Chapter 0

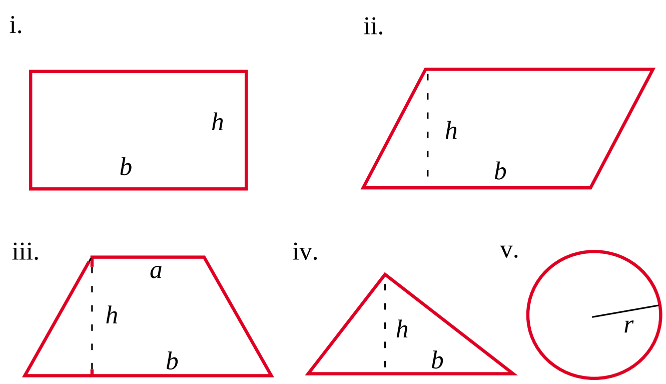
Algebra and Trigonometry Review

0.1 Elementary Geometry Formulas
0.2 Algebra of Fractions
0.3 Algebra of Exponents
0.4 Algebra of Polynomials
0.5 Conic Sections
0.6 Systems of Equations
0.7 Trigonometry Refresher
0.8 Word Problems
0.9 Mathematical Induction

0.1 ELEMENTARY GEOMETRY FORMULAS

Time and again in calculus you will need formulas and techniques from elementary geometry; for your reference we will summarize the most important of these results. You should memorize these formulas, although most you probably already know.

0.1.1 Plane Geometry



i. Rectangle with base $= b$, height $= h$

Area $A = bh$

Perimeter $p = 2b + 2h$

ii. Parallelogram with base $= b$, height $= h$

Area $A = bh$

iii. Trapezoid with bases $= a, b$, height $= h$

Area $A = \frac{1}{2}(a + b)h$

iv. Triangle with base $= b$, height $= h$

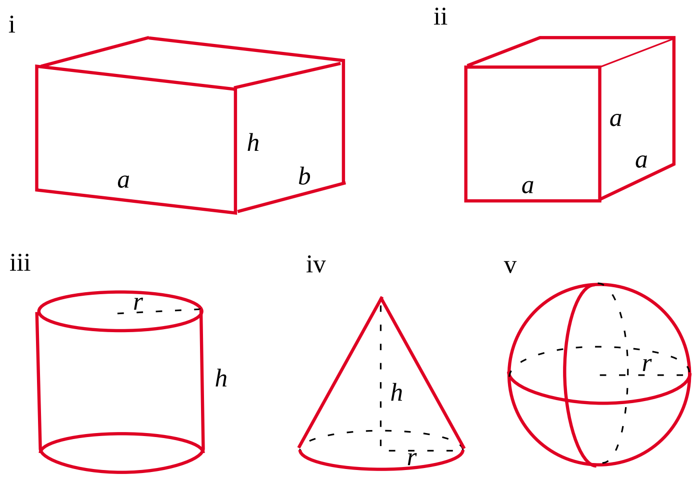
Area $A = \frac{1}{2}bh$

v. Circle with radius $= r$

Area $A = \pi r^2$

Circumference $C = 2\pi r$

0.1.2 Solid Geometry



i. Rectangular box with length $= a$, width $= b$, height $= h$

Volume $V = abh$

Surface area $S = 2ab + 2ah + 2bh$

ii. Cube with side length $= a$ (special case of i)

Volume $V = a^3$

Surface area $S = 6a^2$

iii. Right circular cylinder with radius $= r$, height $= h$

Volume $V = \pi r^2 h$

Surface area $S = 2\pi r^2 + 2\pi rh$

iv. Right circular cone with radius $= r$, height h

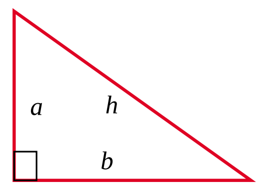
Volume $V = \frac{1}{3}\pi r^2 h$

v. Sphere with radius $= r$

Volume $V = \frac{4}{3}\pi r^3$

Surface area $S = 4\pi r^2$

0.1.3 Pythagorean Theorem

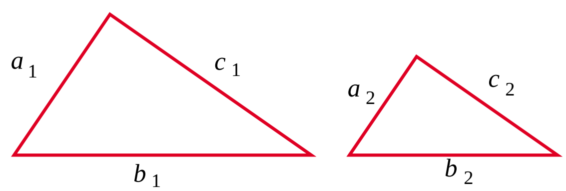


Suppose Δ is a right triangle with hypotenuse $= h$ and remaining sides $= a, b$. Then

$$a^2 + b^2 = h^2$$

The importance of this result cannot be overstated; it occurs over and over again in calculus. You should constantly be on the lookout for this relationship, especially in applied problems of a geometric nature.

0.1.4 Similar Triangles



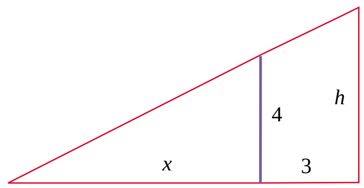
Suppose Δ_1 and Δ_2 are similar triangles (i.e., the three angles of Δ_1 are the same as the three angles of Δ_2) with corresponding side lengths a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Then the corresponding sides are proportional, i.e.,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

The importance of this result also cannot be overstated; however, unlike the Pythagorean theorem, for which people tend to develop a sharp eye, *in applied problems similar triangle relationships are quite commonly overlooked.*

Example

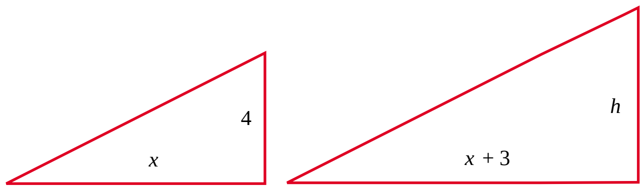
A 4-foot high fence is 3 feet away from the side of a building.



A ladder is propped up on the fence with its foot on the ground and its top against the building side. Express the height h of the top of the ladder as a function of x, the distance of the foot of the ladder from the fence.

Solution

There are two similar triangles to be used:



Thus

$$\frac{h}{4} = \frac{x+3}{x}, \text{ or } h = \frac{4x+12}{x}.$$

0.2 ALGEBRA OF FRACTIONS

Many beginning calculus students are unsure of, or are hesitant in using, the algebraic rules governing fractions. Proficiency in the use of these rules is ***crucial*** for success in calculus.

0.2.1 Multiplication

Multiplying fractions together is very easy. Simply multiply together the two numerators and then the two denominators:

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{ac}{bd} \tag{0.1}$$

As a special case, consider a fraction $\dfrac{c}{d}$ multiplied by an arbitrary number x. Since $x = \dfrac{x}{1}$ we have

$$x \cdot \left(\frac{c}{d}\right) = \left(\frac{x}{1}\right) \cdot \left(\frac{c}{d}\right) = \frac{xc}{1d} = \frac{xc}{d},$$

so x just multiplies the numerator of $\frac{c}{d}$. Seldom do people have trouble with multiplication of fractions.

Example 1

Combine

$$\left(\frac{3+x}{x}\right)\cdot\left(\frac{3-x}{1+x}\right)$$

into one fraction.

Solution

$$\left(\frac{3+x}{x}\right)\cdot\left(\frac{3-x}{1+x}\right)=\frac{(3+x)(3-x)}{x(1+x)}=\frac{9-x^2}{x+x^2}.$$

0.2.2 Addition

This is more complicated than multiplication. Consider first the addition of two fractions with the same denominator. The rule is: *add the numerators together*, i.e.,

$$\frac{a}{h}+\frac{b}{h}=\frac{a+b}{h} \tag{0.2}$$

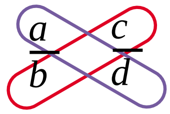
(This makes sense: two-fifths of a pie plus one-fifth of a pie equals three-fifths of a pie.) For fractions with different denominators, say $\frac{a}{b}$ and $\frac{c}{d}$ we must first alter the fractions to have a ***common denominator*** say $h = bd$. We do this by multiplication:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \left(\frac{a}{b}\right)\cdot 1+1\cdot\left(\frac{c}{d}\right)\\ &= \left(\frac{a}{b}\right)\cdot\left(\frac{d}{d}\right)+\left(\frac{b}{b}\right)\cdot\left(\frac{c}{d}\right)\\ &= \frac{ad}{bd}+\frac{bc}{bd}\\ &= \frac{ad+bc}{bd}\end{aligned}$$

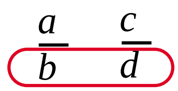
We have thus derived the basic addition formula:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \tag{0.3}$$

The formula is easily remembered as “cross multiplication:”



gives the numerator $ad + bc$



gives the denominator bd

DANGER: Notice that

$$\frac{a}{b} + \frac{c}{d} \text{ does } \textbf{NOT} \text{ equal } \frac{a+c}{b+d}!!$$

Memorize rule (0.3) as though your life depends on it .... ‘cause in calculus it does. As a special case, consider a fraction $\frac{c}{d}$ added to an arbitrary number x. Rule (0.3) still applies by writing $x = \frac{x}{1}$, i.e.,

$$x + \frac{c}{d} = \frac{x}{1} + \frac{c}{d} = \frac{xd + c}{d}$$

Example 2

Combine $\frac{h+3}{1-h} + \frac{h}{1+h}$ into one fraction.

Solution

$$\begin{aligned}\frac{h+3}{1-h} + \frac{h}{1+h} &= \frac{(h+3)(1+h) + (1-h)h}{(1-h)(1+h)} \\ &= \frac{h^2 + 4h + 3 + h - h^2}{1-h^2} \\ &= \frac{5h+3}{1-h^2}\end{aligned}$$

Example 3

Combine $\frac{x+2h}{x-h} + 2$ into one fraction.

Solution

$$\begin{aligned}\frac{x+2h}{x-h}+2 &= \frac{x+2h}{x-h}+\frac{2}{1}\\ &= \frac{(x+2h)+2(x-h)}{x-h}\\ &= \frac{3x}{x-h}\end{aligned}$$

NOTE: In (0.3) we used the denominator $h = bd$ because it was a ***common denominator*** for both a/b and c/d ; however, in many situations, a smaller common denominator will exist, and computations will be greatly simplified if it is used in conjunction with (0.2). The following is a good example.

Example 4

Combine

$$\frac{3-2x^2}{4z^3(x+3)^3(x-1)^2}+\frac{x+1}{2z^3(x+3)^3(x-1)}$$

Solution

We *could* use (0.3) with common denominator

$$h = 8z^6(x+3)^6(x-1)^3\ldots$$

but that would be very silly when we notice that the first denominator is merely $2(x-1)$ times the second. We can, instead, give both fractions a common denominator by multiplying the second fraction by $\frac{2(x-1)}{2(x-1)}$, as we now show:

$$\frac{3-2x^2}{4z^3(x+3)^3(x-1)^2}+\frac{x+1}{2z^3(x+3)^3(x-1)}\cdot\frac{2(x-1)}{2(x-1)}$$

$$\begin{aligned}&= \frac{3-2x^2}{4z^3(x+3)^3(x-1)^2}+\frac{2(x+1)(x-1)}{4z^3(x+3)^3(x-1)^2}\\ &= \frac{(3-2x^2)+(2x^2-2)}{4z^3(x+3)^3(x-1)^2}\\ &= \frac{1}{4z^3(x+3)^3(x-1)^2}\end{aligned}$$

0.2.3 Subtraction

If you know how to add fractions then you know how to subtract them; the "trick" is to convert $-\left(\frac{c}{d}\right)$ into $\frac{-c}{d}$ as follows:

$$\begin{aligned}\frac{a}{b}-\frac{c}{d} &= \frac{a}{b}+\frac{-c}{d}\\ &= \frac{ad+b(-c)}{bd}=\frac{ad-bc}{bd}\end{aligned}$$

Thus our subtraction formula:

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd} \tag{0.4}$$

Example 5

Combine $\frac{x-y}{x+y}-\frac{x+y}{x-y}$ into one fraction.

Solution

$$\begin{aligned}\frac{x-y}{x+y}-\frac{x+y}{x-y} &= \frac{(x-y)(x-y)-(x+y)(x+y)}{(x+y)(x-y)}\\ &= \frac{(x^2-2xy+y^2)-(x^2+2xy+y^2)}{x^2-y^2}\\ &= \frac{-4xy}{x^2-y^2}=\frac{4xy}{y^2-x^2}\end{aligned}$$

0.2.4 Division

Here is the big pitfall. People are often very careless with division of fractions, and consequently make horrendous errors in calculus problems. *Go over these rules carefully*.

Consider the division of $\frac{a}{b}$ by $\frac{c}{d}$. The "trick" is to remember that

$$\frac{c}{d}\cdot\frac{d}{c}=\frac{cd}{cd}=1.$$

Thus

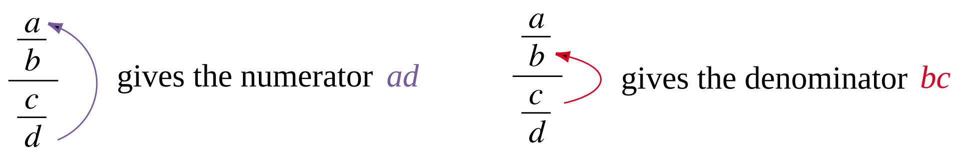
$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \cdot \frac{\left(\frac{d}{c}\right)}{\left(\frac{d}{c}\right)} = \frac{\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)}{\left(\frac{c}{d}\right)\left(\frac{d}{c}\right)}$$

$$= \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}.$$

Our basic division formula is thus

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc} \qquad (0.5)$$

The formula is easily remembered by "swinging arcs."



