial sums, say S_3 and S_4 or S_4 and S_5 so

$$S_2 < S_4 < \cdots < S < \cdots < S_3 < S_1.$$

Thus, the error in using S_n to approximate the true sum S is less than the distance from S_n to S_{n+1}, which is a_{n+1}. Stated symbolically, we have the following result:

Theorem 9.9: Error Bounds for Alternating Series

Let $S_n = \sum_{i=1}^{n}(-1)^{i-1}a_i$ be the n^{th} partial sum of an alternating series and let $S = \lim_{n\to\infty} S_n$. Suppose that $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$. Then

$$|S - S_n| < a_{n+1}.$$

Thus, provided S_n converges to S by the alternating series test, the error in using S_n to approximate S is less than the magnitude of the first term of the series which is omitted in the approximation.

Example 9 Estimate the error in approximating the sum of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ by the sum of the first nine terms.

Solution The ninth partial sum is given by

$$S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{9} = 0.7456\ldots.$$

The first term omitted is $-1/10$, with magnitude 0.1. By Theorem 9.9, we know that the true value of the sum differs from $0.7456\ldots$ by less than 0.1.

Absolute and Conditional Convergence

We say that the series $\sum a_n$ is

- **absolutely convergent** if $\sum a_n$ and $\sum |a_n|$ both converge.
- **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Conditionally convergent series rely on cancelation between positive and negative terms for their convergence.

Example: The series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent because the series converges and the p-series $\sum 1/n^2$ also converges.

Example: The series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is conditionally convergent because the series converges but the harmonic series $\sum 1/n$ diverges.

Exercises and Problems for Section 9.4

Exercises

Use the comparison test to confirm the statements in Exercises 1–3.

1. $\sum_{n=4}^{\infty}\frac{1}{n}$ diverges, so $\sum_{n=4}^{\infty}\frac{1}{n-3}$ diverges.

2. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty}\frac{1}{n^2+2}$ converges.

3. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty}\frac{e^{-n}}{n^2}$ converges.

Use the comparison test to determine whether the series in Exercises 4–9 converge.

4. $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$

5. $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$

6. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

7. $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$

8. $\sum_{n=1}^{\infty} \frac{n\sin n}{n^3+1}.$

9. $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$

Use the ratio test to decide if the series in Exercises 10–15 converge or diverge.

10. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

11. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

12. $\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$

13. $\sum_{n=1}^{\infty} \frac{1}{r^n n!}, r>0$

14. $\sum_{n=1}^{\infty} \frac{1}{ne^n}$

15. $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$

Which of the the series in Exercises 16–19 are alternating?

16. $\sum_{n=1}^{\infty} (-1)^n \left(2-\frac{1}{n}\right)$

17. $\sum_{n=1}^{\infty} \cos(n\pi)$

18. $\sum_{n=1}^{\infty} (-1)^n \cos(n\pi)$

19. $\sum_{n=1}^{\infty} (-1)^n \cos n$

Use the alternating series test to show that the series in Exercises 20–23 converge.

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}$

In Exercises 24–31, use the limit comparison test to determine whether the series converges or diverges.

24. $\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$, by comparing to $\sum_{n=1}^{\infty} \frac{1}{n}$

25. $\sum_{n=1}^{\infty} \left(\frac{1+n}{3n}\right)^n$, by comparing to $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

[Hint: $\lim_{n\to\infty}(1+1/n)^n = e$.]

26. $\sum \frac{1}{n^4-7}$

27. $\sum \frac{n+1}{n^2+2}$

28. $\sum \frac{n^3-2n^2+n+1}{n^4-2}$

29. $\sum \frac{2^n}{3^n-1}$

30. $\sum \frac{1}{2\sqrt{n}+\sqrt{n+2}}$

31. $\sum \left(\frac{1}{2n-1}-\frac{1}{2n}\right)$

Problems

Explain why the comparison test cannot be used to decide if the series in Problems 32–33 converge or diverge.

32. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

33. $\sum_{n=1}^{\infty} \sin n$

Explain why the ratio test cannot be used to decide if the series in Problems 34–35 converge or diverge.

34. $\sum_{n=1}^{\infty} (-1)^n$

35. $\sum_{n=1}^{\infty} \sin n$

Explain why the alternating series test cannot be used to decide if the series in Problems 36–38 converge or diverge.

36. $\sum_{n=1}^{\infty} (-1)^{n-1} n$

37. $\sum_{n=1}^{\infty} (-1)^{n-1} \sin n$

38. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(2-\frac{1}{n}\right)$

Use a computer or calculator to investigate the behavior of the partial sums of the alternating series in Problems 39–41. Which ones converge? Confirm convergence using the alternating series test. If a series converges, estimate its sum.

39. $1-2+3-4+5+\cdots+(-1)^n(n+1)+\cdots$

40. $1-0.1+0.01-0.001+\cdots+(-1)^n 10^{-n}+\cdots$

41. $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}+\cdots$

Determine which of the series in Problems 42–53 converge.

42. $\sum_{n=0}^{\infty} \frac{(0.1)^n}{n!}$

43. $\sum_{n=1}^{\infty} \frac{(n-1)!}{n^2}$

44. $\sum_{n=0}^{\infty} e^{-n}$

45. $\sum_{n=1}^{\infty} e^n$

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n^2}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

50. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$

51. $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\ln n^2}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+2)}}$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$

The series in Problems 54–56 converge by the alternating series test. How many terms give a partial sum, S_n, within 0.01 of the sum, S, of the series?

54. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

55. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^{n-1}$

56. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$

In Problems 57–60 determine whether the series are absolutely convergent, conditionally convergent, or divergent.

