Chapter 11

Series Solutions:
Bessel Functions, Legendre Polynomials

Section 11.1 Aging Springs and Steady Temperatures

Problem 9. Find the coefficient \( a_{100} \) in the series \( \sum_{n=0}^{\infty} a_n x^n \) if it is known that \( a_0 = a_1 = 1 \) and that

\[
\sum_{n=0}^{\infty} [(n+1)^2a_{n+2} - n^2a_{n+1} + (n-1)a_n]x^n = 0
\]

Solution:
We want to find \( a_{100} \) if it is known that \( \sum_{n=0}^{\infty} [(n+1)^2a_{n+2} - n^2a_{n+1} + (n-1)a_n]x^n = 0 \).
The recursion identity here is \( (n+1)^2a_{n+2} - n^2a_{n+1} + (n-1)a_n = 0 \), that is, \( a_{n+2} = (n^2a_{n+1} - (n-1)a_n)/(n+1)^2 \), \( n = 0, 1, 2, \ldots \), where \( a_0 = a_1 = 1 \). A computer may easily be programmed to calculate the coefficients using the identity and the initial data \( a_0 = a_1 = 1 \). In fact, \( a_{100} \approx -1.0629 \times 10^{-6} \).

Section 11.2 Series Solutions Near an Ordinary Point

Problem 5. (Mathieu Equation). The Mathieu equation is \( y'' + (a + b \cos \omega x) y = 0 \). Calculate the first four nonvanishing coefficients in the power series expansion of the solution of the Mathieu equation; use \( a = 1 \), \( b = 2 \), \( \omega = 1 \), and \( y(0) = 1 \), \( y'(0) = 0 \).

Solution:
The Mathieu equation is \( y'' + (a + b \cos \omega x)y = 0 \). Set \( a = 1 \), \( b = 2 \), \( \omega = 1 \), and \( y(0) = 1 \), \( y'(0) = 0 \). Since we only want the first few terms, we shall not attempt to find the general coefficient of the series solution. The initial data imply that \( a_0 = 1 \) and \( a_1 = 0 \) if \( y = a_0 + a_1 x + a_2 x^2 + \cdots \). Since \( \cos x \) has Taylor series \( 1 - x^2/2! + x^4/4! - x^6/6! + \cdots \) about \( x_0 = 0 \), we see that inserting the series for \( y \) and for \( \cos x \) in the ODE and using item 6(ii) of Appendix B.2 to determine the first few terms of the product series for \( (2 \cos x)y(x) \), we have that

\[
0 = y'' + (1 + 2 \cos x) y = (3 + 2a_2) + 6a_3 x + (12a_4 + 3a_2 - 1)x^2 + (3a_3 + 20a_5)x^3 + (30a_6 + 3a_4 - a_2 + \frac{1}{12})x^4 + \cdots
\]

So \( a_2 = -3/2 \), \( a_3 = 0 \), \( a_4 = 13/48 \), \( a_5 = 0 \), \( a_6 = -23/288 \), and the series for the solution is

\[
y = 1 - \frac{3}{2} x^2 + \frac{13}{48} x^4 - \frac{23}{288} x^6 + \cdots
\]
Section 11.3 Legendre Polynomials

**Problem 1.** (*Legendre’s ODE and the Wronskian Method*). Legendre’s equation of orders 0 and 1 have respective solutions \( P_0 = 1 \) and \( P_1 = x \). Use the Wronskian Reduction Method (Problem 6, Section 3.7) to find a second independent solution for each case in terms of elementary functions. Plot the second solution for \(|x| < 1\).

**Solution:**
Assume that \( u(x) \) is a solution of the ODE \( y'' + a(x)y' + b(x)y = 0 \) and that \( u(x) \) is not the trivial solution \( y = 0 \), all \( x \). Then a second solution \( v(x) \), \(|x| < 1\), independent of \( u(x) \) can be found by solving the ODE

\[
W[u, v](x) = uv' - u'v = e^{-\int a(s)ds}
\]

In Legendre’s ODEs of order 0 and 1, we have that \( a(x) = -2x/(1-x^2) \), \( u_0(x) = 1 \), and \( u_1(x) = x \), respectively. So \( v_0' = e^{\int 2s/(1-s^2)ds} = C_1/(1-x^2) \), and (using an integral table)

\[
v_0 = \frac{C_1}{2} \ln \frac{1+x}{1-x} + C_2, \quad (C_1 > 0)
\]

Also, \( xv_1' - v_1 = C_1/(1-x^2) \), \( v_1' - v_1/x = C_1/x(1-x^2) \).