The formula can also be remembered by "invert the denominator, then multiply," i.e.,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

All fraction divisions can be handled by rule (0.5); however, it is easy to be careless in applying the formula in "less obvious" situations. For example, people tend to confuse the two expressions

$$\frac{\frac{a}{h}}{d} \text{ and } \frac{a}{\frac{h}{d}}$$

These are very different, as we now show.

$$\frac{\frac{a}{h}}{d} = \frac{\left(\frac{a}{h}\right)}{\left(\frac{d}{1}\right)} = \frac{a \cdot 1}{dh} = \frac{a}{dh},$$

unequal expressions

$$\frac{\frac{a}{d}}{h} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{h}{d}\right)} = \frac{ad}{1 \cdot h} = \frac{ad}{h},$$

[redacted] highlighted; keep in mind, however, that

$$= \frac{a}{dh} \tag{0.6}$$

$$= \frac{ad}{h} \tag{0.7}$$

$$1)(n+1)(x-1)^{n+1}$$
$$^2(x-1)^n$$

$$^2 - 1)(x-1)$$
$$n^2$$

$$1 - \quad 1$$

0.6 : $\dfrac{\left(\frac{a}{h}\right)}{d} = \dfrac{a}{dh}$

0.7 : $\dfrac{a}{\left(\frac{h}{d}\right)} = \dfrac{ad}{h}$

EXERCISES

In Problems 1 through 8 combine the given expressions into one simple fraction.

1. $\dfrac{2}{3} - \dfrac{4}{x}$
2. $\frac{1}{3}(a+1) + \dfrac{2}{a-1}$
3. $\dfrac{\left(\frac{x}{x+1}\right)}{3} + 1$
4. $\dfrac{x}{\left(\frac{x+1}{3}\right)} + 1$
5. $\dfrac{5-3x^2}{3(a+1)^2(x-1)^2} + \dfrac{x+2}{(a+1)^2(x-1)}$
6. $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1}$
7. $\dfrac{\frac{-2}{h}}{h+1} + \dfrac{2}{h}$
8. $\dfrac{\dfrac{1}{x+1} - \dfrac{1}{x}}{h}$
9. Solve for x: $\dfrac{x+1}{x-3} - \dfrac{10}{x+3} = 1$

ANSWERS

1. $\dfrac{2x-12}{3x}$
2. $\dfrac{a^2+5}{3a-3}$
3. $\dfrac{4x+3}{3x+3}$
4. $\dfrac{4x+1}{x+1}$
5. $\dfrac{3x-1}{3(a+1)^2(x-1)^2}$
6.

$$\begin{aligned}\frac{x+1}{x-1} + \frac{x-1}{x+1} &= \frac{(x+1)^2 + (x-1)^2}{(x-1)(x+1)} \\ &= \frac{x^2+2x+1+x^2-2x+1}{x^2-1} \\ &= \frac{2x^2+2}{x^2-1}\end{aligned}$$

7.

$$\frac{\left(\frac{-2}{h}\right)}{h+1}+\frac{2}{h} = \frac{\left(\frac{-2}{h}\right)h+2(h+1)}{(h+1)h} \underset{0.5}{=} \frac{-2+2h+2}{(h+1)h}$$
$$= \frac{2h}{(h+1)h} = \frac{2}{h+1}$$

8.

$$\frac{\frac{1}{x+1}-\frac{1}{x}}{h} \underset{0.4}{=} \frac{\left(\frac{x-(x+1)}{x(x+1)}\right)}{h} = \frac{-1}{x(x+1)h}$$
$$= -\frac{1}{x(x+1)}$$

9.

$$\begin{aligned}
\frac{x+1}{x-3}-\frac{10}{x+3} &= 1 \\
\frac{(x+1)(x+3)-10(x-3)}{(x-3)(x+3)} &\underset{0.4}{=} 1 \\
\frac{x^2+4x+3-10x+30}{x^2-9} &= 1, \\
x^2-6x+33 &= x^2-9 \\
-6x+42 &= 0 \\
x &= 7
\end{aligned}$$

0.3 ALGEBRA OF EXPONENTS

The rules governing exponents are crucial for calculus and its applications, and yet many calculus students are unsure of them. What follows is a brief description of the algebra of exponents. The examples and exercises which we present are similar to the more difficult problems you will see in calculus.

Suppose a is any *positive real* number, and r is any *rational* number, i.e., $r = m/n$, the quotient of two *integers* m and n, where $n = 0$. We wish to recall the definition of a^r, read "a raised to the r-th power." We do this in the following stages:

0.3.1 What You Should Already Know

Suppose n is a *positive integer*. Then a^n is simply n copies of a multiplied together, i.e.,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a \cdot a}_{n \text{ copies of } a}$$

Examples:

$$2^3 = 2 \cdot 2 \cdot 2 = 8, \quad \left(\frac{3}{4}\right)^2 = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}.$$

Also recall that any number raised to the "zero-th" power is defined to be 1, i.e.,

$$a^0 = 1.$$

Examples:

$$3^0 = 1, \quad \pi^0 = 1, \quad \left(\tfrac{1}{2}\right)^0 = 1$$

You are probably also comfortable with the "n-th root" of a positive number a: it is that positive real number $a^{1/n}$(also written $\sqrt[n]{a}$) which, when raised to the n-th power, gives back a, i.e.,

$$\left(a^{1/n}\right)^n = a$$

In particular:

$$\left(a^{1/2}\right)^2 = a, \quad \left(a^{1/3}\right)^3 = a.$$

As an example, $8^{1/3}$ is that positive nmber which, when raised to the 3rd power, gives 8. Since $2^3 = 8$, then 2 is the number we are looking for, i.e., $8^{1/3} = 2$.

0.3.2 What You Might Have Forgotten

We are ready for the major definition: what is $a^{m/n}$, where m and n are any two *positive integers*? Well, we know how to define the n-th root of a (from previous paragraph) and we know how to raise any number to the m-th power; thus the expression $(a^{1/n})^m$ makes sense. *This is our definition for $a^{m/n}$*, i.e.,

DEFINITION Suppose a is any positive real number, and m, n are any positive integers. Then

$$a^{m/n} = \left(a^{1/n}\right)^m$$

Examples of this definition are

$$8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$$

$$2^{3/2} = \left(2^{1/2}\right)^3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 2\sqrt{2}$$

$$\left(\tfrac{1}{16}\right)^{3/4} = \left[\left(\tfrac{1}{16}\right)^{1/4}\right]^3 = \left[\tfrac{1}{2}\right]^3 = \tfrac{1}{8}.$$

This definition allows us to raise any positive real number a (the "base") to any *non-negative* rational power $r = m/n$ (the "exponent"). To allow *negative* rational powers we use the familar "invert the base" rule:

$$a^{-r} = \left(\frac{1}{a}\right)^r = \frac{1}{a^r} \tag{0.8}$$

As examples of this rule,

$$8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}$$

$$2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

0.3.3 Rules of Exponent Algebra

MEMORIZE CAREFULLY!

Addition and subtraction of exponents

$$a^{r+s} = a^r a^s \text{ and } a^{r-s} = \frac{a^r}{a^s} = \frac{1}{a^{s-r}} \tag{0.9}$$

Examples:

$$2^{1+r} = 2(2)^r$$

$$4^{(1/2-r)} = \frac{4^{1/2}}{4^r} = \frac{2}{4^r}.$$

Multiplication and division of exponents

$$\begin{gathered} a^{rs} = (a^r)^s = (a^s)^r \\ \text{and} \\ a^{r/s} = \left(a^{1/s}\right)^r = (a^r)^{1/s} \end{gathered} \tag{0.10}$$

Examples:

$$2^{2r} = \left(2^2\right)^r = 4^r$$

$$8^{-r/3} = \left(8^{1/3}\right)^{-r} = 2^{-r} = \frac{1}{2^r}.$$

Multiplication and division of bases

$$(ab)^r = a^r b^r \text{ and } \left(\tfrac{a}{b}\right)^r = \frac{a^r}{b^r} \tag{0.11}$$

Examples:

$$(4a)^{1/2} = 4^{1/2}a^{1/2} = 2\sqrt{a}$$

$$\left(\frac{a^2}{8}\right)^{2/3} = \frac{\left(a^2\right)^{2/3}}{8^{2/3}} = \frac{a^{4/3}}{4}$$

NON-FORMULAS

Addition and subtraction of bases

$$(a+b)^r = ? \text{ and } (a-b)^r = ? \tag{0.12}$$

There are **NO** simple formulas in these cases. For instance, $(a+b)^r$ does **NOT** equal a^r+b^r. ***This represents a very common error, so be careful!*** There are a number of observations to be made here.

1. Memorizing these rules is not difficult if you keep in mind the integer exponent case. For example, suppose you need to simplify $\left(3^{r+1}\right)^{r-1}$ but cannot recall if the rule for $(a^r)^s$ is a^{rs} or a^{r+s}. Then check the formulas in a simple case, say with $a=3$, $r=2$, and $s=3$:

$$\left.\begin{aligned} (a^r)^s &= \left(3^2\right)^3 = 9^3 &&= 729 \\ a^{rs} &= 3^{2\cdot 3} = 3^6 &&= 729 \end{aligned}\right\} \text{EQUAL}$$

$$a^{r+s} = 3^{2+3} = 3^5 = 243$$

   This should pretty quickly make you remember $(a^r)^s = a^{rs}$. Thus

$$\left(3^{r+1}\right)^{r-1} = 3^{(r+1)(r-1)} = 3^{r^2-1}.$$

2. Although a^r has only been defined for a a *positive* real number, in *some* cases a^r does make sense when a is negative. For example, $(-2)^3 = -8$, and thus $(-8)^{1/3} = -2$. On the other hand, $(-8)^{1/2}$ is *not defined* (unless we allow ourselves to use complex numbers). In general, $a^{m/n}$ will be defined for negative a if n is an *odd* integer, but not if n is an *even* integer. For these reasons (and others) rules 0.9, 0.10, 0.11 are not always valid for negative bases *and must be handled with care!* As an example, when a is negative (say $a=-2$) it is NOT true that $\left(a^2\right)^{1/2} = a^{2(1/2)} = a$; instead $\left(a^2\right)^{1/2} = -a$. See Examples 5 and 6 below.

3. A major goal of calculus is to define a^r for r any *real* number, not just for r a *rational* number. This is done in Chapter 7 of the text. *It is then amazing but true that all the rules* 0.8, 0.9, 0.10, 0.11 *remain valid.*

4. Rule (0.8) is quite useful in allowing us to eliminate negative exponents in fractions. Observe its use in the following simplification:

$$\frac{(x+2)^{-3}(x-2)^3}{(x^2-4)^{-2}} \underset{0.8}{=} \frac{(x-2)^3(x^2-4)^2}{(x+2)^3}$$

where the terms with negative exponents have "switched" between the numerator and denominator

$$= \frac{(x-2)^3\,((x-2)(x+2))^2}{(x+2)^3}$$

(by factoring (x^2-4))

$$\underset{0.11}{=} \frac{(x-2)^3(x-2)^2(x+2)^2}{(x+2)^3}$$

$$\underset{0.9}{=} \frac{(x-2)^5(x+2)^2}{(x+2)^3}$$

(by combining the two $(x-2)$ terms)

$$= \frac{(x-2)^5}{(x+2)}$$

(by cancelling $(x+2)^2$ from both the numerator and the denominator)

n-th Roots An important special case of exponents occurs when $r = 1/n$ for n a positive integer. In that case we often will use the notation

$$a^{1/n} = \sqrt[n]{a},$$

and refer to $\sqrt[n]{a}$ as the *n-th root of* a. (When $n = 2$ we simply write $\sqrt{a}$ in place of $\sqrt[2]{a}$.)

From its definition the basic formulas

$$\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a \tag{0.13}$$

easily follow for any *positive* real number a.

All the rules 0.8, 0.9, 0.10, 0.11 are of course valid for $\sqrt[n]{a}$; it is, however, instructive to write 0.11 down in the n-th root notation:

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \text{ and } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \tag{0.14}$$

It is also useful to emphasize again the lack of any "0.12 rules," i.e., there are NO simple formulas for

$$\sqrt[n]{a+b} \text{ or } \sqrt[n]{a-b}$$

Don't dream anything up for these expressions!

0.3.4 Numerous Examples

Example 1

Simplify $\dfrac{x(x+1)^{1/3}-(x+1)^{4/3}}{x^2-1}$

Solution

Using $4/3 = 1+(1/3)$, 0.9 allows us to rewrite our expression as

$$\frac{x(x+1)^{1/3}-(x+1)(x+1)^{1/3}}{x^2-1} = \frac{(x+1)^{1/3}(x-(x+1))}{(x+1)(x-1)}$$

by factoring $(x+1)^{1/3}$ from both terms in the numerator

$$\underset{0.9}{=} \frac{(x+1)^{1/3}(-1)}{(x+1)^{1/3}(x+1)^{2/3}(x-1)}$$

since $1/3+2/3=1$,

$$= -\frac{1}{(x+1)^{2/3}(x-1)}$$

by cancelling $(x+1)^{1/3}$ from both the numerator and the denominator.

Example 2

Simplify $\dfrac{(x-2)^{-1/3}+(x-2)^{2/3}}{x-1}$

Solution

This is a fairly common type of expression. As in the previous example we have the same base raised to different fractional powers. In general the way to proceed is to *factor out the lowest fractional power*—in this case $(x-2)^{-1/3}$—from both terms in the numerator:

$$\frac{(x-2)^{-1/3}+(x-2)^{2/3}}{x-1} \underset{0.9}{=} \frac{(x-2)^{-1/3}+(x-2)^{-1/3}(x-2)^{1}}{x-1}$$

since $2/3 = -1/3+1$

$$= (x-2)^{-1/3}\frac{[1+(x-2)]}{x-1}$$

by factoring $(x-2)^{-1/3}$ from both terms in the numerator

$$= \frac{1}{(x-2)^{1/3}} \cdot \left(\frac{x-1}{x-1}\right) = \frac{1}{\sqrt[3]{x-2}}$$

Example 3

Simplify $\sqrt[3]{8a^6(x-h)^4} + 2a^2h\sqrt[3]{x-h}$.