57. $\displaystyle\sum \frac{(-1)^n}{2^n}$

58. $\displaystyle\sum \frac{(-1)^n}{2n}$

59. $\displaystyle\sum (-1)^n \left(1+\frac{1}{n^2}\right)$

60. $\displaystyle\sum \frac{(-1)^n}{n^4+7}$

61. Suppose $0 \le b_n \le 2^n \le a_n$ and $0 \le c_n \le 2^{-n} \le d_n$ for all n. Which of the series $\sum a_n$, $\sum b_n$, $\sum c_n$, and $\sum d_n$ definitely converge, which definitely diverge?

62. Given two convergent series $\sum a_n$ and $\sum b_n$, we know that the term-by-term sum $\sum (a_b + b_n)$ converges. What about the series formed by taking the products of the terms $\sum a_n \cdot b_n$? This problem shows that it may or may not converge.

(a) Show that if $\sum a_n = \sum 1/n^2$ and $\sum b_n = \sum 1/n^3$, then $\sum a_n \cdot b_n$ converges.

(b) Explain why $\sum (-1)^n/\sqrt{n}$ converges.

(c) Show that if $a_n = b_n = (-1)^n/\sqrt{n}$, then $\sum a_n \cdot b_n$ does not converge.

63. Suppose that $b_n > 0$ and $\sum b_n$ converges. Show that if $\lim_{n\to\infty} a_n/b_n = 0$ then $\sum a_n$ converges.

64. Suppose that $b_n > 0$ and $\sum b_n$ diverges. Show that if $\lim_{n\to\infty} a_n/b_n = \infty$ then $\sum a_n$ diverges.

65. A series $\sum a_n$ of positive terms (that is, $a_n > 0$) can be used to form another series $\sum b_n$ where each term b_n is the average of the first n terms of the original series, that is, $b_n = (a_1 + a_2 + \cdots + a_n)/n$. Show that $\sum b_n$ does not converge (even if $\sum a_n$ does). [Hint: Compare $\sum b_n$ to a multiple of the harmonic series.]

66. Show that if $\sum |a_n|$ converges, then $\sum (-1)^n a_n$ converges.

67. **(a)** For a series $\sum a_n$, show that $0 \le a_n + |a_n| \le 2|a_n|$.
(b) Use part (a) to show that if $\sum |a_n|$ converges, then $\sum a_n$ converges.

Problems 68–69 introduce the *root test* for convergence. Given a series $\sum a_n$ of positive terms (that is, $a_n > 0$) such that the root $\sqrt[n]{a_n}$ has a limit r as $n \to \infty$,

- if $r < 1$, then $\sum a_n$ converges
- if $r > 1$, then $\sum a_n$ diverges
- if $r = 1$, then $\sum a_n$ could converge or diverge.

(This test works since $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ tells us that the series is comparable to a geometric series with ratio r.) Use this test to determine the behavior of the series.

68. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$

69. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+1}{3n^2}\right)^n$

9.5 POWER SERIES AND INTERVAL OF CONVERGENCE

In Section 9.2 we saw that the geometric series $\sum ax^n$ converges for $-1 < x < 1$ and diverges otherwise. This section studies the convergence of more general series constructed from powers. Chapter 10 shows how such power series are used to approximate functions such as e^x, $\sin x$, $\cos x$, and $\ln x$.

A **power series** about $x = a$ is a sum of constants times powers of $(x - a)$:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n.$$

We think of a as a constant. For any fixed x, the power series $\sum C_n(x-a)^n$ is a series of numbers like those considered in Section 9.3. To investigate the convergence of a power series, we consider the partial sums, which in this case are the polynomials $S_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n$. As before, we consider the sequence[7]

$$S_0(x),\ S_1(x),\ S_2(x),\ \ldots\ , S_n(x), \ldots.$$

For a fixed value of x, if this sequence of partial sums converges to a limit L, that is, if $\lim_{n\to\infty} S_n(x) = L$, then we say that the power series **converges** to L for this value of x.

A power series may converge for some values of x and not for others.

Example 1 Determine whether the power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges or diverges for

(a) $x = -1$ (b) $x = 3$

Solution (a) Substituting $x = -1$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n.$$

This is a geometric series with ratio $-1/2$, so the series converges to $1/(1-(-\frac{1}{2})) = 2/3$.

(b) Substituting $x = 3$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n.$$

This is a geometric series with ratio greater than 1, so it diverges.

Example 2 Find an expression for the general term of the series and use it to write the series using $\sum$ notation:

$$\frac{(x-2)^4}{4} - \frac{(x-2)^6}{9} + \frac{(x-2)^8}{16} - \frac{(x-2)^{10}}{25} + \cdots.$$

Solution The series is about $x = 2$ and the odd terms are zero. We use $(x-2)^{2n}$ and begin with $n = 2$. Since the series alternates and is positive for $n = 2$, we multiply by $(-1)^n$. For $n = 2$, we divide by 4, for $n = 3$ we divide by 9, and in general, we divide by n^2. One way to write this series is

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-2)^{2n}}{n^2}.$$

Numerical and Graphical View of Convergence

Consider the series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots.$$

To investigate the convergence of this series, we look at the sequence of partial sums graphed in Figure 9.10. The graph suggests that the partial sums converge for x in the interval $(0, 2)$. In Examples 4 and 6, we show that the series converges for $0 < x \leq 2$. This is called the *interval of convergence* of this series.

[7]Here we have chosen to call the first term in the sequence $S_0(x)$ rather than $S_1(x)$ to correspond to the power of $(x-a)$.

At $x = 1.4$, which is inside the interval, the series appears to converge quite rapidly:

$$S_5(1.4) = 0.33698\ldots \qquad S_7(1.4) = 0.33653\ldots$$
$$S_6(1.4) = 0.33630\ldots \qquad S_8(1.4) = 0.33645\ldots$$

Table 9.1 shows the results of using $x = 1.9$ and $x = 2.3$ in the power series. For $x = 1.9$, which is inside the interval of convergence but close to an endpoint, the series converges, though rather slowly. For $x = 2.3$, which is outside the interval of convergence, the series diverges: the larger the value of n, the more wildly the series oscillates. In fact, the contribution of the twenty-fifth term is about 28; the contribution of the hundredth term is about $-2{,}500{,}000{,}000$. Figure 9.10 shows the interval of convergence and the partial sums.

