THE DISCRIMINANT

We saw in Section 11.5 that the graph of a quadratic equation in x and y is often a conic section. We were able to determine the type of conic section by using rotation of axes to put the equation either in standard form, or in the form of a translated conic. The next result shows that it is possible to determine the nature of the graph directly from the equation itself.

H.1 THEOREM. *Consider a second-degree equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

(*a*) *If* $B^2 - 4AC < 0$, *the equation represents an ellipse, a circle, a point, or else has no graph.*

(*b*) *If* $B^2 - 4AC > 0$, *the equation represents a hyperbola or a pair of intersecting lines.*

(*c*) *If* $B^2 - 4AC = 0$, *the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.*

The quantity $B^2 - 4AC$ in this theorem is called the ***discriminant*** of the quadratic equation. To see why this theorem is true, we need a fact about the discriminant. It can be shown (Exercise 33 of Section 11.5) that if the coordinate axes are rotated through any angle θ, and if

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{2}$$

is the equation resulting from (1) after rotation, then

$$B^2 - 4AC = B'^2 - 4A'C' \tag{3}$$

In other words, the discriminant of a quadratic equation is not altered by rotating the coordinate axes. For this reason the discriminant is said to be ***invariant*** under a rotation of coordinate axes. In particular, if we choose the angle of rotation to eliminate the cross-product term, then (2) becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{4}$$

and since $B' = 0$, (3) tells us that

$$B^2 - 4AC = -4A'C' \tag{5}$$

PROOF OF (*a*). If $B^2 - 4AC < 0$, then from (5), $A'C' > 0$, so (4) can be divided through by $A'C'$ and written in the form

$$\frac{1}{C'}\left(x'^2 + \frac{D'}{A'}x'\right) + \frac{1}{A'}\left(y'^2 + \frac{E'}{C'}y'\right) = -\frac{F'}{A'C'}$$

Since $A'C' > 0$, the numbers A' and C' have the same sign. We assume that this sign is positive, since Equation (4) can be multiplied through by -1 to achieve this, if necessary. By completing the squares, we can rewrite the last equation in the form

$$\frac{(x'-h)^2}{(\sqrt{C'})^2} + \frac{(y'-k)^2}{(\sqrt{A'})^2} = K$$

There are three possibilities: $K > 0$, in which case the graph is either a circle or an ellipse, depending on whether or not the denominators are equal; $K < 0$, in which case there is no graph, since the left side is nonnegative for all x' and y'; or $K = 0$, in which case the graph is the single point (h, k), since the equation is satisfied only by $x' = h$ and $y' = k$. The proofs of parts (b) and (c) require a similar kind of analysis. ■

▶ **Example 1** Use the discriminant to identify the graph of

$$8x^2 - 3xy + 5y^2 - 7x + 6 = 0$$

Solution. We have

$$B^2 - 4AC = (-3)^2 - 4(8)(5) = -151$$

Since the discriminant is negative, the equation represents an ellipse, a point, or else has no graph. (Why can't the graph be a circle?) ◀

In cases where a quadratic equation represents a point, a line, a pair of parallel lines, a pair of intersecting lines, or has no graph, we say that equation represents a ***degenerate conic section***. Thus, if we allow for possible degeneracy, it follows from Theorem H.1 that *every quadratic equation has a conic section as its graph.*

EXERCISE SET H   C CAS

1–5 Use the discriminant to identify the graph of the given equation.

1. $x^2 - xy + y^2 - 2 = 0$

2. $x^2 + 4xy - 2y^2 - 6 = 0$

3. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$

4. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

5. $34x^2 - 24xy + 41y^2 - 25 = 0$

6. Each of the following represents a degenerate conic section. Where possible, sketch the graph.
(a) $x^2 - y^2 = 0$
(b) $x^2 + 3y^2 + 7 = 0$
(c) $8x^2 + 7y^2 = 0$
(d) $x^2 - 2xy + y^2 = 0$
(e) $9x^2 + 12xy + 4y^2 - 36 = 0$
(f) $x^2 + y^2 - 2x - 4y = -5$

7. Prove parts (b) and (c) of Theorem H.1.

C **8.** Consider the conic whose equation is

$$x^2 + xy + 2y^2 - x + 3y + 1 = 0$$

(a) Use the discriminant to identify the conic.
(b) Graph the equation by solving for y in terms of x and graphing both solutions.
(c) Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.

C **9.** Consider the conic whose equation is

$$2x^2 + 9xy + y^2 - 6x + y - 4 = 0$$

(a) Use the discriminant to identify the conic.
(b) Graph the equation by solving for y in terms of x and graphing both solutions.
(c) Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.