Solution

The trick in this type of an expression is to "pull" as much out of the cube root as possible. We start by examining the first term:

$$\begin{aligned}\sqrt[3]{8a^6(x-h)^4} &\underset{0.14}{=} \sqrt[3]{2^3}\sqrt[3]{a^6}\sqrt[3]{(x-h)^4} \\ &\underset{0.13}{=} 2a^{6/3}(x-h)^{4/3} \\ &\underset{0.9}{=} 2a^2(x-h)\sqrt[3]{x-h}\end{aligned}$$

since $4/3 = 1 + 1/3$,

Thus our full expression becomes

$$\begin{aligned}\sqrt[3]{8a^6(x-h)^4} + 2a^2h\sqrt[3]{x-h} &= 2a^2(x-h)\sqrt[3]{x-h} + 2a^2h\sqrt[3]{x-h} \\ &= 2a^2\sqrt[3]{x-h}\,[((x-h)) + h]\end{aligned}$$

by factoring $2a^2\sqrt[3]{x-h}$ from both terms,

$$= 2a^2x\sqrt[3]{x-h}$$

Example 4

Remove all square roots from the denominator of

$$\frac{h}{\sqrt{x+h} - \sqrt{x}}$$

Solution

We must *rationalize the denominator*. This is done by using a variant of the difference of squares law:

$$\begin{aligned}(\sqrt{c} - \sqrt{d})(\sqrt{c} + \sqrt{d}) &= (\sqrt{c})^2 - (\sqrt{d})^2 \\ &= c - d\end{aligned}$$

(The terms $\sqrt{c}+\sqrt{d}$ and $\sqrt{c}-\sqrt{d}$ are called *algebraic conjugates* of each other.) In the case at hand we have

$$\begin{aligned}\frac{h}{\sqrt{x+h}-\sqrt{x}} &= \frac{h}{\sqrt{x+h}-\sqrt{x}}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{h(\sqrt{x+h}+\sqrt{x})}{(x+h)-x}\\ &= \frac{h(\sqrt{x+h}+\sqrt{x})}{h}\\ &= \sqrt{x+h}+\sqrt{x}\end{aligned}$$

***NOTE*:** This same technique can be used to *rationalize the numerator* of a fraction. You are asked to do this in Exercise 6.

Example 5

Is it always true that $\sqrt{a^2}=a$?

***Answer*:** No. The equation is true only when a is positive. In general the correct formula is

$$\sqrt{a^2}=|a|$$

(Recall that the *absolute value of* a, written $|a|$, is defined to be the distance of a to zero, or intuitively, the "positive" part of a, e.g., $|3.2|=3.2$, $|-4|=4$). Many calculus errors are made by forgetting these absolute value signs!

Example 6

Determine all x values for which

$$\sqrt{(x-1)^2}=1$$

Solution

From the previous example we know our equation is equivalent to

$$|x-1|=1$$

Thus $x-1=1$ or $x-1=-1$. Hence $x=0$ or 2.

Summary of Rules

Suppose $a > 0$ and $b > 0$. Then:

0.8 $a^{-r} = \dfrac{1}{a^r}$

0.9 $a^{r+s} = a^r a^s$ and $a^{r-s} = \dfrac{a^r}{a^s} = \dfrac{1}{a^{s-r}}$

0.10 $a^{rs} = (a^r)^s = (a^s)^r$ and $a^{r/s} = \left(a^{1/s}\right)^r = (a^r)^{1/s}$

0.11 $(ab)^r = a^r b^r$ and $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$

0.13 $\left(\sqrt[n]{a}\right)^n = a$ and $\sqrt[n]{a^n} = a$

0.14 $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ and $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

EXERCISES

Simplify each of the following expressions. These are complicated, so don't get discouraged if each one takes some time to solve!

1. $\left[(x+1)^{-1/6}\right]^3 - (x+1)^{1/2}$
2. $\left(\dfrac{x+h}{x-h}\right)^{-2/3} \cdot \dfrac{x+h}{\sqrt[3]{x^2-h^2}}$
3. $\dfrac{\left(\frac{4a^3}{9}\right)^{1/2} - \frac{2}{3}\sqrt{a}}{a-1}$
4. $\dfrac{\left(8a^3b^{-1/2}x\right)^{2/3}}{a^3\left(b^2\right)^{1/3}\left(4a^{-4}x^{1/3}\right)^{1/2}}$
5. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$ (Rationalize the denominator)
6. $\dfrac{\sqrt{x+h+3}-\sqrt{x+3}}{h}$ (Rationalize the numerator)

ANSWERS

(with some intermediate steps provided)

1. $(x+1)^{-1/2} - (x+1)^{1/2}$

$$= (x+1)^{-1/2}\,[1-(x+1)] = -\frac{x}{\sqrt{x+1}}$$

2. $\left(\dfrac{x-h}{x+h}\right)^{2/3} \cdot \dfrac{x+h}{(x+h)^{1/3}(x-h)^{1/3}}$

$$= \frac{(x-h)^{2/3-1/3}}{(x+h)^{2/3+1/3-1}} = \sqrt[3]{x-h}$$

3. $\dfrac{\frac{2}{3}a^{3/2} - \frac{2}{3}a^{1/2}}{a-1}$

$$= \frac{2}{3}a^{1/2}\left(\frac{a-1}{a-1}\right) = \frac{2}{3}a^{1/2}$$

4. $\dfrac{4a^2b^{-1/3}x^{2/3}}{a^3b^{2/3}(2a^{-2}x^{1/6})}$

$$= \frac{2a^{2-3+2}x^{\frac{2}{3}-\frac{1}{6}}}{b^{\frac{2}{3}+\frac{1}{3}}} = \frac{2a}{b}\sqrt{x}$$

5. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}} \cdot \dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

$$= \frac{(a+b)+2\sqrt{a^2-b^2}+(a-b)}{(a+b)-(a-b)} = \frac{a+\sqrt{a^2-b^2}}{b}$$

6. $\dfrac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \cdot \dfrac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}}$

$$= \frac{(x+h+3)-(x+3)}{h(\sqrt{x+h+3}+\sqrt{x+3})} = \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}}$$

0.4 ALGEBRA OF POLYNOMIALS

0.4.1 Multiplication

Most calculus students are comfortable with the multiplication of simple expressions, e.g.,

$$(a+b)(c+d) = ac + ad + bc + bd$$

For multiplication of longer expressions a person can always resort, if confused, to "long multiplication," as we now illustrate.

Example 1

Multiply $3x + 4y - 1$ by $x - 2y + 2$.

Solution

A "long multiplication" would be written out as follows

$$\begin{array}{rrrrrrl}
 & & & 3x & +4y & -1 & \\
 & & & x & -2y & +2 & \\
\hline
 & & & 6x & +8y & -2 & \leftarrow \text{first expression multiplied by 2} \\
 & -6xy & -8y^2 & & +2y & & \leftarrow \text{first expression multiplied by } -2y \\
3x^2 & +4xy & & -x & & & \leftarrow \text{first expression multiplied by } x \\
3x^2 & -2xy & -8y^2 & +5x & +10y & -2 & \leftarrow \text{adding the previous three lines gives the answer.}
\end{array}$$

The procedure is just that of ordinary multiplication of numbers; the only difference is in the placement of "like terms" underneath each other to aid in the final addition.

Certain combinations of terms appear so often that you are well-advised to memorize them:

$$\begin{aligned}
(a+b)(a-b) &= a^2 - b^2 \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}$$

The second and third of these equations are special cases of the *binomial formula:*

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \\ &\quad \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n\end{aligned}$$

The symbol $\binom{n}{k}$ is read "n choose k" and is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $m!$ (read "m factorial") is defined to be

$$0! = 1$$

$$m! = 1 \cdot 2 \cdot 3 \ldots \cdot (m-1) \cdot m \text{ for } m = 1, 2, 3, \ldots$$

■

Example 2

Expand $(x+2)^4$ by the binomial formula.

Solution

$$\begin{aligned}(x+2)^4 &= x^4 + \binom{4}{1}x^3 \cdot 2^1 + \binom{4}{2}x^2 \cdot 2^2 + \binom{4}{3}x^1 \cdot 2^3 + 2^4 \\ &= x^4 + 4 \cdot x^3 \cdot 2 + 6 \cdot x^2 \cdot 4 + 4 \cdot x \cdot 8 + 16 \\ &= x^4 + \frac{4!}{1!3!}x^3 \cdot 2 + \frac{4!}{2!2!}x^2 \cdot 4 + \frac{4!}{3!1!}x \cdot 8 + 16 \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16.\end{aligned}$$

■

0.4.2 Division

The division of one algebraic expression by another frequently gives people difficulty. However, the technique of "long division" is quite important and is similar to the usual "long division" of decimal numbers.

Example 3

Divide $x^3 + x^2 + x - 3$ by $x - 1$.

Solution

We'll go through this division very slowly to see precisely what is happening. We start by arranging the divisor $(x-1)$ and the dividend (x^3+x^2+x-3) in the usual way

$$x-1\overline{\big)\,x^3+x^2+x-3}$$

Notice that both terms are written with the powers of x in *descending order* (i.e., x^3+x^2+x-3, not $x+x^3-3+x^2$).

1. Take the first term (x) in the divisor and divide it into the first term (x^3) in the dividend; the result is x^2, which we write above the x^3 as shown

$$\begin{array}{r|l} & x^2 \\ \hline x-1 & x^3+x^2+x-3 \end{array}$$

   Now multiply the divisor $(x-1)$ by x^2, place the result (x^3-x^2) under the dividend (with correct positioning of powers of x), and subtract

$$\begin{array}{rl} & x^2 \\ x-1 & \overline{)\,x^3+x^2+x-3} \\ & \underline{x^3-x^2} \\ & \qquad 2x^2+x-3 \end{array}$$

2. The bottom term so obtained will be referred to as the "new dividend." We operate on it just as we did on the original dividend: divide the x in the divisor into the $2x^2$ of the new dividend, and place the resulting $2x$ above the division sign as shown:

$$\begin{array}{rl} & x^2+2x \\ x-1 & \overline{)\,x^3+x^2+x-3} \\ & \underline{x^3-x^2} \\ & \qquad 2x^2+x-3 \end{array}$$

   Then multiply the divisor $(x-1)$ by $2x$, place the result $(2x^2-2x)$ under the new dividend, and subtract

$$\begin{array}{rl} & x^2+2x \\ x-1 & \overline{)\,x^3+x^2+x-3} \\ & \underline{x^3-x^2} \\ & \qquad 2x^2+x-3 \\ & \qquad \underline{2x^2-2x} \\ & \qquad\qquad 3x-3 \end{array}$$

3. The bottom line so obtained is an "even newer dividend"! and we're sure you can guess what to do with it! But if you're still not sure: divide the x in the divisor into the $3x$ of the "even newer dividend," and place the resulting 3 above the division sign. Then multiply the divisor $(x-1)$ by 3, and subtract the result from the "even newer dividend"

$$
\begin{array}{r|l}
 & x^2 + 2x + 3 \\
\hline
x - 1 & x^3 + x^2 + x - 3 \\
 & \underline{x^3 - x^2} \\
 & \quad 2x^2 + x - 3 \\
 & \quad \underline{2x^2 - 2x} \\
 & \qquad\quad 3x - 3 \\
 & \qquad\quad \underline{3x - 3} \\
 & \qquad\qquad\quad 0
\end{array}
$$

In this example we were lucky: we obtained a zero remainder in our subtraction, and thus our division is done. The result is that $x^3 + x^2 + x - 3$, when divided by $x - 1$, yields $x^2 + 2x + 3$, i.e., $x - 1$ divides $x^3 + x^2 + x - 3$ "evenly" and

$$\frac{x^3 + x^2 + x - 3}{x - 1} = x^2 + 2x + 3$$

■

Example 4

Divide $x^3 + 2x^2 + 3x + 2$ by $x^2 + 1$.

Solution

Again we set up the divisor and dividend with the terms in decreasing powers of x.

$$x^2 + 1 \overline{\smash{)}\, x^3 + 2x^2 + 3x + 2}$$

i. Divide x^2 into x^3 to get x; multiply $x^2 + 1$ by x and (with correct positioning of powers of x) subtract the result from the dividend:

$$
\begin{array}{r|l}
 & x \\
\hline
x^2 + 1 & x^3 + 2x^2 + 3x + 2 \\
 & \underline{x^3 \qquad\quad + x} \\
 & \qquad 2x^2 + 2x + 2
\end{array}
$$

ii. Divide x^2 into $2x^2$ to obtain 2; multiply x^2+1 by 2 and subtract the result of the "new dividend."

$$\begin{array}{r|l} & x + 2 \\ \hline x^2+1 & x^3 + 2x^2 + 3x + 2 \\ & \underline{x^3 \qquad\quad + x} \\ & \qquad 2x^2 + 2x + 2 \\ & \qquad \underline{2x^2 \qquad\ + 2} \\ & \qquad\qquad +2x \end{array}$$

iii. Ahh... now we have a major difference from Example 3. We cannot obtain a *positive* power of x by dividing x^2 into $2x$. Thus our division is finished, but we did not end up with a zero remainder—we have a remainder of $2x$ (i.e., x^2+1 does not divide x^3+2x^2+3x+2 "evenly"). To see what we do with it, consider an ordinary division:

$$\begin{array}{r|r} & 16 \\ \hline 7 & 115 \\ & \underline{70} \\ & 45 \\ & \underline{42} \\ & 3 \end{array}$$

This yields

$$\frac{115}{7} = 16 + \frac{3}{7}$$

We treat our remainder term in the same way as in this numerical example, i.e., our division computation yields

$$\frac{x^3+2x^2+3x+2}{x^2+1} = x+2+\frac{2x}{x^2+1}$$

To check the accuracy of this answer combine the terms on the right-hand side of the equation (Rule (0.3), *Companion* Section 0.2) to obtain

$$\begin{aligned} x+2+\frac{2x}{x^2+1} &= \frac{(x+2)(x^2+1)+2x}{x^2+1} \\ &= \frac{(x^3+2x^2+x+2)+2x}{x^2+1} \\ &= \frac{x^3+2x^2+3x+2}{x^2+1} \end{aligned}$$

The last fraction is what we started with, and so our answer checks out as correct.