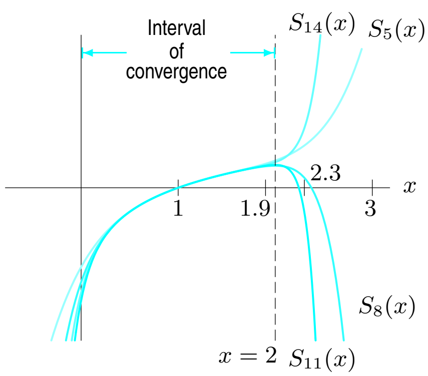


Figure 9.10: Partial sums for series in Example 4 converge for $0 < x < 2$

Table 9.1 *Partial sums for series in Example 4 with $x = 1.9$ inside interval of convergence and $x = 2.3$ outside*

n	$S_n(1.9)$	n	$S_n(2.3)$
2	0.495	2	0.455
5	0.69207	5	1.21589
8	0.61802	8	0.28817
11	0.65473	11	1.71710
14	0.63440	14	−0.70701

Notice that the interval of convergence, $0 < x \leq 2$, is centered on $x = 1$. Since the interval extends one unit on either side, we say the *radius of convergence* of this series is 1.

Intervals of Convergence

Each power series falls into one of the three following cases, characterized by its *radius of convergence*, R.

- The series converges only for $x = a$; the **radius of convergence** is defined to be $R = 0$.
- The series converges for all values of x; the **radius of convergence** is defined to be $R = \infty$.
- There is a positive number R, called the **radius of convergence**, such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. See Figure 9.11.
- The **interval of convergence** is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.

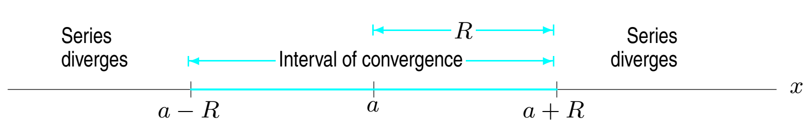


Figure 9.11: Radius of convergence, R, determines an interval, centered at $x = a$, in which the series converges

There are some series whose radius of convergence we already know. For example, the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

converges for $|x| < 1$ and diverges for $|x| \geq 1$, so its radius of convergence is 1. Similarly, the series

$$1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots + \left(\frac{x}{3}\right)^n + \cdots$$

converges for $|x/3| < 1$ and diverges for $|x/3| \geq 1$, so its radius of convergence is 3.

The next theorem gives a method of computing the radius of convergence for many series. To find the values of x for which the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ converges, we use the ratio test. Writing $a_n = C_n(x-a)^n$ and assuming $C_n \neq 0$ and $x \neq a$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|} = \lim_{n\to\infty} \frac{|C_{n+1}||x-a|}{|C_n|} = |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}.$$

Case 1. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite. Then the ratio test shows that the power series converges only for $x = a$. The radius of convergence is $R = 0$.

Case 2. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$. Then the ratio test shows that the power series converges for all x. The radius of convergence is $R = \infty$.

Case 3. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where $\lim_{n\to\infty} |C_{n+1}|/|C_n| = K$. In Case 1, K does not exist; in Case 2, $K = 0$. Thus, we can assume K exists and $K \neq 0$, and we can define $R = 1/K$. Then we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a| = \frac{|x-a|}{R},$$

so the ratio test tells us that the power series:

- Converges for $\dfrac{|x-a|}{R} < 1$; that is, for $|x-a| < R$
- Diverges for $\dfrac{|x-a|}{R} > 1$; that is, for $|x-a| > R$.

The results are summarized in the following theorem.

Theorem 9.10: Method for Computing Radius of Convergence

To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$, use the ratio test with $a_n = C_n(x-a)^n$.

- If $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite, then $R = 0$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$, then $R = \infty$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where K is finite and nonzero, then $R = 1/K$.

Note that the ratio test does not tell us anything if $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ fails to exist, which can occur, for example, if some of the C_ns are zero.

A proof that a power series has a radius of convergence and of Theorem 9.10 is given in the online theory supplement. To understand these facts informally, we can think of a power series as being like a geometric series whose coefficients vary from term to term. The radius of convergence depends on the behavior of the coefficients: if there are constants C and K such that for larger and larger n,

$$|C_n| \approx CK^n,$$

then it is plausible that $\sum C_n x^n$ and $\sum CK^n x^n = \sum C(Kx)^n$ converge or diverge together. The geometric series $\sum C(Kx)^n$ converges for $|Kx| < 1$, that is, for $|x| < 1/K$. We can find K using the ratio test, because

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|C_{n+1}||(x-a)^{n+1}|}{|C_n||(x-a)^n|} \approx \frac{CK^{n+1}|(x-a)^{n+1}|}{CK^n|(x-a)^n|} = K|x-a|.$$

Example 3 Show that the following power series converges for all x:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Solution Because $C_n = 1/n!$, none of the C_ns are zero and we can use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

This gives $R = \infty$, so the series converges for all x. We see in Chapter 10 that it converges to e^x.

Example 4 Determine the radius of convergence of the series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots.$$

What does this tell us about the interval of convergence of this series?

Solution The general term of the series is $(x-1)^n/n$ if n is odd and $-(x-1)^n/n$ if n is even, so $C_n = (-1)^{n-1}/n$, and we can use the ratio test. We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-1| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x-1| \lim_{n\to\infty} \frac{|(-1)^n/(n+1)|}{|(-1)^{n-1}/n|} = |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1|.$$

Thus, $K = 1$ in Theorem 9.10, so the radius of convergence is $R = 1$. The power series converges for $|x-1| < 1$ and diverges for $|x-1| > 1$, so the series converges for $0 < x < 2$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = 0$ and $x = 2$. We see in Chapter 10 that the series converges to $\ln x$ for $0 < x \le 2$.