\[
v_1 = e^{\int (1/s)ds}[C_3 + \int e^{-\int (1/s)ds} \frac{C_1}{s(1-s^2)} ds]
\]

\[
= x[C_3 + C_1 \int \frac{1}{s^2(1-s^2)} ds]
\]

\[
= x[C_3 + C_1 \left(-\frac{1}{x} + \frac{1}{2} \ln \frac{1+x}{1-x}\right)] \quad (C_1 > 0)
\]

where we have used a table of integrals. So, second independent solutions for Legendre’s equations of order 0 and 1 on the interval \(|x| < 1\) are, respectively,

\[
v_0 = \frac{C_1}{2} \ln \frac{1+x}{1-x} + C_2 \quad \text{and} \quad v_1 = x[C_3 + C_1 \left(-\frac{1}{x} + \frac{1}{2} \ln \frac{1+x}{1-x}\right)]
\]

where \( C_1, C_2 \) and \( C_3 \) are constants and \( C_1 > 0 \). See Graph 1 and Graph 2, respectively, for graphs of \( v_0 \) and \( v_1 \), where \( C_1 = 1, \ C_2 = 0, \ C_3 = 0 \).
Section 11.4 Regular Singular Points

Problem 3. (Euler ODEs). Find the general, real-valued solution (for \( x > 0 \)) of each of the following equations. Plot some solutions of each equation.

\((a)\) \( x^2 y'' - 6y = 0 \)
\((b)\) \( x^2 y'' + xy' - 4y = 0 \)
\((c)\) \( x^2 y'' + xy' + 9y = 0 \)
\((d)\) \( x^2 y'' + xy'/2 - y/2 = 0 \)
\((e)\) \( xy'' - y' + (5/x)y = 0 \)
\((f)\) \( x^2 y'' + 7xy' + 9y = 0 \)

Solution:
Each of the equations may be written as an Euler equation \( x^2 y'' + p_0 xy' + q_0 y = 0 \), and the solutions are determined by the roots \( r_1 \) and \( r_2 \) of the indicial polynomial \( r^2 + (p_0 - 1)r + q_0 \). The three types of solutions are given in (13) in the text. In each case we list the indicial polynomial, its roots, and then the general solution; \( c_1 \) and \( c_2 \) denote arbitrary constants. It is assumed throughout that \( x > 0 \), and absolute value signs around \( x \) are not needed. See the appropriate figures for plots of some solutions of each equation. For negative \( x \), replace \( x \) by \( |x| \) in all formulas.

\((a)\) \( r^2 - r - 6; \quad r_1 = -2, \quad r_2 = 3; \quad y = c_1 x^{-2} + c_2 x^3 \).

\((c)\) \( r^2 + 9; \quad r_1 = 3i, \quad r_2 = -3i; \quad y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x). \)

\((e)\) \( r^2 - 2r + 5; \quad r_1 = 1 + 2i = \bar{r}_2; \quad y = x[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]. \)

Section 11.5 Series Solutions Near Regular Singular Points, I

Problem 3. (Frobenius Series). Solve the ODE \( 3x^2 y'' + 5xy' - e^x y = 0 \) by expanding \( e^x \) in a Taylor series about \( x_0 = 0 \) and recalling the formula for the product of two series (Appendix B.2). You only need to find the first four terms in the Frobenius series explicitly.

Solution:
The ODE is \( 3x^2 y'' + 5xy' - e^x y = 0 \). The Taylor series for \( e^x \) about \( x_0 = 0 \) is \( 1 + x + x^2/2! + x^3/3! + \cdots \). So \( p_0 = 5/3, \quad q_0 = -1/3 \) for the ODE in nonstandard form

\[ x^2 y'' + 5xy'/3 - e^x y'/3 = x^2 y'' + 5xy'/3 + (-1/3 - x/3 - \cdots) y = 0 \]
The indicial polynomial is $r^2 + 2r/3 - 1/3$ and $r_1 = 1/3$, $r_2 = -1$. There are independent solutions of the form

$$y_1 = x^{1/3} + a_1 x^{4/3} + a_2 x^{7/3} + \cdots, \quad y_2 = x^{-1} + b_1 + b_2 x + \cdots$$

where we have set $a_0 = b_0 = 1$ in each series. Although we could use Frobenius Theorem I to determine the coefficients, it is probably simpler to make a direct substitution of $y_1$ and then $y_2$ into the ODE. For $y_1$ we have that

$$3x^2 y_1'' + 5xy_1' - e^x y_1 = (7a_1 - 1)x^{4/3} + (20a_2 - a_1 - 1/2)x^{7/3} + (39a_3 - a_2 - a_1/2 - 1/6)x^{10/3} + \cdots = 0$$

So $a_1 = 1/7$, $a_2 = 9/280$, $a_3 = 227/32760$ and

$$y_1 = x^{1/3} + x^{4/3}/7 + 9x^{7/3}/280 + 227x^{10/3}/32760 + \cdots$$

In the same way we see that

$$y_2 = x^{-1} - 1 - x/8 - 11x^2/360 + \cdots$$

### Section 11.6 Bessel Functions

**Problem 5.** Plot $J_0$, $J_1$, $J_2$, and $J_3$. Then plot $J_{1/3}$, $J_{4/3}$, $J_{7/3}$, $0 \leq x \leq 20$.

**Solution:**
See Figs. 5 for plots of $J_0$, $J_1$, $J_2$, $J_3$ (Graph 1) and $J_{1/3}$, $J_{4/3}$, $J_{7/3}$ (Graph 2).