Example 5

Divide $2t^2 - t + t^3$ by $t + 1$

Solution

Without the running commentary the solution would look like this

$$\begin{array}{r|l} & t^2 + t - 2 \\ \hline t+1 & t^3 + 2t^2 - t \\ & \underline{t^3 + \;\; t^2} \\ & \qquad t^2 - t \\ & \qquad \underline{t^2 + t} \\ & \qquad\quad -2t \\ & \qquad\quad \underline{-2t} \;\; \underline{-2} \\ & \qquad\qquad\quad +2 \end{array}$$

Thus

$$\frac{t^3 + 2t^2 - t}{t+1} = t^2 + t - 2 + \frac{2}{t+1}.$$

■

0.4.3 Polynomials

A ***polynomial in*** x is simply an addition of non-negative integer powers of x multiplied by constants. Six examples of polynomials are

$$x^2 + 5x - 3 \qquad \sqrt{2}x + 5 \qquad x^3$$

$$\pi x^8 + 2 \qquad 3 \qquad \tfrac{1}{2}x - \tfrac{3}{2}$$

Three examples of *non*-polynomials are

$$\begin{array}{ll} x^2 + \sqrt{x} & \left(\sqrt{x} = x^{1/2} \text{is a } \textit{non-integer} \text{ power of} x\right), \\ 3 + 1/x & \left(1/x = x^{-1} \text{is a } \textit{negative} \text{ power of} x\right), \\ x^3 + \sin x & (\sin x \text{ is } \textbf{\textit{not a power}} \text{ of } x) \end{array}$$

The general expression for any polynomial in x is

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}\ldots, a_1$ and a_0 are simply constants, called the ***coefficients*** of $p(x)$. Polynomials are the most elementary functions of a single variable; one major goal of calculus is to "approximate" other functions by polynomials (Taylor series, as done in Chapter 11).

The ***degree*** of a polynomial is the highest power of x which it contains. For example, x^4-1 is of degree 4, while $\sqrt{3}$ is of degree 0. Polynomials of low degrees have special names

degree	**form**	**name**
0	a	constant
1	$ax+b$	linear term $(a\neq 0)$
2	ax^2+bx+c	quadratic term $(a\neq 0)$
3	ax^3+bx^2+cx+d	cubic term $(a\neq 0)$

The multiplication of two polynomials will always yield a polynomial. For example,

$$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd,$$

$$(x^2-x)(3x^3-x^2+1)=3x^5-4x^4+x^3+x^2-x$$

Such computations are easily carried out by "long multiplication" as discussed earlier; however, for most calculus applications, *the reverse procedure of factoring is much more important.*

0.4.4 Factoring

To *factor* a polynomial means to break it down into a product of polynomials of smaller degree. Examples of factored polynomials are as follows:

$$\begin{aligned} 2x^2+x-1 &= (2x-1)(x+1) \\ x^2-2 &= (x-\sqrt{2})(x+\sqrt{2}) \end{aligned}$$

$$2x^3-\frac{17}{3}x^2-\frac{5}{3}x+2=(x-3)(2x-1)(x+2/3)$$

$$x^3-1=(x-1)(x^2+x+1)$$

Factoring a polynomial is not always a pleasant or easy operation, but it is important. We will thus study the procedure in some depth, first for quadratic terms, and then for general polynomials.

0.4.5 Factoring Quadratic Terms

Sometimes the factorization of a quadratic term can be arrived at "by inspection," as the following example illustrates.

Example 6

Factor the quadratic term $2x^2 + x - 6$.

Solution

If we assume that $2x^2 + x - 6$ factors into linear terms with *integer* coefficients, then we would have

$$2x^2 + x - 6 = (2x + a)(x + b)$$

where $2x$ and x are necessary to obtain the $2x^2$ term, and a and b are two integers which need to be determined. Multiplying out the right-hand side of this equation shows

$$2x^2 + x - 6 = 2x^2 + (a + 2b)x + ab$$

and thus we must choose a and b so that $ab = -6$ and $a + 2b = 1$. The first condition gives you a small number of (integer) possibilities to test, i.e., those pairs of integers whose product is the constant term -6:

a	1	−1	2	−2	−6	6	−3	3
b	−6	6	−3	3	1	−1	2	−2

However, only $a = -3$, $b = 2$ satisfy the second condition, i.e., $a + 2b = 1$. Thus

$$2x^2 + x - 6 = (2x - 3)(x + 2).$$

■

Factorization of polynomials is related to the *roots* of polynomials: a number r is a root of $p(x)$ if $p(r) = 0$. If a quadratic term can be factored as follows:

$$p(x) = ax^2 + bx + c = a(x - r_1)(x - r_2) \tag{0.15}$$

then quite clearly $p(r_1) = p(r_2) = 0$, so that r_1 and r_2 are roots of $p(x)$. Surprisingly, the converse of this statement is also true: if r_1 and r_2 are the roots of $ax^2 + bx + c$, then Equation (0.15) must hold true. Thus, *factoring a quadratic term is equivalent to finding its roots*. This is fortunately made easy by the:

The Quadratic Formula

The roots r_1 and r_2 of the quadratic term

$$ax^2 + bx + c \quad (a \neq 0)$$

are equal to

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Thus our two roots r_1 and r_2 are given by

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and, when combined with Equation (0.15), show that any quadratic term can be factored as follows:

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

If you take a moment and multiply out the right-hand side of this equation, you will indeed obtain $ax^2 + bx + c$ as claimed (this actually *proves* both Equation (0.15) and the quadratic formula).

Example 7

Factor the quadratic term $3x^2 - 5x + 1$.

Solution

According to the quadratic formula, the roots are

$$\frac{5 \pm \sqrt{25 - 12}}{6} = \frac{5 \pm \sqrt{13}}{6}.$$

Thus

$$3x^2 - 5x + 1 = 3\left(x - \frac{5 + \sqrt{13}}{6}\right)\left(x - \frac{5 - \sqrt{13}}{6}\right)$$

You can always *check* a factorization by multiplying the terms out to see if you obtain what you started with. In this example a check would go as follows:

$$\begin{aligned}
3\left(x-\frac{5+\sqrt{13}}{6}\right)\left(x-\frac{5-\sqrt{13}}{6}\right) &= 3\left[\begin{array}{c} x^2-\left(\frac{5+\sqrt{13}}{6}+\frac{5-\sqrt{13}}{6}\right)x \\ +\left(\frac{5+\sqrt{13}}{6}\right)\left(\frac{5-\sqrt{13}}{6}\right)\end{array}\right] \\
&= 3\left[x^2-\frac{10}{6}x+\frac{25-13}{36}\right] \\
&= 3\left[x^2-\frac{5}{3}x+\frac{1}{3}\right] \\
&= 3x^2-5x+1
\end{aligned}$$

as desired. ■

Example 8

Factor the quadratic term x^2-2x+2.

Solution

The roots are

$$\frac{2\pm\sqrt{4-8}}{2}=1\pm i,$$

where i is the "imaginary" number $\sqrt{-1}$.

Thus $x^2-2x+2=(x-1-i)(x-1+i)$. While this is a perfectly correct factorization using *complex numbers* (i.e., numbers containing i), it is ***not*** a factorization using only *real numbers*. Since in elementary calculus we do not wish to deal with complex numbers, a quadratic such as x^2-2x+2 which does not factor into real linear terms will be called an *irreducible quadratic term*. We will not use the complex factorization of such a term. ■

0.4.6 Factoring General Polynomials

It is a theorem of algebra that any polynomial can be factored into real linear and irreducible quadratic terms. *It is, however, entirely another matter to actually determine what these factors are!* (This is not an uncommon situation in mathematics. Frequently, we can prove that something exists, but we do not have an effective, surefire way to compute what that "something" is.) Generally the factorization (when computable!) is done in stages: a given polynomial is factored into the product of two lower degree polynomials, each of which is then further factored. . . , etc.,. . . until you can go no farther. Many times the start of such a procedure is “by inspection,” i.e., there are certain commonly occurring factorizations which a person should remember. The most important of these are as follows:

$$a^2 - b^2 = (a-b)(a+b) \tag{0.16}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2) \tag{0.17}$$

$$a^3 + b^3 = (a+b)(a^2-ab+b^2) \tag{0.18}$$

Example 9

Factor the cubic term $8x^3 - 27$.

Solution

You should recognize this term as a difference of cubes, i.e.,

$$8x^3 - 27 = (2x)^3 - 3^3$$

Thus 0.17 applies to give

$$8x^3 - 27 = (2x-3)(4x^2+6x+9)$$

Since the quadratic formula shows that the quadratic term $4x^2+6x+9$ is irreducible—i.e., it has complex roots because

$$\sqrt{b^2-4ac} = \sqrt{36-144} = \sqrt{-108} = i\sqrt{108}$$

—we have obtained as complete a factorization as is possible. ■

Example 10

Factor the 5^{th} degree polynomial $x^5 - 16x$.

Solution

If a polynomial in x has no constant term, then you have one factor for free (!), namely “x”. Thus

$$\begin{aligned} x^5 - 16x &= x(x^4-16) \\ &= x((x^2)^2 - 4^2) \end{aligned}$$

a difference of squares!,

$$= x(x^2-4)(x^2+4)$$

by 0.16,

$$= x(x-2)(x+2)(x^2+4)$$

■

Equations (0.16), (0.17), (0.18), are simply specific cases of the following general rules:

For any positive integer n,

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1} \right) \tag{0.19}$$

For any *odd* positive integer m,

$$a^m + b^m = (a+b)\left(a^{m-1} - a^{m-2}b + \ldots - ab^{m-2} + b^{m-1} \right) \tag{0.20}$$

Memorizing these rules is probably not necessary, but at least knowing that $(a-b)$ is always a factor of $a^n - b^n$ and $(a+b)$ is always a factor of $a^m + b^m$ (m odd) can be useful.

When faced with a factoring problem to which (0.16)-(0.20) do not apply (a common occurrence...), one generally attempts to use the following result:

The Factor Theorem

> The linear term $x - r$ is a factor of a polynomial $p(x)$ if and only if r is a root of $p(x)$, i.e., $p(r) = 0$.

This is very much like the quadratic situation which we discussed earlier. *However...* unlike the quadratic case, there is no simple formula which gives the roots of a polynomial of degree greater than 2. We are thus again reduced to "inspection" methods for finding roots for polynomials.

Example 11

Factor the 3rd degree polynomial $x^3 - 5x^2 + 6x - 2$.

Solution

If you try some small integer values for x you will find that $x = 1$ is a root of $p(x)$ and hence $x - 1$ is a factor (we will give below a good method for "guessing" at the roots of a polynomial). The other factor is now obtained by long division:

$$\begin{array}{r|l} & x^2 - 4x + 2 \\ \hline x - 1 & x^3 - 5x^2 + 6x - 2 \\ & x^3 - x^2 \\ \hline & -4x^2 + 6x - 2 \\ & -4x^2 + 4x \\ \hline & 2x - 2 \end{array}$$

Thus $x^3 - 5x^2 + 6x - 2 = (x-1)(x^2 - 4x + 2)$. The roots of the quadratic factor are now found to be

$$\frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$$

Thus $x^3 - 5x^2 + 6x - 2$

$$= (x-1)(x - 2 - \sqrt{2})(x - 2 + \sqrt{2})$$

Finding the exact roots of an arbitrary polynomial can oftentimes be impossible; numerical approximations via a computer or hand calculator are then called for. However, many situations are helped along by the following result:

The Rational Root Test

> Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with *integer* coefficients, and $r = p/q$ is a rational number where p/q is expressed in lowest terms. Then $r = p/q$ can be a root of $p(x)$ only if p divides the constant term a_0 and q divides the "leading coefficient" a_n.

Notice that this only says "$r = p/q$ *can be* a root!" it does ***not*** say "$r = p/q$ *is* a root!"

Example 12

Factor the 3rd degree polynomial $3x^3 - 8x^{12} + x + 2$.

Solution

Suppose this polynomial has a rational root $r = p/q$; then p and q are integers such that p divides 2 and q divides 3, i.e., the possibilities are

$$p = \pm 1, \pm 2, \quad q = \pm 1, \pm 3$$

Hence there are a total of eight *possible* rational roots:

$$\pm 1,\ \pm 2,\ \pm 1/3,\ \pm 2/3$$

By plugging these into the polynomial we find that only $r = 2/3$ actually is a root. Thus $x - 2/3$ is a factor, and long division produces

$$3x^3 - 8x^2 + x + 2 = 3(x - 2/3)(x^2 - 2x - 1)$$

The roots of the quadratic factor are now found to be $1 \pm \sqrt{2}$. Thus

$$3x^3 - 8x^2 + x + 2 = 3\,(x - 2/3)\left(x - 1 - \sqrt{2}\right)\left(x - 1 + \sqrt{2}\right)$$

■

0.4.7 Completing the Square

With quadratic terms (especially irreducible terms) it is often important to ***complete the square***. Here is how this works on a quadratic term whose x^2 coefficient is 1.

$$x^2 + bx + c = x^2 + bx \underbrace{+ \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2}_{*} + c$$

* add in and subtract out the square of one half the x-coefficient:

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

Example 13

Complete the square in $x^2 + 3x + 4$.

Solution

$$\begin{aligned} x^2 + 3x + 4 &= x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4 \\ &= \left(x + \frac{3}{2}\right) + \frac{7}{4} \end{aligned}$$

■

Example 14

Complete the square in $3x^2 - 2x + 1$.

Solution

If the x^2 coefficient is not 1, then first factor this coefficient out of the whole expression.

$$\begin{aligned}
3x^2 - 2x + 1 &= 3\left[x^2 - \frac{2}{3}x + \frac{1}{3}\right] \\
&= 3\left[x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right)^2 + \frac{1}{3}\right] \\
&= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right] \\
&= 3\left(x - \frac{1}{3}\right)^2 + \frac{2}{3}
\end{aligned}$$

■

Although the usefulness of this operation may not be immediately apparent, its value lies in "eliminating" the x term (i.e., x to the first power). See, for example, Section 9.4 in Anton. Also see its use in Section 0.5 (Conic Sections) of *The Companion*.