The ratio test requires $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ to exist for $a_n = C_n(x-a)^n$. What happens if some of the coefficients C_n are zero? Then we use the fact that an infinite series can be written in several ways and pick one in which the terms are nonzero. For example, we think of the power series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

as the series with $a_1 = x$ and $a_2 = -x^3/3!, \ldots$, so $a_n = (-1)^{n-1}x^{2n-1}/(2n-1)!$. With this choice of a_n, all $a_n \neq 0$, so we can use the ratio test.[8]

[8] We do not take $a_1 = x, a_2 = 0, a_3 = -x^3/3!, a_4 = 0, \ldots$ because then $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ does not exist.

Example 5 Find the radius and interval of convergence of the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\cdots$$

Solution We take

$$a_n=(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!},$$

so that, replacing n by $n+1$, we have

$$a_{n+1}=(-1)^{(n+1)-1}\frac{x^{2(n+1)-1}}{(2\,(n+1)-1)!}=(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Thus,

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\left|(-1)^n\,\frac{x^{2n+1}}{(2n+1)!}\right|}{\left|(-1)^{n-1}\,\frac{x^{2n-1}}{(2n-1)!}\right|}=\left|\frac{(-1)^n x^{2n+1}(2n-1)!}{(-1)^{n-1}x^{2n-1}(2n+1)!}\right|=\left|\frac{(-1)\,x^2}{(2n+1)\,2n}\right|=\frac{x^2}{(2n+1)\,2n}.$$

Because

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{x^2}{(2n+1)\,2n}=0,$$

we have $\lim_{n\to\infty}|a_{n+1}|/|a_n|=0<1$ for all x. Thus, the ratio test guarantees that the power series converges for every x. The radius of convergence is infinite and the interval of convergence is all x. We see in Chapter 10 that the series converges to $\sin x$.

What Happens at the Endpoints of the Interval of Convergence?

The ratio test does not tell us whether a power series converges at the endpoints of its interval of convergence, $x=a\pm R$. There is no simple theorem that answers this question. Since substituting $x=a\pm R$ converts the power series to a series of numbers, the tests in Sections 9.3 and 9.4 are often useful. See Examples 6 and 7.

Example 6 Determine the interval of convergence of the series

$$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+(-1)^{n-1}\frac{(x-1)^n}{n}+\cdots.$$

Solution In Example 4 on page 467, we showed that this series has $R=1$; it converges for $0<x<2$ and diverges for $x<0$ or $x>2$. We need to determine whether it converges at the endpoints of the interval of convergence, $x=0$ and $x=2$. At $x=2$, we have the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n}+\cdots.$$

This is an alternating series with $a_n=1/(n+1)$, so by the alternating series test (Theorem 9.8), it converges. At $x=0$, we have the series

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots-\frac{1}{n}-\cdots.$$

This is the negative of the harmonic series, so it diverges. Therefore, the interval of convergence is $0<x\le 2$. The right endpoint is included and the left endpoint is not.

Example 7 Find the radius and interval of convergence of the series

$$1+2^2x^2+2^4x^4+2^6x^6+\cdots+2^{2n}x^{2n}+\cdots.$$

Solution If we take $a_n = 2^n x^n$ for n even and $a_n = 0$ for n odd, $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ does not exist. Therefore, for this series we take

$$a_n = 2^{2n}x^{2n},$$

so that, replacing n by $n+1$, we have

$$a_{n+1} = 2^{2(n+1)}x^{2(n+1)} = 2^{2n+2}x^{2n+2}.$$

Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|2^{2n+2}x^{2n+2}|}{|2^{2n}x^{2n}|} = |2^2x^2| = 4x^2.$$

We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 4x^2.$$

The ratio test guarantees that the power series converges if $4x^2 < 1$, that is, if $|x| < \frac{1}{2}$. The radius of convergence is $\frac{1}{2}$. The series converges for $-\frac{1}{2} < x < \frac{1}{2}$ and diverges for $x < \frac{1}{2}$ or $x > -\frac{1}{2}$. At $x = \pm\frac{1}{2}$, all the terms in the series are 1, so the series diverges (by Theorem 9.2 Property 3). Thus, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

Exercises and Problems for Section 9.5

Exercises

Which of the series in Exercises 1–4 are power series?

1. $x - x^3 + x^6 - x^{10} + x^{15} - \cdots$

2. $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots$

3. $1 + x + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$

4. $x^7 + x + 2$

Find an expression for the general term of the series in Exercises 5–10. Give the starting value of the index (n or k for example).

5. $\frac{1}{2}x + \frac{1\cdot 3}{2^2\cdot 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3 + \cdots$

6. $px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$

7. $1 - \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} - \frac{(x-1)^6}{6!} + \cdots$

8. $(x-1)^3 - \frac{(x-1)^5}{2!} + \frac{(x-1)^7}{4!} - \frac{(x-1)^9}{6!} + \cdots$

9. $\frac{x-a}{1} + \frac{(x-a)^2}{2\cdot 2!} + \frac{(x-a)^3}{4\cdot 3!} + \frac{(x-a)^4}{8\cdot 4!} + \cdots$

10. $2(x+5)^3 + 3(x+5)^5 + \frac{4(x+5)^7}{2!} + \frac{5(x+5)^9}{3!} + \cdots$

Use the ratio test to find the radius of convergence of the power series in Exercises 11–21.