### Section 11.7 Series Solutions Near Regular Singular Points, II

**Problem 1.** (Frobenius’s Theorem II: Cases II, III), Check that 0 is a regular singular point of each equation and find a basis for the solution space on the interval $(0, \infty)$. [Hint: In parts (a), (b), (c), the solution $y_1$ is easily found in closed form. Then use the Wronskian Reduction Method of Problem 6 of Section 3.7 to find a second independent solution.]

- (a) $xy'' + (1 + x)y' + y = 0$
- (b) $x^2 y'' + x(x - 1)y' + (1 - x)y = 0$
- (c) $xy'' - xy' + y = 0$
- (d) $xy'' - x^2 y' + y = 0$
Solution:
It is straightforward to show that 0 is a regular singular point of each ODE; the details are omitted.

(a) The ODE is \( xy'' + (1 + x)y' + y = 0 \). The equation in standard form is \( x^2 y'' + x(1 + x)y' + xy = 0 \). So, \( p_0 = 1, q_0 = 0 \) and the indicial polynomial is \( r^2 \) with roots \( r_1 = r_2 = 0 \). We are in Case II of Frobenius Theorem II. First, there is a solution of the form \( y_1 = \sum_{n=0}^{\infty} a_n x^n \) with \( a_0 = 1 \). Using the techniques of Section 11.5, we see that the recursion formula in this case is

\[
a_{n+1} = -\frac{a_n}{n+1}, \quad n = 0, 1, 2, \ldots
\]

and a solution is

\[
y_1 = \sum_{n=0}^{\infty} (-1)^n x^n / n! = e^{-x}
\]

We have by the Wronskian Reduction Method of Problem 6, Section 3.7, that \( e^{-x} y_2' + e^{-x} y_2 = e^{-x}/x \), where the right-hand side is \( \exp[-\int x a(s) ds] \), \( a(s) = s^{-1} + 1 \), for the normalized ODE, \( y'' + x^{-1}(1 + x)y' + x^{-1}y = 0 \). So, \( y_2' + y_2 = 1/x \), or \( (y_2 e^x)' = e^x/x \).

(c) The ODE is \( xy'' - xy' + y = 0 \). Here the ODE in standard form is \( x^2 y'' - x^2 y' + xy = 0 \), and the indicial polynomial is \( r^2 - r \) and \( r_1 = 1, r_2 = 0 \). We are in Case III of Frobenius Theorem II. We may use the methods of Section 11.5 to find a solution \( y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad a_0 = 1 \). The recursion formula is

\[
a_{n+1} = \frac{(1 - n)a_n}{n(n + 1)}, \quad n \geq 0
\]

So \( a_n = 0 \) for \( n \geq 1 \) and

\[
y_1 = x
\]

is a solution. We have by the Wronskian Reduction Method of Problem 6, Section 3.7 that \( xy_2' - y_2 = e^x, \ y_2' - y_2/x = e^x/x, \ (y_2/x)' = e^x/x^2 \) and

\[
y_2 = x \int e^x/s^2 ds = x[-1/x + \ln x + \cdots + x^{n-1}/(n!(n-1)) + \cdots]
\]

where we replaced \( e^x \) by its Maclaurin series, divided each term by \( s^2 \), and then integrated term by term.
Section 11.8 Steady Temperatures in Spheres and Cylinders

Problem 3. Solve the boundary value problems below:

(a) Solve boundary problem (16) if \( f(r) = 1, \ 0 \leq r \leq 1. \)

(b) Solve the boundary value problem that arises from (16) if the side-wall temperature condition \( u(1, \theta, z) = 0 \) is replaced by the perfect-insulation condition \( u_r(1, \theta, z) = 0, \ 0 \leq z \leq a, \ -\pi \leq \theta \leq \pi. \) [Hint: Use (19) in Theorem 11.6.2 with \( p = 0, \) and use the identity below in Problem 4 with \( a = x_n^*, \) where \( x_n^* \) is a positive zero of \( J_0' \) (and so of \( J_1' \)).]

Solution:

(a) We are to solve boundary problem (16) in a cylinder if \( f(r) = 1, \ 0 \geq r \geq 1. \) If \( f(r) = 1, \ 0 \leq r \leq 1, \) then with \( x_n, n = 1, 2, ..., \) denoting the consecutive positive zeros of \( J_0(x), \) we have from (20) that

\[
A_n = \frac{2}{[J_1(x_n)]^2 \sinh(x_n a)} \int_0^1 r J_0(x_n r) \, dr = \frac{2}{[J_1(x_n)]^2 \sinh(x_n a) k_n^2} \int_0^1 (r J_1(x_n r))' \, dr
\]

\[
= \frac{2}{[J_1(x_n)]^2 \sinh(x_n a) x_n^2} (r J_1(x_n r)) \Big|_0^1 = \frac{2}{J_1(x_n) \sinh(x_n a) x_n^2}
\]

where we have used the Bessel identity, \( (x J_1(x))' = x J_0(x). \) So

\[
u(r, z) = \sum_{n=1}^{\infty} \frac{2}{x_n^2} \cdot \frac{\sinh(x_n (a - z))}{\sinh(x_n a)} \cdot \frac{J_0(x_n r)}{J_1(x_n)}
\]