EXERCISES

1. Multiply the following expressions.

   (a) $\left(\frac{2}{3}x + 4y\right)\left(\frac{1}{2}x - 3y + 1\right)$

   (b) $(2x + a)(ax - 1)(2x - a)$

   (c) $(2x + 1)^4$

2. Divide the following expressions.

   (a) $\dfrac{2x^3 - x^2 - 3x + 14}{x + 2}$

   (b) $\dfrac{x^3 + \frac{7}{8}}{2x - 1}$

   (c) $\dfrac{2x^4 + x^3 + 1}{x^2 - 2}$

3. Factor the following polynomials.

   (a) $3x^2 + 11x - 4$

(b) $2x^2 + 2x - 1$

(c) $2x^2 + 2x + 1$

(d) $x^4 - 8x$

(e) $2x^3 - 4x^2 + 3x - 1$

(f) $10x^4 + 41x^3 + 12x^2 - 7x - 2$

4. Complete the square in 3(a), 3(b), 3(c).

ANSWERS

1(a) $\frac{1}{3}x^2 + \frac{2}{3}x - 12y^2 + 4y$

(b) $4ax^3 - 4x^2 - a^3x + a^2$

(c) $16x^4 + 32x^3 + 24x^2 + 8x + 1$

2(a) $2x^2 - 5x + 7$

(b) $\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{1}{2x-1}$

(c) $2x^2 + x + 4 + \frac{2x+9}{x^2-2}$

3(a) $(3x - 1)(x + 4)$

(b) $2\left(x + \frac{1}{2} - \frac{1}{2}\sqrt{3}\right)\left(x + \frac{1}{2} + \frac{1}{2}\sqrt{3}\right)$

(c) $2x^2 + 2x + 1$ (irreducible quadratic term)

(d) $x(x-2)\left(x^2 + 2x + 4\right)$

(e) $(x-1)\left(2x^2 - 2x + 1\right)$

(f) Method:

$$(2x+1)(5x-2)\left(x + 2 - \sqrt{3}\right)\left(x + 2 + \sqrt{3}\right)$$

The only possible rational roots for the original polynomial are

$$\pm 2,\ \pm 1,\ \pm\tfrac{1}{2},\ \pm\tfrac{2}{5},\ \pm\tfrac{1}{5},\ \pm\tfrac{1}{10}.$$

Running down through the list in the order given will find $r = -\frac{1}{2}$ as the first of these numbers which is a root. Hence $x + \frac{1}{2}$ will be a factor, but for convenience we use $2x+1$. (If $ax+b$ is a factor of a given polynomial $p(x)$, then any constant multiple of $ax+b$ is also a factor of $p(x)$. Long division yields

$$5x^3 + 18x^2 - 3x - 2.$$

Continuing through our list of possible roots will also yield $r + \frac{2}{5}$ as a root. Hence $x - \frac{2}{5}$ will be a factor, but again for convenience we use $5x - 2$. Long division yields $x^2 + 4x + 1$, to which the quadratic formula applies, giving our final answer.

4 a) $3\left(x + \frac{11}{6}\right)^2 - \frac{169}{12}$

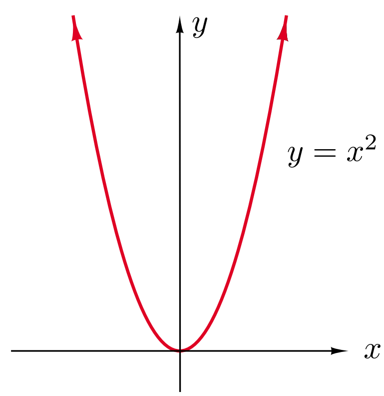
b) $2\left(x + \frac{1}{2}\right)^2 - \frac{3}{2}$

c) $2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$

0.5 CONIC SECTIONS

In this section we review the most basic properties of the four *conic section* curves: the circle, ellipse, parabola and hyperbola. They are known as conic sections because they can all be obtained by slicing a cone with a plane. Pictures of such slices are given in Anton's Figure 12.4.1. A much more detailed and sophisticated study of conic sections is undertaken in Chapter 12 of Anton.

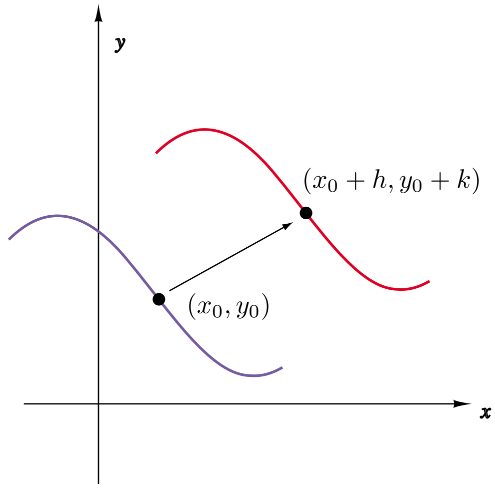
0.5.1 The Parabola



First consider the simplest equation of a conic section:

$$y = x^2.$$

Its graph is an upward-turning parabola with vertex at $(0, 0)$. Multiplying x^2 by a non-zero constant α will only change the flatness or steepness of the curve and, in the case of negative α, will make the curve turn downward.



Then the following are equivalent statements:

i. (x_0, y_0) lies on the curve A

ii. $y_0 = f(x_0)$

iii. $(y_0 + k) - k = f((x_0 + h) - h)$

iv. $(x_0 + h,\ y_0 + k)$ lies on the curve B

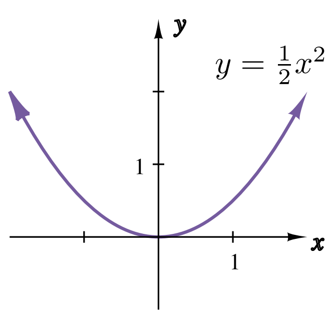
Thus curve B is obtained by moving curve A to the right by h units and up by k units, as the translation principles claim.

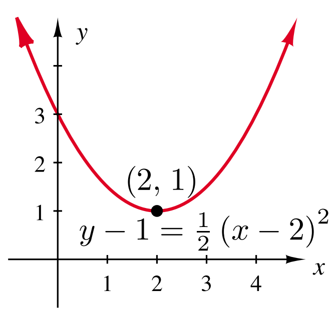
Example 1

Sketch the graph of $y - 1 = \frac{1}{2}(x - 2)^2$.

Solution

We have only to translate the graph of $y = \frac{1}{2}x^2$ up by $k = 1$ and to the right by $h = 2$ units.





Example 2

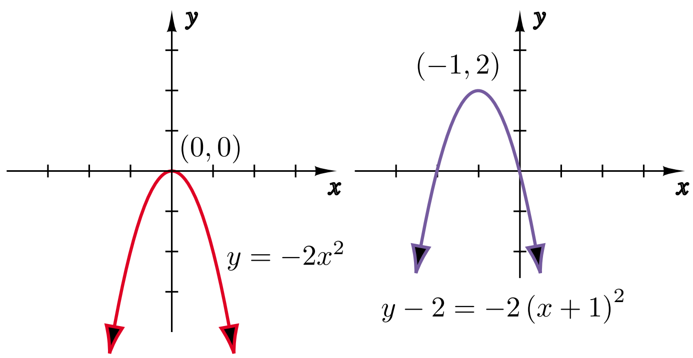
Sketch the graph of $y = -2x^2 - 4x$.

Solution

We must first put the equation into the standard form (0.22); this requires completing the square (see Section 0.4.7 of *The Companion*).

$$\begin{aligned} y &= -2(x^2+2x) \\ &= -2(x^2+2x+1-1) \\ &= -2(x+1)^2+2 \end{aligned}$$

Thus $y-2=-2(x+1)^2$, so we must translate the graph of $y=-2x^2$ up by $k=2$ units and to the left by $h=1$ unit (or, if you prefer, to the right by $h=-1$ unit).



As Example 2 illustrates, any equation of the form

$$y = ax^2+bx+c, \;\; a \neq 0$$

is the equation of an upward or downward turning parabola, and can be put in the form (0.22) by completing the square. To get rightward or leftward turning parabolas we simply reverse the roles of x and y, as the next example shows.

Example 3

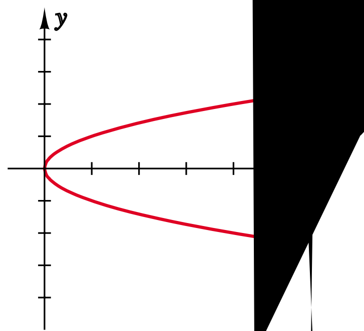
Sketch the graph of $x-y^2+2y=1$.

Solution

Rearranging terms we obtain

$$x = y^2-2y+1=(y-1)^2$$

an equation whose graph is the translation upward by $k=1$ unit of the graph of $x=y^2$. This is a rightward turning parabola.


y

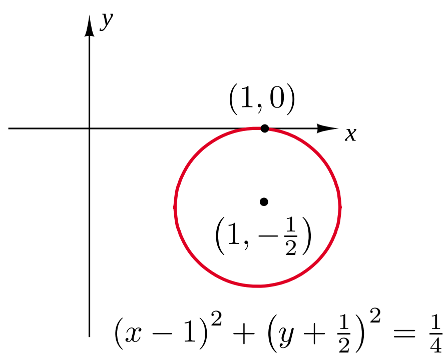
Group the x terms together and the y terms together:

$$\left(x^2 - 2x\right) + \left(y^2 + y + 1\right) = 0$$

Then complete the square in each:

$$\begin{aligned}\left(x^2 - 2x + 1\right) - 1 + \left(y^2 + y + \frac{1}{4}\right) - \frac{1}{4} + 1 &= 0\\(x-1)^2 - 1 + \left(y + \frac{1}{2}\right)^2 - \frac{1}{4} + 1 &= 0\\(x-1)^2 + \left(y + \frac{1}{2}\right)^2 &= \frac{1}{4}\end{aligned}$$

We thus have a circle of radius $r = \frac{1}{2}$ centered on the point $\left(1, -\frac{1}{2}\right)$.

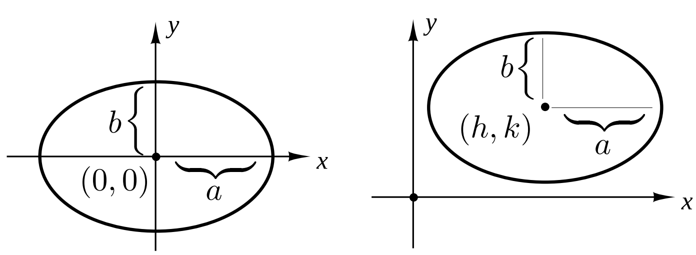


0.5.3 The Ellipse

An ellipse[1] centered on (0, 0) has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{0.25}$$

where a and b are positive numbers. The constants a and b are the length of the *semi-major* and *semi-minor* axes (the larger number representing the semi-major axis). The four "extreme points," $(\pm a, 0)$ and $(0, \pm b)$, are the *vertices* of the ellipse.



[1]For simplicity of development, the use of a and b for the ellipse differ *slightly* from that of Anton's Section 12.4. In our usage, a denotes the term dividing the x term; in Anton, a always denotes the larger of the constants a and b.

The translation principles then tell us

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{0.26}$$

is the equation of an ellipse centered on (h, k) and with $a, b > 0$ the lengths of the semi-major and semi-minor axes. (See the figure above.) If $a = b$, then our ellipse is a circle with radius $r = a$.

Example 5

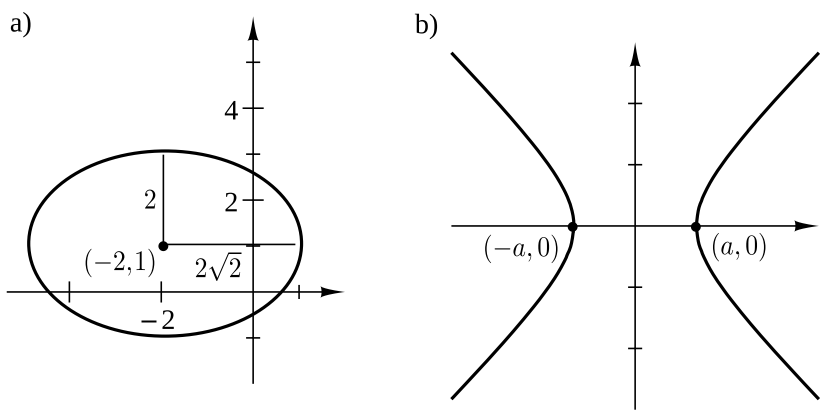
Sketch the graph of $x^2 + 4x + 2y^2 - 4y - 2 = 0$.

Solution

Completing the squares (*Companion* Section 0.4.7) in both x and y yields

$$(x+2)^2 + 2(y-1)^2 = 8, \quad \text{or} \quad \frac{(x+2)^2}{8} + \frac{(y-1)^2}{4} = 1$$

Thus we have an ellipse centered on $(-2, 1)$, with the semi-major and semi-minor axes of lengths $2\sqrt{2}$ and 2 (since we have obtained Equation (0.26) with $a = 2\sqrt{2}$ and $b = 2$). See Figure a).



0.5.4 The Hyperbola

Hyperbolas are the most interesting (and the most complicated) of the conic sections. For starters let's recall the hyperbolas centered on $(0, 0)$ which are given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{0.27}$$

where a and b are positive numbers. Notice the minus sign!

It is easy to see that the x-intercept points are $(\pm a, 0)$; these are called the *vertices* of the hyperbola. There are no y-intercepts. Moreover, solving for y in terms of x we obtain

$$\begin{aligned}\frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1\\ \frac{y}{b} &= \pm\sqrt{\frac{x^2}{a^2} - 1}\\ y &= \pm b\sqrt{\frac{x^2}{a^2} - 1}\end{aligned}$$

Thus, when x becomes large, the "1" in the radical sign becomes insignificant when compared with x^2/a^2. Therefore, for large values of x we obtain

$$y \cong \pm b\sqrt{\frac{x^2}{a^2}} = \pm\frac{b}{a}x$$

The lines $y = \pm\frac{b}{a}x$ are the *asymptotes* of the hyperbola; as x gets large the hyperbola is approximated very well by these lines (i.e., the hyperbola approaches these lines but never quite touches them). A handy way to remember the asymptote formulas is to take the hyperbola equation, replace the "1" with a "0", and solve for y:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is changed to } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Solving for y yields[2]

$$y = \pm\frac{b}{a}x$$

the correct asymptote equations.