11. $\sum_{n=0}^{\infty}(5x)^n$

12. $\sum_{n=0}^{\infty}n^3x^n$

13. $\sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^n+n}$

14. $\sum_{n=1}^{\infty}\frac{2^n(x-1)^n}{n}$

15. $x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \cdots$

16. $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \cdots$

17. $1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \frac{32x^5}{5!} + \cdots$

18. $\frac{x}{3} + \frac{2x^2}{5} + \frac{3x^3}{7} + \frac{4x^4}{9} + \frac{5x^5}{11} + \cdots$

19. $1 + 2x + \frac{4!x^2}{(2!)^2} + \frac{6!x^3}{(3!)^2} + \frac{8!x^4}{(4!)^2} + \frac{10!x^5}{(5!)^2} + \cdots$

20. $3x + \frac{5}{2}x^2 + \frac{7}{3}x^3 + \frac{9}{4}x^4 + \frac{11}{5}x^5 + \cdots$

21. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

22. **(a)** Determine the radius of convergence of the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots.$$

What does this tell us about the interval of convergence of this series?

(b) Investigate convergence at the end points of the interval of convergence of this series.

23. Show that the series $\sum_{n=1}^{\infty}\frac{(2x)^n}{n}$ converges for $|x| < 1/2$. Investigate whether the series converges for $x = 1/2$ and $x = -1/2$.

Problems

In Problems 24–27, find the interval of convergence.

24. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

25. $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^n}{n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2x^{2n}}{2^{2n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-5)^n}{2^n n^2}$

28. For constant p, find the radius of convergence of the binomial power series:[9]

$$1+px+\frac{p(p-1)x^2}{2!}+\frac{p(p-1)(p-2)x^3}{3!}+\cdots.$$

29. Show that if $C_0+C_1x+C_2x^2+C_3x^3+\cdots$ converges for $|x|<R$ with R given by the ratio test, then so does $C_1+2C_2x+3C_3x^2+\cdots$.

30. The series $\sum C_nx^n$ converges at $x=-5$ and diverges at $x=7$. What can you say about its radius of convergence?

31. The series $\sum C_n(x+7)^n$ converges at $x=0$ and diverges at $x=-17$. What can you say about its radius of convergence?

32. Suppose that the power series $\sum C_nx^n$ converges when $x=-4$ and diverges when $x=7$. Which of the following are true, false or not possible to determine?

- **(a)** The power series converges when $x=10$.
- **(b)** The power series converges when $x=3$.
- **(c)** The power series diverges when $x=1$.
- **(d)** The power series diverges when $x=6$.

33. If $\sum C_n(x-3)^n$ converges at $x=7$ and diverges at $x=10$, what can you say about the convergence at $x=11$? At $x=5$? At $x=0$?

34. Bessel functions are important in such diverse areas as describing planetary motion and the shape of a vibrating drumhead. The Bessel function of order 0 is defined by

$$J(x)=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{2^{2n}(n!)^2}.$$

- **(a)** Find the domain of $J(x)$ by finding the interval of convergence for this power series.
- **(b)** Find $J(0)$.
- **(c)** Find the partial sum polynomials S_0, S_1, S_2, S_3, S_4.
- **(d)** Estimate $J(1)$ to three decimal places.
- **(e)** Use your answer to part (d) to estimate $J(-1)$.

35. For all x-values for which it converges, the function f is defined by the series

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

- **(a)** What is $f(0)$?
- **(b)** What is the domain of f?
- **(c)** Assuming that f' can be calculated by differentiating the series term-by-term, find the series for $f'(x)$. What do you notice?
- **(d)** Guess what well-known function f is.

36. From Example 5 on page 468, we know the following series converges for all x. We define $g(x)$ to be its sum:

$$g(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

- **(a)** Is $g(x)$ odd, even, or neither? What is $g(0)$?
- **(b)** Assuming that derivatives can be computed term-by-term, show that $g''(x)=-g(x)$.
- **(c)** Guess what well-known function g might be. Check your guess using $g(0)$ and $g'(0)$.

37. The functions $p(x)$ and $q(x)$ are defined by the series

$$p(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\quad q(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

Assuming that these series converge for all x and that multiplication can be done term-by-term:

- **(a)** Find the series for $(p(x))^2+(q(x))^2$ up to the term in x^6.
- **(b)** Guess what well-known functions p and q could be.

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Sequences**
  Recursive definition, monotone, bounded, convergence
- **Geometric series**
  Finite sum, infinite sum
- **Harmonic series**
- **Alternating series**
- **Tests for convergence of series**
  Integral test, p-series, comparison test, limit comparison test, ratio test, alternating series test
  Absolute and conditional convergence
- **Power series**
  Ratio test for radius of convergence, interval of convergence

[9]For an explanation of the name, see Section 10.2.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER NINE

Exercises

Do the sequences in Exercises 1–4 converge or diverge? If a sequence converges, find its limit.

1. $\dfrac{3+4n}{5+7n}$

2. $(-1)^n \dfrac{(n+1)}{n}$

3. $\sin\left(\dfrac{\pi}{4}n\right)$

4. $\dfrac{1}{n} + \ln n$

5. Find the sum of the series $b^5 + b^6 + b^7 + b^8 + b^9 + b^{10}$.

6. Find the sum of the series $(0.5)^3 + (0.5)^4 + \cdots + (0.5)^k$.

7. Find the sum: $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + 5}{4^n}$

Use the integral test to decide whether the series in Exercises 8–11 converge or diverge.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2n}{n^3+n^2+1}$

10. $\displaystyle\sum_{n=0}^{\infty} ne^{-n^2}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

Use the comparison test to confirm the statements in Exercises 12–13.

12. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges, so $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2}{3n^2+4}\right)^n$ converges.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sin^2 n}$ diverges.

In Exercises 14–17 use the limit comparison test to determine whether the series converges or diverges.

14. $\displaystyle\sum \frac{\sqrt{n-1}}{n^2+3}$

15. $\displaystyle\sum \frac{n^3-2n^2+n+1}{n^5-2}$

16. $\displaystyle\sum \sin\frac{1}{n^2}$

17. $\displaystyle\sum \frac{1}{\sqrt{n^3-1}}$

Use the ratio test to decide if the series in Exercises 18–19 converge or diverge.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n!}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$

Use the alternating series test to decide which of the series in Exercises 20–21 converge.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$

Determine which of the series in Exercises 22–41 converge.

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

24. $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-2}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+1}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^2 2^n}$

31. $\displaystyle\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$

32. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(2n+1)!}$

33. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{\sqrt{n}}$

34. $\displaystyle\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sin n}$

36. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n-5)^3}$

37. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-3}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^3}$

39. $\displaystyle\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

41. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

Find the radius of convergence in Exercises 42–45.

42. $\displaystyle\sum_{n=0}^{\infty} nx^n$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^n}{(n!)^2}$

44. $\displaystyle\sum_{n=0}^{\infty} (2^n+n^2)x^n$

45. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!+1}$

Problems

In Problems 46–47 determine whether the series are absolutely convergent, conditionally convergent, or divergent.

46. $\sum \frac{(-1)^n}{n^{1/2}}$

47. $\sum (-1)^n \frac{n}{n+1}$

48. For $r > 0$, how does the convergence of the following series depend on r?

$$\sum_{n=1}^{\infty} \frac{n^r + r^n}{n^r r^n}$$

In Problems 49–51, find the interval of convergence for the power series.

49. $\sum_{n=1}^{\infty} \frac{x^n}{3^n n^2}$

50. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{5^n}$

51. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

52. Suppose that $\sum_{n=0}^{\infty} C_n (x-2)^n$ converges when $x = 4$ and diverges when $x = 6$. Which of the following are true, false or not possible to determine? Give reasons for your answers.

(a) The power series converges when $x = 7$.
(b) The power series diverges when $x = 1$.
(c) The power series converges when $x = 0.5$.
(d) The power series diverges when $x = 5$.
(e) The power series converges when $x = -3$.

53. For all the t-values for which it converges, the function h is defined by the series

$$h(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}.$$

(a) What is the domain of h?
(b) Is h odd, even, or neither?
(c) Assuming that derivatives can be computed term-by-term, show that

$$h''(t) = -h(t).$$

54. A new car costs \$30,000; it loses 10% of its value each year. Maintenance is \$500 the first year and increases by 20% annually.

(a) Find a formula for l_n, the value lost by the car in year n.
(b) Find a formula for m_n, the maintenance expenses in year n.
(c) In what year do maintenance expenses first exceed the value lost by the car?

Problems 55–57 are about *bonds*, which are issued by a government to raise money. An individual who buys a \$1000 bond gives the government \$1000 and in return receives a fixed sum of money, called the *coupon*, every six months or every year for the life of the bond. At the time of the last coupon, the individual also gets the \$1000, or *principal* back.

55. What is the present value of a \$1000 bond which pays \$50 a year for 10 years, starting one year from now? Assume interest rate is 6% per year, compounded annually.

56. What is the present value of a \$1000 bond which pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 4% per year, compounded annually.

57. **(a)** What is the present value of a \$1000 bond which pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 5% per year, compounded annually.
(b) Since \$50 is 5% of \$1000, this bond is often called a 5% bond. What does your answer to part (a) tell you about the relationship between the principal and the present value of this bond when the interest rate is 5%?
(c) If the interest rate is more than 5% per year, compounded annually, which is larger: the principal or the present value of the bond? Why do you think the bond is then described as *trading at a discount*?
(d) If the interest rate is less than 5% per year, compounded annually, why is the bond described as *trading at a premium*?

58. Cephalexin is an antibiotic with a half-life in the body of 0.9 hours, taken in tablets of 250 mg every six hours.

(a) What percentage of the cephalexin in the body at the start of a six-hour period is still there at the end (assuming no tablets are taken during that time)?
(b) Write an expression for Q_1, Q_2, Q_3, Q_4, where Q_n mg, is the amount of cephalexin in the body right after the n^{th} tablet is taken.
(c) Express Q_3, Q_4 in closed-form and evaluate them.
(d) Write an expression for Q_n and put it in closed-form.
(e) If the patient keeps taking the tablets, use your answer to part (d) to find the quantity of cephalexin in the body in the long run, right after taking a pill.

59. Around January 1, 1993, Barbra Streisand signed a contract with Sony Corporation for \$2 million a year for 10 years. Suppose the first payment was made on the day of signing and that all other payments are made on the first day of the year. Suppose also that all payments are made into a bank account earning 4% a year, compounded annually.

(a) How much money was in the account
  (i) On the night of December 31, 1999?
  (ii) On the day the last payment is made?

(b) What was the present value of the contract on the day it was signed?

60. Before World War I, the British government issued what are called *consols*, which pay the owner or his heirs a fixed amount of money every year forever. (Cartoonists of the time described aristocrats living off such payments as "pickled in consols.") What should a person expect to pay for consols which pay £10 a year forever? Assume the first payment is one year from the date of purchase and that interest remains 4% per year, compounded annually. (£ denotes pounds, the British unit of currency.)

61. This problem illustrates how banks create credit and can thereby lend out more money than has been deposited. Suppose that initially \$100 is deposited in a bank. Experience has shown bankers that on average only 8% of the money deposited is withdrawn by the owner at any time. Consequently, bankers feel free to lend out 92% of their deposits. Thus \$92 of the original \$100 is loaned out to other customers (to start a business, for example). This \$92 will become someone else's income and, sooner or later, will be redeposited in the bank. Then 92% of \$92, or $\$92(0.92) = \$84.64$, is loaned out again and eventually redeposited. Of the \$84.64, the bank again loans out 92%, and so on.

(a) Find the total amount of money deposited in the bank as a result of these transactions.

(b) The total amount of money deposited divided by the original deposit is called the *credit multiplier*. Calculate the credit multiplier for this example and explain what this number tells us.