Example 6

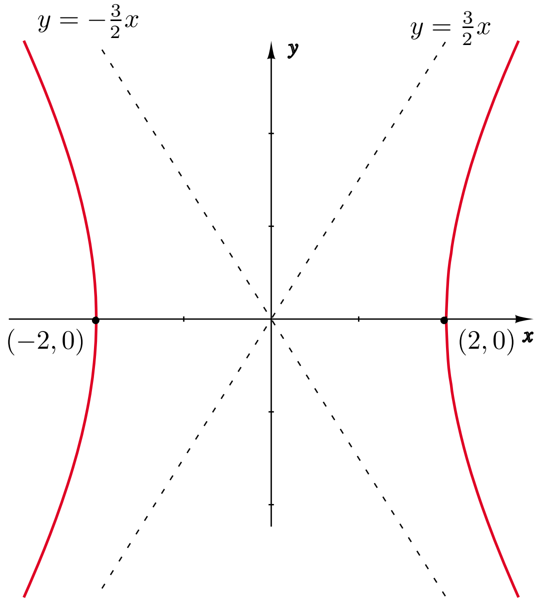
Sketch the graph of $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$.

Solution

In this case $a = 2$ and $b = 3$. Thus the vertices are $(\pm 2, 0)$, and the asymptotes are the lines

$$\frac{x^2}{4} - \frac{y^2}{9} = 0, \text{ i.e., } y = \pm\frac{3}{2}x.$$

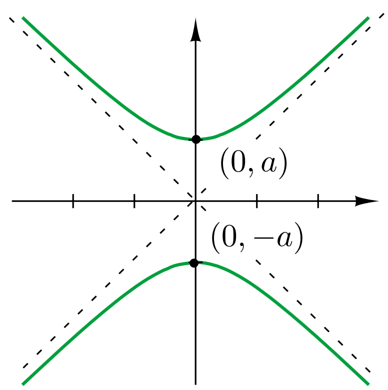
[2]Note that the asymptotes lie along the diagonals of the rectangle with vertices $x = \pm a$ and $y = \pm b$.



$$\frac{x^2}{4}-\frac{y^2}{9}=1$$

The hyperbolas given by (0.27) all turn outward to the *left* and *right*; to obtain hyperbolas turning *upward* and *downward* we need to switch the roles of x and y by considering equations of the form

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \qquad (0.28)$$



Notice the minus sign!

Here there are no x-intercepts, while the y-intercepts (the vertices) are $(0,\pm a)$. The asymptotes are given by changing the "1" to a "0", so $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=0$, which yields

$$y=\pm\frac{a}{b}x$$

Example 7

Sketch the graph of $y^2-4x^2=2$.

Solution

We first must place the equation into the standard form (0.28). To do so divide by 2 to obtain

$$\frac{y^2}{2} - 2x^2 = 1$$

which is equivalent to

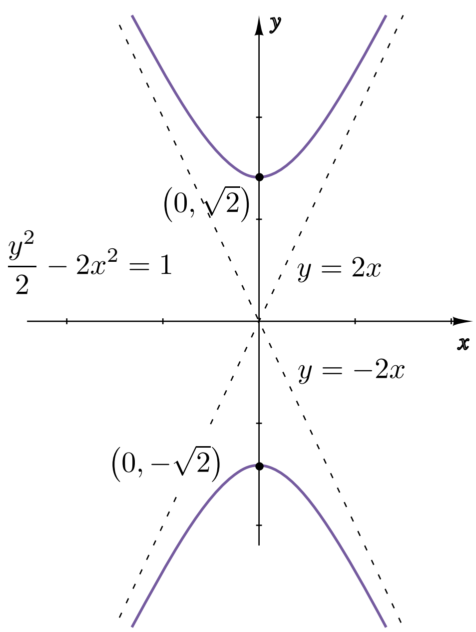
$$\frac{y^2}{2} - \frac{x^2}{\left(\frac{1}{2}\right)} = 1$$

by fraction rule (0.7). Thus $a = \sqrt{2}$ and $b = \sqrt{1/2} = \sqrt{2}/2$, giving vertices $(0, \pm\sqrt{2})$ and asymptotic lines

$$\frac{y^2}{2} - \frac{x^2}{\left(\frac{1}{2}\right)} = 0,$$

i.e.,

$$y = \pm 2x.$$



As done previously for the parabola, circle and ellipse, we use the translation principles to find the equations of hyperbolas with centers other than (0, 0):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad (0.29)$$

is the equation of a *"left-right" hyperbola* centered on (h, k) with vertices $(h \pm a, k)$ and asymptotes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0, \text{ i.e.,}$$

$$y - k = \pm \frac{b}{a}(x-h);$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \qquad (0.30)$$

is the equation of an *"up-down" hyperbola* centered on (h, k) with vertices $(h, k \pm a)$ and asymptotes

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 0,$$

i.e.,

$$y - k = \pm \frac{a}{b}(x-h)$$

An important note concerning memorization: You really need ***not*** memorize Equations (0.22),(0.24), (0.26), (0.29), (0.30) for conics with center (h, k) if you remember the simplier Equations (0.21), (0.23), (0.25), (0.27), (0.28) for conics with center $(0, 0)$ *along with the translation principles!*

Example 8

Sketch the graph of $x^2 - 4y^2 + x + 8y + \frac{1}{4} = 0$

Solution

As in Examples 4 and 5 we must first complete the squares in both the x and y terms

$$\begin{aligned}
\left(x^2+x\right)-4\left(y^2-2y\right)+\frac{1}{4} &= 0\\
\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}-4\left(y^2-2y+1\right)+4+\frac{1}{4} &= 0\\
\left(x+\frac{1}{2}\right)^2-4\left(y-1\right)^2+4 &= 0\\
4\left(y-1\right)^2-\left(x+\frac{1}{2}\right)^2 &= 4\\
(y-1)^2-\frac{\left(x+\frac{1}{2}\right)^2}{4} &= 1
\end{aligned}$$

Thus we are in standard form (0.30) (or standard form (0.28) translated by $\left(-\frac{1}{2},1\right)$) with $a=1$ and $b=2$; this is an up-down hyperbola with

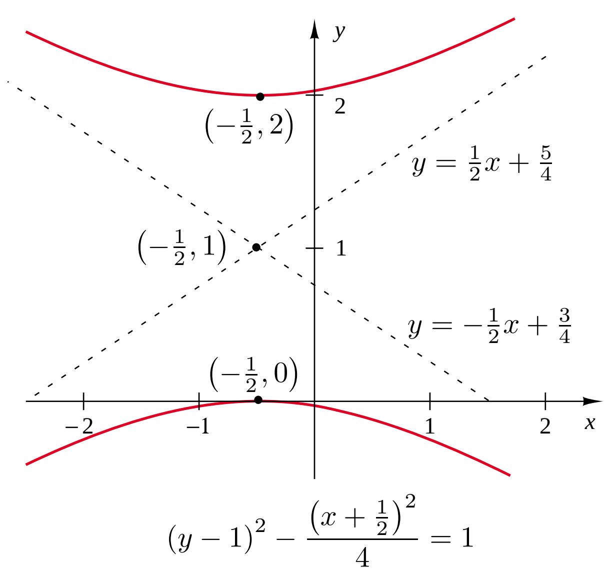
center: $\left(-\frac{1}{2},1\right)$,

vertices: $\left(-\frac{1}{2},1\pm 1\right)=\left(-\frac{1}{2},0\right),\left(-\frac{1}{2},2\right)$, and

asymptotes: $y-1=\pm\frac{1}{2}\left(x+\frac{1}{2}\right)$,

i.e.,

$$y=\frac{1}{2}x+\frac{5}{4} \text{ and } y=-\frac{1}{2}x+\frac{3}{4}$$

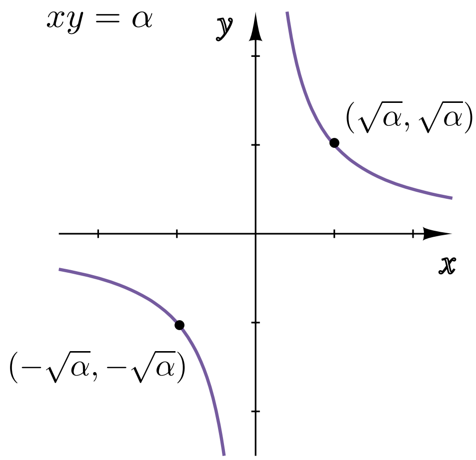


0.5.5 "Mixed-Term" Hyperbolas

There is another very standard way in which hyperbolas commonly arise:

$$y = \frac{\alpha}{x} \text{ or } xy = \alpha \tag{0.31}$$

where α is a non-zero constant.



The center is at $(0, 0)$, the asymptotes are $x = 0$ and $y = 0$ (the coordinate axes), and the hyperbolas fall in the 1st and 3rd quadrants if $\alpha > 0$, or the 2nd and 4th quadrants if $\alpha < 0$. By the translation principles the equation

$$y - k = \frac{\alpha}{x-h} \text{ or } (x-h)(y-k) = \alpha \tag{0.32}$$

gives hyperbolas with center (h, k) and asymptotes $x = h$ and $y = k$.

Example 9

Sketch the graph of $xy + 2x - 4y - 7 = 0$.

Solution

We must transform our equation into standard form (0.32). Notice that (0.32) can be rewritten as

$$xy - kx - hy + hk - \alpha = 0$$

Thus in our case we must have $k = -2$, $h = 4$ and $hk - \alpha = -7$, i.e., $\alpha = 7 + (-2)(4) = -1$. Our equation then becomes

$$(x-4)(y+2) = -1$$

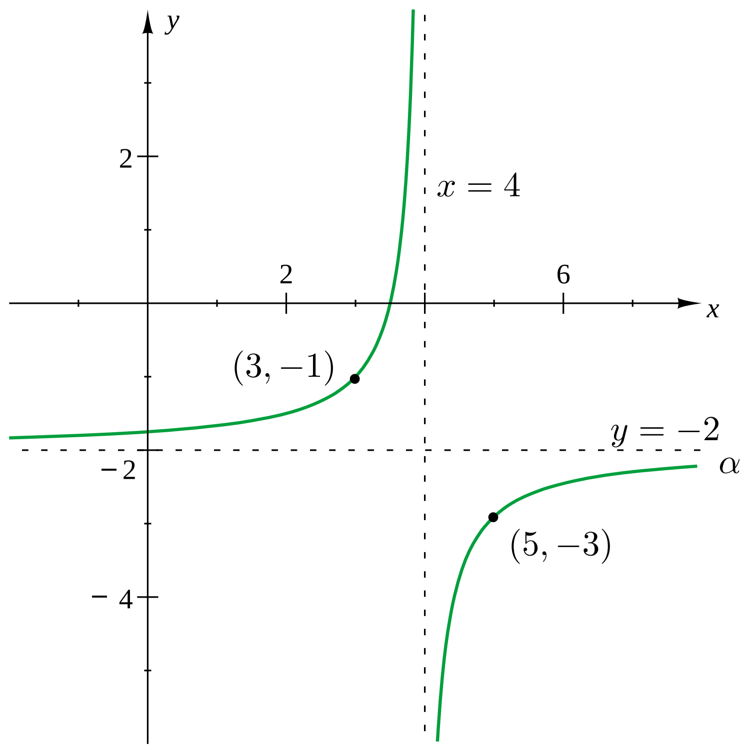
which is a hyperbola with center $(4, -2)$ and asymptotes $x = 4$ and $y = -2$. The *vertices* are seen to be

$$(4-1, -2+1) = (3, -1)$$

and

$$(4+1, -2-1) = (5, -3).$$

$$xy + 2x - 4y - 7 = 0$$



0.5.6 "Degenerate" Cases

Minor changes in the equations for conic sections can often produce surprisingly radical changes in the graphs themselves. In some cases perfectly respectable conic sections can degenerate into pairs of lines, single points, or the "empty set" (i.e., there are no values of x and y which satisfy the equation under consideration). For example,

$$\frac{(x+2)^2}{8} + \frac{(y-1)^2}{4} = 1$$

is the equation for an ellipse; change the 1 to a 0 however, and the graph becomes the one point $(-2, 1)$. If instead you change the 1 to -1, then the graph is the empty set (a sum of positive numbers can never be negative). As a final example,

$$(y-1)^2 - \frac{\left(x+\frac{1}{2}\right)^2}{4} = 1$$

is the equation for a hyperbola; change the 1 to a 0 however, and the graph becomes the two lines

$$y = \frac{1}{2}x + \frac{5}{4}$$

and

$$y = -\frac{1}{2}x + \frac{3}{4}.$$

(When these two lines are graphed, they form an "x".)

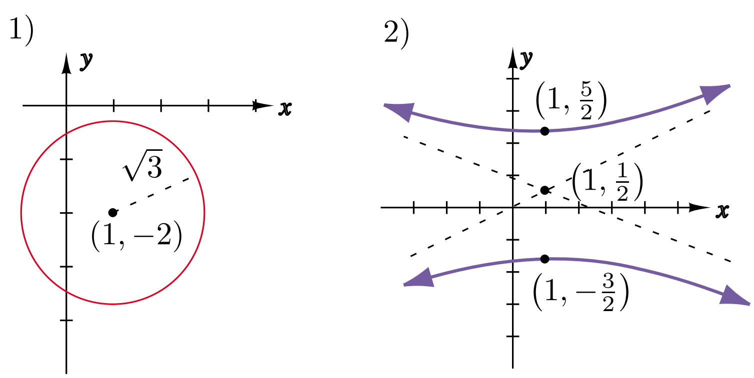
EXERCISES

Identify each of the following graphs by putting each equation into one of the standard forms discussed in the section. Specify the center, and if appropriate, the vertices, the radius, the lengths of the semi-major and semi-minor axes, or the asymptotes. Sketch the curve. (Note: A few "degenerate" cases are also included to keep you alert.)