62. In theory, drugs that decay exponentially always leave a residue in the body. However, in practice, once the drug has been in the body for 5 half-lives, it is regarded as being eliminated.[10] If a patient takes a tablet of the same drug every 5 half-lives forever, what is the upper limit to the amount of drug that can be in the body?

63. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$ to within 0.01 of the actual sum of the series.

64. Is it possible to construct a convergent alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ for which $0 < a_{n+1} < a_n$ but $\displaystyle\lim_{n\to\infty} a_n \neq 0$?

65. Show that if $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ diverges. [Hint: Assume that $\sum(a_n + b_n)$ converges and consider $\sum(a_n + b_n) - \sum a_n$.]

Suppose that $\sum a_n$ converges with $a_n > 0$ for all n. Decide if the series in Problems 66–70 converge, diverge or if there is not enough information provided.

66. $\sum a_n/n$ **67.** $\sum 1/a_n$ **68.** $\sum na_n$

69. $\sum (a_n + a_n/2)$ **70.** $\sum a_n^2$

71. Does $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n}\right)$ converge or diverge? Does $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n}\right)$ converge or diverge? Is the statement "If $\sum a_n$ and $\sum b_n$ diverge, then $\sum(a_n + b_n)$ may or may not diverge" true?

72. Estimate the sum of the first 100,000 terms of the harmonic series,

$$\sum_{k=1}^{100{,}000} \frac{1}{k},$$

to the closest integer. [Hint: Use left- and right-hand sums of the function $f(x) = 1/x$ on the interval from 1 to 100,000 with $\Delta x = 1$.]

73. Although the harmonic series does not converge, the partial sums grow very, very slowly. Take a right-hand sum approximating the integral of $f(x) = 1/x$ on the interval $[1, n]$, with $\Delta x = 1$, to show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} < \ln n.$$

If a computer could add a million terms of the harmonic series each second, estimate the sum after one year.

74. Is the following argument true or false? Give reasons for your answer.

> Consider the infinite series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$. Since $\displaystyle\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ we can write this series as
>
> $$\sum_{n=2}^{\infty}\frac{1}{n-1} - \sum_{n=2}^{\infty}\frac{1}{n}.$$
>
> For the first series $a_n = 1/(n-1)$. Since $n-1 < n$ we have $1/(n-1) > 1/n$ and so this series diverges by comparison with the divergent harmonic series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$. The second series is the divergent harmonic series. Since both series diverge, their difference also diverges.

[10] http://dr.pierce1.net/PDF/half_life.pdf, accessed on May 10, 2003.

CHECK YOUR UNDERSTANDING

Decide if the statements in Problems 1–46 are true or false. Give an explanation for your answer.

1. You can tell if a sequence converges by looking at the first 1000 terms.
2. If the terms s_n of a convergent sequence are all positive then $\lim_{n\to\infty} s_n$ is positive.
3. If the sequence s_n of positive terms is unbounded, then the sequence has a term greater than a million.
4. If the sequence s_n of positive terms is unbounded, then the sequence has an infinite number of terms greater than a million.
5. If a sequence s_n is convergent, then the terms s_n tend to zero as n increases.
6. If a series converges, then the sequence of partial sums of the series also converges.
7. A monotone sequence can not have both positive and negative terms.
8. If a monotone sequence of positive terms does not converge, then it has a term greater than a million.
9. If the terms s_n of a sequence alternate in sign, then the sequence converges.
10. If all terms s_n of a sequence are less than a million, then the sequence is bounded.
11. A geometric series in x is a power series.
12. $\sum_{n=1}^{\infty}(x-n)^n$ is a power series.
13. If the power series $\sum C_n x^n$ converges for $x=2$, then it converges for $x=1$.
14. If the power series $\sum C_n x^n$ converges for $x=1$, then the power series converges for $x=2$.
15. If the power series $\sum C_n x^n$ does not converge for $x=1$, then the power series does not converge for $x=2$.
16. If $0 \le a_n \le b_n$ and $\sum a_n$ converges, then $\sum b_n$ converges.
17. If $0 \le a_n \le b_n$ and $\sum a_n$ diverges, then $\sum b_n$ diverges.
18. If $b_n \le a_n \le 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
19. If $\sum a_n$ converges, then $\sum |a_n|$ converges.
20. If $\sum |a_n + b_n|$ converges, then $\sum |a_n|$ and $\sum |b_n|$ converge.
21. If $\sum a_n$ converges, then $\lim_{n\to\infty} |a_{n+1}|/|a_n| \ne 1$.
22. $\sum_{n=0}^{\infty}(-1)^n\cos(\pi n)$ is an alternating series.
23. $\sum_{n=1}^{\infty}(1+(-1)^n)$ is a series of nonnegative terms.
24. The series $\sum_{n=0}^{\infty}(-1)^n 2^n$ converges.
25. The series $\sum_{n=1}^{\infty} 2^{(-1)^n}$ converges.
26. If $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.
27. If an alternating series converges by the alternating series test, then the error in using the first n terms of the series to approximate the entire series is less in magnitude than the first term omitted.
28. If an alternating series converges, then the error in using the first n terms of the series to approximate the entire series is less in magnitude than the first term omitted.
29. If $\sum |a_n|$ converges, then $\sum(-1)^n|a_n|$ converges.
30. To find the sum of the alternating harmonic series $\sum(-1)^{n-1}/n$ to within 0.01 of the true value, we can sum the first 100 terms.
31. $\sum C_n(x-1)^n$ and $\sum C_n x^n$ have the same radius of convergence.
32. If $\sum C_n x^n$ and $\sum B_n x^n$ have the same radius of convergence, then the coefficients, C_n and B_n, must be equal.
33. If a series $\sum a_n$ converges, then the terms, a_n, tend to zero as n increases.
34. If the terms, a_n, of a series tend to zero as n increases, then the series $\sum a_n$ converges.
35. If $\sum a_n$ does not converge and $\sum b_n$ does not converge, then $\sum a_n b_n$ does not converge.
36. If $\sum a_n b_n$ converges, then $\sum a_n$ and $\sum b_n$ converge.
37. If $\sum a_n$ is absolutely convergent, then it is convergent.
38. If $\sum a_n$ is conditionally convergent, then it is absolutely convergent.
39. If a power series converges at one endpoint of its interval of convergence, then it converges at the other.
40. If $a_n > 0.5b_n > 0$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges.
41. A power series always converges at at least one point.
42. If the power series $\sum C_n x^n$ converges at $x=10$, then it converges at $x=-9$.
43. If the power series $\sum C_n x^n$ converges at $x=10$, then it converges at $x=-10$.
44. $-5 < x \le 7$ is a possible interval of convergence of a power series.
45. $-3 < x < 2$ could be the interval of convergence of $\sum C_n x^n$.
46. If $-11 < x < 1$ is the interval of convergence of $\sum C_n(x-a)^n$, then $a=-5$.

PROJECTS FOR CHAPTER NINE

1. A Definition of e

We show that the sequence $s_n = \left(1 + \dfrac{1}{n}\right)^n$ converges; its limit can be used to define e.

(a) For a fixed integer $n > 0$, let $f(x) = (n+1)x^n - nx^{n+1}$. For $x > 1$, show f is decreasing and that $f(x) < 1$. Hence, for $x > 1$,

$$x^n(n + 1 - nx) < 1.$$

(b) Substitute the following x-value into the inequality from part (a)

$$x = \frac{1 + 1/n}{1 + 1/(n+1)},$$

and show that

$$x^n \left(\frac{n+1}{n+2}\right) < 1.$$

(c) Use the inequality from part (b) to show that $s_n < s_{n+1}$ for all $n > 0$. Conclude the sequence is increasing.

(d) Substitute $x = 1 + 1/2n$ into the inequality from part (a) to show that

$$\left(1 + \frac{1}{2n}\right)^n < 2.$$

(e) Use the inequality from part (d) to show $s_{2n} < 4$. Conclude the sequence is bounded.

(f) Use parts (c) and (e) to show that the sequence has a limit.

2. Probability of Winning in Sports

In certain sports, winning a game requires a lead of two points. That is, if the score is tied you have to score two points in a row to win.

(a) For some sports (e.g. tennis), a point is scored every play. Suppose your probability of scoring the next point is always p. Then, your opponent's probability of scoring the next point is always $1 - p$.

(i) What is the probability that you win the next two points?

(ii) What is the probability that you and your opponent split the next two points, that is, that neither of you wins both points?

(iii) What is the probability that you split the next two points but you win the two after that?

(iv) What is the probability that you either win the next two points or split the next two and then win the next two after that?

(v) Give a formula for your probability w of winning a tied game.

(vi) Compute your probability of winning a tied game when $p = 0.5$; when $p = 0.6$; when $p = 0.7$; when $p = 0.4$. Comment on your answers.

(b) In other sports (e.g. volleyball), you can score a point only if it is your turn, with turns alternating until a point is scored. Suppose your probability of scoring a point when it is your turn is p, and your opponent's probability of scoring a point when it is her turn is q.

(i) Find a formula for the probability S that you are the first to score the next point, assuming it is currently your turn.

(ii) Suppose that if you score a point, the next turn is yours. Using your answers to part (a) and your formula for S, compute the probability of winning a tied game (if you need two points in a row to win).

- Assume $p = 0.5$ and $q = 0.5$ and it is your turn.
- Assume $p = 0.6$ and $q = 0.5$ and it is your turn.

3. Quinine

Malaria is a parasitic infection transmitted by mosquito bites, mainly in tropical areas of the world. The disease has existed since ancient times, and currently there are hundreds of millions of cases each year, with millions of deaths. Around 1630, Jesuits in Peru introduced the bark of the cinchona tree to the West as the first treatment for malaria. The drug quinine is the active ingredient in the bark, and it is still used today. However, the concentration in the body of quinine, and of most drugs, must be kept within certain parameters. If the concentration is too low, the drug is ineffective. If the concentration is too high, toxic side effects can result.

Suppose you are a doctor who prescribes quinine for a 70 kg malaria patient. At 8 am each day, she is to receive 50 mg of the drug.[11] To be effective, the average concentration in the body must be at least 0.4 mg/kg. However, concentrations above 3.0 mg/kg can be fatal. The half-life of quinine in the body is 11.5 hours.

(a) What is the continuous rate of decay of quinine (in units of $\%$/min)?
(b) How much quinine is in the patient's system just before and just after the first day's dose? After the second day's dose?
(c) How much is in her system after a few more doses? After the n^{th} dose? Explain what happens in the long run.
(d) Graph the concentration of the drug versus time showing what the steady-state looks like. Find a formula for the repeating function.
(e) What is the average (over time) of the concentration of quinine in the patient's body?
(f) Is this treatment both effective and safe?
(g) Suppose instead that the patient were to receive two doses of 25 mg each day, one at 8 am and one at 8 pm. Is this a safe and effective treatment?
(h) Determine the average value of an exponentially decaying function between two points (x_0, y_0) and (x_1, y_1).
(i) Notice that the average concentration of quinine is the same for the 50 mg once per day and 25 mg twice per day treatments. Use part (h) to explain why. Would the average concentration be the same for a 100 mg dose once every two days?
(j) Suppose the original quinine regimen is stopped. How long after the last dose will the amount of quinine be less than 10^{-10} times the patient's body mass?

[11] This is a simplified model; actual treatments involve different drugs and more complicated dosage regimens. See, for example, *The Pharmacological Basis of Therapeutics*, 9th Ed., ed. Joel G. Hardman, Alfred Goodman Gilman, and Lee E. Limbird, (New York: McGraw Hill, 1996).