1. $x^2 + y^2 - 2x + 4y + 2 = 0$
2. $4x^2 - 36y^2 - 8x + 36y + 31 = 0$
3. $x\left(\frac{1}{2}x - \sqrt{2}\right) + y(y+2) + 1 = 0$
4. $x^2 - y^2 + 4x + 2y + 3 = 0$
5. $3x^2 = y + 3x - 2$
6. $y + 2x = xy$
7. $2x^2 + y^2 - 4x + 6y + 15 = 0$
8. $8x^2 = 4y(y-1) + 17$

ANSWERS

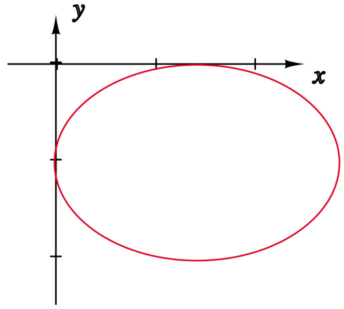
1. $(x-1)^2 + (y+2)^2 = 3$; a circle with center $(1, -2)$ and radius $\sqrt{3}$.



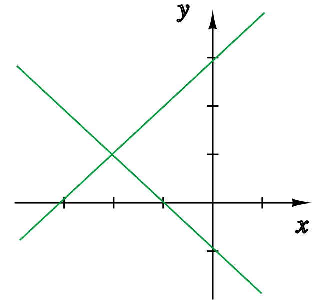
2. $\frac{(y-\frac{1}{2})^2}{4} - \frac{(x-1)^2}{9} = 1$; an "up-down" hyperbola with $a = 2$, $b = 3$, center $\left(1, \frac{1}{2}\right)$, vertices $\left(1, -\frac{3}{2}\right)$, $\left(1, \frac{5}{2}\right)$ and asymptotes $y = \frac{2}{3}x - \frac{1}{6}$ and $y = -\frac{2}{3}x + \frac{7}{6}$ as in Figure 2 above.

3. $\frac{\left(x-\sqrt{2}\right)^2}{2} + (y+1)^2 = 1$; an ellipse with semi-major axis length $a = \sqrt{2}$ and semi-minor axis length $b = 1$; center $(\sqrt{2}, -1)$, and vertices $(0, -1)$, $(2\sqrt{2}, -1)$, $(\sqrt{2}, -2)$, and $(\sqrt{2}, 0)$ as in Figure 3.

3)

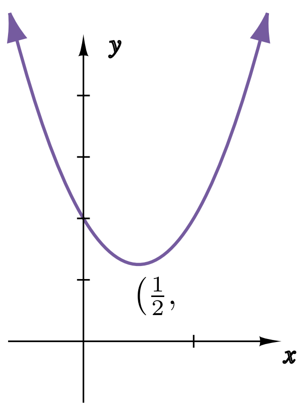


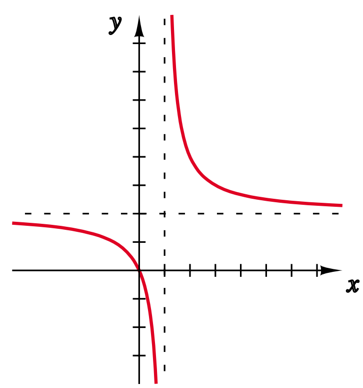
4)



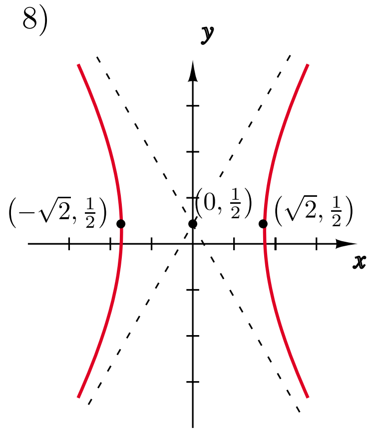
4. $(x+2)^2 - (y-1)^2 = 0$; a "degenerate" hyperbola which reduces to two lines, $y = x + 3$ and $y = -x - 1$ as in Figure 4 above.

5. $y - \frac{5}{4} = 3\left(x - \frac{1}{2}\right)^2$ a steep upward turning parabola with vertex $\left(\frac{1}{2}, \frac{5}{4}\right)$.





8)



0.6 SYSTEMS OF EQUATIONS

0.6.1 Two Simultaneous Equations

Given two equations in variables x and y, it is often times necessary to find those values of x and y which satisfy *both* of the equations *simultaneously*; this is naturally referred to as *solving a system of simultaneous equations*.

Example 1

Solve the system of equations

$$\left\{ \begin{array}{r} x + y = 3 \\ 2x^2 + y^2 = 6 \end{array} \right\}$$

Solution

A common procedure in solving simultaneous equations is to use one equation to solve for one variable in terms of the other. This is called the *method of substitution*. In the case at hand we can use the first equation to solve for y in terms of x,

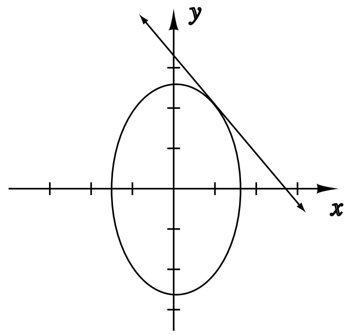
$$y = 3 - x$$

and then plug this into the second equation (i.e., *substitute* $3 - x$ for y) to obtain

$$\begin{aligned} 2x^2 + (3 - x)^2 &= 6 \\ 2x^2 + 9 - 6x + x^2 &= 6 \\ 3x^2 - 6x + 3 &= 0 \\ x - 2x + 1 &= 0 \\ (x - 1)^2 &= 0 \\ x &= 1 \end{aligned}$$

Thus $x = 1$ and $y = 3 - 1 = 2$; testing this back in the original two equations shows that $(x, y) = (1, 2)$ is indeed a solution.

Solving simultaneous equations, while an algebraic operation, has an important geometric meaning: the (x, y) solution values which one finds are the *intersection points* of the *graphs* of the two given equations. Thus, in Example 1, the line $x+y = 3$ intersects the ellipse $2x^2 + y^2 = 6$ in the one point $(1, 2)$, as shown.



The following list generalizes the procedures used to solve

Example 1 Method of substitution for solving two simultaneous equations in x and y

i. Use one of the equations to solve for y in terms of x, obtaining $y = f(x)$.

ii. Use $y = f(x)$ to eliminate y from the remaining equation, i.e., *substitute* $f(x)$ for y in the remaining equation. The result is an equation containing only x.

iii. Solve the new equation for specific values of x.

iv. Find the corresponding values of y by using $y = f(x)$.

v. Check all your (x, y) values back in the original two equations. (It is *not uncommon* to find that incorrect solutions have slipped in during the solution process.) If feasible, sketch the graphs of the two equations and compare their intersection points with the (x, y) solutions you discovered algebraically.

NOTE: The roles of x and y can be (and, in some instances, *must* be) reversed in this process, i.e., first solve for x in terms of y to get $x = g(y)$, etc.

Example 2

Solve the system of equations

$$\left\{ \begin{array}{c} \dfrac{2 - xy}{y + 1} = 1 \\ 2x^2 + 5x - (3x + 5)y + 3y^2 = 0 \end{array} \right\}$$

Solution

i. Solving the first equation for y in terms of x yields

$$y = \frac{1}{(x+1)}$$

ii. Plugging this into the second equation will yield

$$2x^2 + 5x - \frac{(3x+5)}{(x+1)} + \frac{3}{(x+1)^2} = 0$$

which simplifies to

$$2x^4 + 9x^3 + 9x^2 - 3x - 2 = 0.$$

iii. By the Rational Root Test (*Companion* Section 0.4.6) the only possible rational roots $x = p/q$ are those for which p divides -2 and q divides 2, i.e.,

$$x = \pm 2,\ \pm 1 \text{ or } \pm 1/2.$$

Only two of these check out to be roots: $x = -2$ and $x = 1/2$. Long division (*Companion* Section 0.4.2) of our polynomial by the product

$$2(x+2)(x-1/2) = 2x^2 + 3x - 2$$

yields $x^2 + 3x + 1$. By the Quadratic Formula (*Companion* Section 0.4.5) the roots of this term are $-3/2 \pm \sqrt{5}/2$. Thus all the possible x values are

$$-2,\ 1/2 - 3/2 \pm \sqrt{5}/2.$$

iv. From $y = \frac{1}{(x+1)}$ we see that our possible (x, y) values are

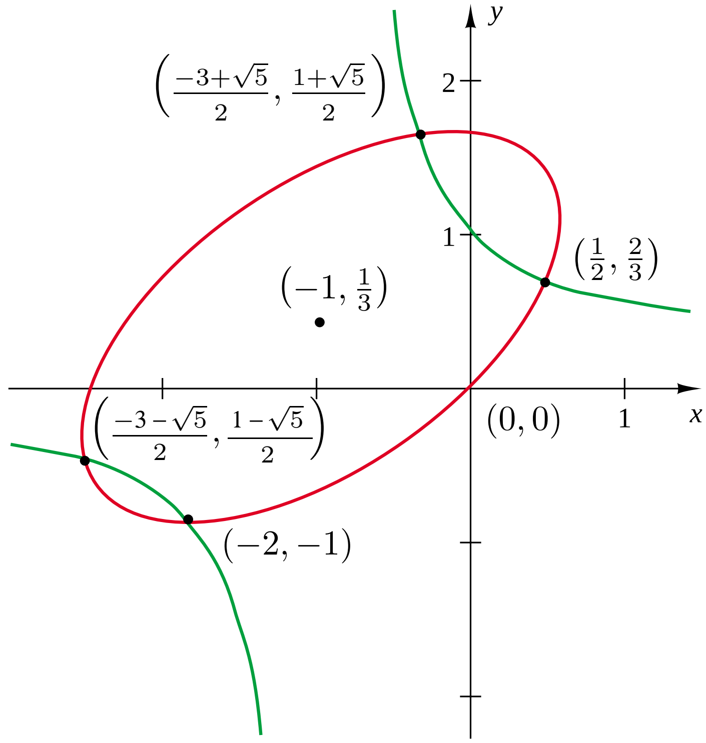
$$(-2, -1), (1/2, 2/3),$$

and

$$\left(-\frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{-1}{2} + \frac{\sqrt{5}}{2}\right), \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{-1}{2} - \frac{\sqrt{5}}{2}\right)$$

v. The first of these points does ***not*** check out in the original equations since it would require a division by zero. The remaining three points check out properly.

You would ordinarily ***not*** graph these two equations to check their solutions because of the complexity of the second function. We do, however, show the two graphs below so that you can see our solution values as points of intersection of the two graphs. The graph of our first equation can be shown to be a "mixed term" hyperbola with the point $(-2, -1)$ deleted (*Companion* Section 0.5.5). The second equation requires advanced techniques from Anton's Section 12.4 for its analysis: it is a tilted ellipse with center $(-1, 1/3)$.



The method just given for solving two simultaneous equations can run into difficulties in two places: in Step **i** it may be very difficult (or impossible) to use one equation to solve for y in terms of x, or in Step **iii**, the one variable equation may be difficult (or impossible) to solve for x. In such cases approximate solutions might be sought by numerical methods (which we do not discuss here) or slightly more round-about solution methods might be employed. This later situation is quite common when Step **i** proves unpleasant, as illustrated in the next example.

Example 3

Solve the system of equations

$$\left\{ \begin{array}{r} xy + y^2 - x^2 = 5 \\ 2xy + y^2 - x^2 = 2 \end{array} \right\}$$

Solution

Although either equation can be used to solve for y in terms of x, the result is somewhat unpleasant. It is much easier to notice that subtracting the first equation from the second will yield $xy = -3$, i.e., $y = -3/x$. This is a common type of modification of Step **i**. The rest of the steps now proceed in the standard way:

$$xy + y^2 - x^2 = 5 \text{ becomes } -3 + \frac{9}{x^2} - x^2 = 5$$

which is $x^4 + 8x^2 - 9 = 0$. This factors into

$$0 = \left(x^2 - 1\right)\left(x^2 + 9\right) = (x - 1)(x + 1)\left(x^2 + 9\right)$$

so that the only possible x values are 1 and -1; the corresponding y-values are -3 and 3 respectively. Both solutions $(1, -3)$ and $(-1, 3)$ test out correctly in the original equations. ■

0.6.2 Three Simultaneous Equations

There are times when you will run into *three* simultaneous equation in *three* unknowns, say x, y and z. The method of substitution generalizes to this situation in a straightforward manner. Choose one of the equations to solve for z in terms of x and y, and use this result to eliminate z from the remaining two equations. You will then have reduced to a system of two equations in two unknowns, and the procedures of the previous section apply.

Example 4

Solve the system of equations

$$\left\{\begin{array}{c} x + y + y^2 z = 2 \\ 2x = -yz \\ xy^2 z = 2 \end{array}\right\}$$

Solution

There are numerous ways to begin; we will use the second equation to solve for z in terms of x and y:

$$z = \frac{-2x}{y}$$

The remaining two equations then become

$$\left\{\begin{array}{r} x + y - 2xy = 2 \\ -2x^2 y = 2 \end{array}\right\}$$

Solve the second of these equations for y in terms of x:

$$y = -\frac{1}{x^2}$$

The remaining equation becomes

$$x - 1/x^2 + 2/x = 2$$

which simplies to

$$x^3 - 2x^2 + 2x - 1 = 0 \tag{0.33}$$

By the Rational Root Test (*Companion* Section 0.4.6) the only possible rational roots for this cubic are $x = \pm 1$; in fact, $x = 1$ does work, while $x = -1$ does not. Dividing our cubic by $x - 1$ yields

$$x^2 - x + 1$$

an irreducible quadratic term (no real roots).Thus $x = 1$ is the only solution to Equation (0.33). Tracing back we find

$$y = -1/1^2 = -1$$

$$z = -2(1)/(-1) = 2$$

Thus $x = 1$, $y = -1$ and $z = 2$ give the only possible solution for our system of equations. Checking these values in our equations show that indeed they do give a solution.

Solving simultaneous equations can be very difficult and (in the general case) no one method can be given which always works. Many times clever tricks need to be employed. . . , but we've done enough on the general case for now. We move on, instead, to the special (but important) case of *linear* equations.

0.6.3 Systems of Linear Equations

An equation of the form

$$ax + by = c$$

where a, b and c are constants, is called a *linear equation* in the variables x and y. As long as either a or b is non-zero, the graph of this equation is a line. Systems of linear equations occur quite often in calculus and, in contrast to the general situation discussed in Section One, there are specific techniques for dealing with linear systems which apply in *all* situations.

Example 5

Solve the system of linear equations

$$\left\{ \begin{array}{r} 2x + 3y = 4 \\ x - 2y = -5 \end{array} \right\}$$

Solution

As these equations represent two non-parallel lines in the plane (unequal slopes), there must be one and only one intersection point (i.e., the system has *exactly one* solution). We find this solution in two different ways:

1st method:

The *method of substitution* described in the previous sections will always work with linear systems. In the case at hand the first equation yields

$$y = 4/3 - (2/3)x$$

which when placed into the second equation will give

$$x - 2\left(\frac{4}{3} - \frac{2}{3}x\right) = -5$$

This solves to $x = -1$, and hence $y = 2$. The point $(-1, 2)$ checks out correctly in both of the original equations.

2nd method:

This is called the ***elimination method:*** we *eliminate* one variable by adding together carefully chosen multiples of the two given equations. In the case at hand multiplying the second equation by -2 and adding it to the first eliminates x quite neatly:

$$\begin{array}{r} 2x + 3y = 4 \\ -2x + 4y = +10 \\ \hline 0 + 7y = 14 \end{array}$$

so that $y = 2$. The first equation then gives $x = -1$.

You might, with good reason, question why we bothered to give the *elimination method* in Example 5 when *substitution* worked so well itself. The answer lies in generalization to more complicated linear systems, say 3 linear equations in 3 unknowns, or 4 linear equations in 4 unknowns. In these (very common) situations, both of the given methods will work, but the elimination method is much faster and less prone to careless errors (and it is programmable on a computer or hand calculator).

Example 6

Solve the system of linear equations

$$\left\{ \begin{array}{r} x + 2y - z = -3 \\ 2x - y + 3z = 9 \\ -3x + y - z = -6 \end{array} \right\}$$

Solution

We use the 1st equation to eliminate x from all the equations below it (i.e., from Equations 2 and 3):

-2 times the 1st equation added to the 2nd yields

$$\begin{array}{r} -2x - 4y + 2z = 6 \\ 2x - y + 3z = 9 \\ \hline -5y + 5z = 15 \end{array}$$

or $-y = z = 3$

3 times the 1st equation added to the 3rd yields

$$\begin{array}{r} 3x + 6y - 3z = -9 \\ -3x + y - z = -6 \\ \hline 7y - 4z = -15 \end{array}$$

We thus have a new system of linear equations:

$$\left\{ \begin{array}{r} x + 2y - z = -3 \\ -y + z = 3 \\ 7y - 4z = -15 \end{array} \right\}$$

in which the x variable has been eliminated from all but the 1st equation. We now use the 2nd equation to eliminate y from all the equations below it (i.e., from Equation 3):

7 times the 2nd equation added to the 3rd yields

$$\begin{array}{r} -7y + 7x = 21 \\ 7y - 4z = -15 \\ \hline 3z = 6 \end{array}$$

or $z = 2$.

This yields a third system of linear equations,

$$\left\{ \begin{array}{r} x + 2y - z = -3 \\ -y + z = 3 \\ z = 2 \end{array} \right\}$$

where each successive equation has one less variable, until only one variable appears. But thus the last variable has been solved for, and *back substitution* up through the system gives all the solutions:

3rd equation:	$z = 2$	gives $z = 2$
2nd equation:	$-y + z = 3$ $-y + 2 = 3$	gives $y = -1$
1st equation:	$x + 2y - z = -3$ $x + 2(-1) - (2) = -3$	gives $x = 1$

The solution $(2, -1, 1)$ checks out in the three original equations. ■

The following list generalizes the procedures used to solve Example 6.

Elimination method for solving three simultaneous linear equations in x, y and z:

i. Use the 1st equation to eliminate x from the 2nd and 3rd equations. Do this by adding suitable multiples of the 1st equation to suitable multiples of the 2nd and 3rd equations.

ii. Use the 2nd equation (new version) to eliminate y from the 3rd equation (new version). Do this by adding a suitable multiple of the 2nd equation to a suitable multiple of the 3rd equation.

iii. Solve the 3rd equation for z, solve the 2nd equation for y, and solve the 1st equation for x.

iv. Check your answer in the original equations.

Some comments are in order concerning this method:

a) The order of the equations and the order of the elimination process (x, then y, then z) are not sacred and can be changed around for convenience in solving a specific system. In fact, sometimes they *must* be switched around.

b) The generalization of this method to systems of n linear equations in n unknowns should be fairly clear.

c) There are systems of linear equations with *no solutions*; a simple example is

$$\left\{ \begin{array}{c} x + y = 1 \\ x + y = 0 \end{array} \right\}$$

These equations represent two parallel lines, and hence there cannot be a simultaneous solution. In such cases the elimination method reveals the problem by producing, at some stage, a nonsense equation such as $0 = 1$.

d) There are systems of linear equations with more than one solution (in fact, with an *infinite number* of solutions); a simple example is

$$\left\{\begin{array}{r} x + y = 1 \\ 2x + 2y = 2 \end{array}\right\}$$

In such cases the method indicates this occurrence by a "simultaneous elimination," e.g., when eliminating y you find that x has also been eliminated. Why this leads to an infinite number of solutions is illustrated in Example 7.

These last two points illustrate an important principle:

> There are only three possibilities for a system of two linear equations in two unknowns:

1. it has *exactly one* solution
2. it has an *infinite number* of solutions
3. it has *no solutions*.

It can be very helpful to have these three options in mind.

Example 7

Solve the system of linear equations

$$\left\{\begin{array}{r} x + 2y = -1 \\ 2x - y + 5z = 3 \\ 2x + 4z = 2 \end{array}\right\}$$

Solution

Use the 1st equation to eliminate x in the 2nd and 3rd equations. This gives

$$\left\{\begin{array}{r} x + 2y = -1 \\ -5y + 5z = 5 \\ -4y + 4z = 4 \end{array}\right\}$$

or

$$\left\{\begin{array}{lll} x + 2y & & = -1 \\ & y - z & = -1 \\ & y - z & = -1 \end{array}\right\}$$

Then using the 2nd equation to eliminate y in the 3rd equation will yield

$$\left\{\begin{array}{r} x + 2y = -1 \\ y - z = -1 \\ 0 = 0 \end{array}\right\}$$

This is an example of the last bulleted comment above; we cannot solve for z since z can take on any value. However, once z is specified, then

$$y = z - 1 \text{ and } x = -2y - 1 = -2z + 1$$

Any solution of the form $(-2z+1, z-1, z)$, where z is *any* number, checks out correctly in the original set of equations. This is termed the *general solution* for our system of equations; *specific solutions* are obtained by fixing a value for z. For example, taking $z = 1$ gives the specific solution

$$(-2(1)+1, 1-1, 1) = (-1, 0, 1)$$

EXERCISES

Solve each of the following system of equations

1. $\left\{\begin{array}{rl} 2x+4y & =-1 \\ 3x+y & = 1 \end{array}\right\}$

2. $\left\{\begin{array}{ll} xy & =2 \\ xy^2+x^2y & =6 \end{array}\right\}$

3. $\left\{\begin{array}{rl} xy & =x-y \\ 4y+2xy+x^2 & =4 \end{array}\right\}$

4. $\left\{\begin{array}{ll} x^3+z^2xz & =3 \\ x^3-z+xz & =1 \end{array}\right\}$

5. $\left\{\begin{array}{rl} A-B-5C & =3 \\ A+B+C & =3 \\ 2A+B+3C & =-2 \end{array}\right\}$

6. $\left\{\begin{array}{rl} x-y+z & =-1 \\ x+y-z & =4 \\ 3x-y+z & =1 \end{array}\right\}$

7. $\left\{\begin{array}{rl} x-y-z & =0 \\ 2x+3y+6z & =3 \\ x-2y+z & =0 \end{array}\right\}$

8. $\left\{\begin{array}{rl} x+y-(3/2)z & =0 \\ -2x-z & =-2 \\ x+y-z & =1 \end{array}\right\}$

9. $\left\{\begin{array}{rl} x+3y+2xz & =-1 \\ 3xyz & =-4 \\ x^2+3yz & =-3 \end{array}\right\}$

ANSWERS

1. $(x, y) = (1/2, -1/2)$

2. $(x, y) = (1, 2)$ or $(2, 1)$

3. $(x, y) = (-2, 2)$ or $(1, 1/2)$

4. $(x, z) = (1, 1), (1, -2)$ $\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}, -2\right)$ or $\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}, -2\right)$

5. $(A, B, C) = (-1, 6, -2)$

6. **No solutions.**

7. $(x, y, z) = (1/2, 1/3, 1/6)$

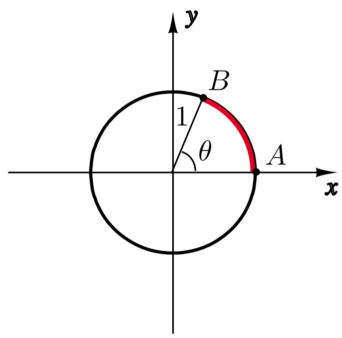
8. $(x, y, z) = \left(1 - \frac{1}{2}z, -1 + 2z, z\right)$; z is arbitrary

9. $(x, y, z) = (1, 2/3, -2)$ or $(1, -4/3, 1)$

0.7 TRIGONOMETRY REFRESHER

Appendix E in Anton's text is an extensive trigonometry review; this section concentrates more on the most important trigonometry results and/or those results which tend to give students the most trouble. Students with serious deficiencies in this topic should go through Anton's material (and perhaps top it off with this refresher).

0.7.1 Radian Measure



Consider the unit circle in the xy-plane with center $(0, 0)$:

$$x^2 + y^2 = 1$$

Take any angle θ measured *counterclockwise* from the x-axis. The length of that portion of the circle determined by θ (i.e., in the picture the circular arc from A to B) is called the *radian measure* of θ.

Since the circumference of the unit circle is 2π, the radian measure of an angle of 360° is 2π. The conversion from degrees to radians is always given by this proportion, i.e.,

$$\frac{\theta \text{ in radians}}{\theta \text{ in degrees}} = \frac{2\pi}{360^\circ} = \frac{\pi}{180^\circ} \tag{0.34}$$

Listing some common angles in both degrees and radians gives

degs.	···	rads.	degs.	···	rads.
30°	···	$\pi/6$	45°	···	$\pi/4$
60°	···	$\pi/3$	90°	···	$\pi/2$

(0.35)

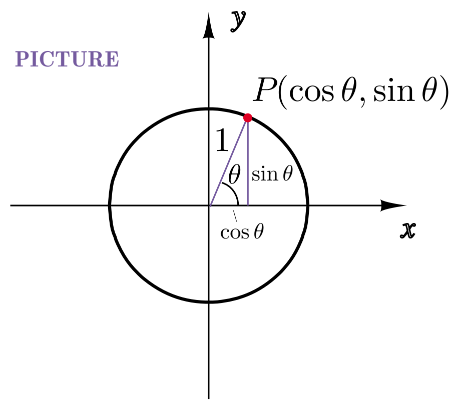
Angles measured in the *clockwise* direction from the x-axis are given *negative* radian measure. Thus, for example,

$$-30^\circ \ldots -\pi/6 \quad -180^\circ \ldots -\pi$$

People are frequently mystified as to why this bizarre radian measure is introduced for measuring angles when measurement by degrees seems so simple. Isn't trigonometry complicated enough without radian measure to further cloud the issue? The fact is, however, that radian measure is really a *necessity* for the *calculus* of trigonometric functions. Without it many of our important calculus formulas would need unpleasant changes in them! but we can't prove that to you until "differentiation" is developed in Chapter 3 of the text.

0.7.2 The Trigonometric Functions

THE BASIC PICTURE



As in the previous section consider the unit circle in the xy-plane with center (0, 0),

$$x^2 + y^2 = 1$$

and take any angle θ measured counterclockwise from the x-axis. The (x, y)-coordinates of the point P on the unit circle corresponding to θ are defined to be the *cosine of* θ and the *sine of* θ respectively[3], i.e.,

$$P = (x, y) = (\cos\theta, \sin\theta)$$

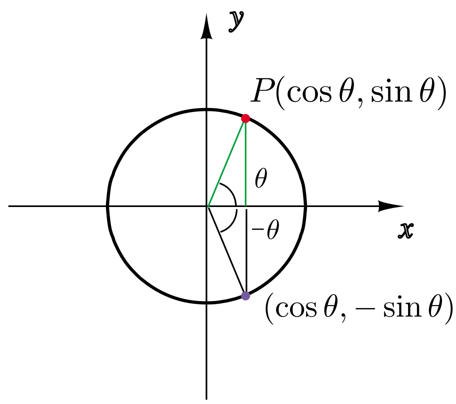
Since the point P is the *same* for both the angle θ and the angle $\theta+2\pi$ we immediately see that the sine and cosine are *periodic with period* 2π, i.e.,

$$\begin{aligned} \cos(\theta + 2\pi) &= \cos\theta \\ \sin(\theta + 2\pi) &= \sin\theta \end{aligned} \tag{0.36}$$

[3]These definitions for $\sin\theta$ and $\cos\theta$ are the same as the usual ones involving "opposite, adjacent, and hypotenuse" —as is established in Anton's Appendix E.

for all angles θ (when measured in radians of course!) Also, as seen from the figure, flipping from θ to $-\theta$ *does not change* the x-coordinate of P (the cosine) but *negates* the y-coordinate of P (the sine); thus

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta \tag{0.37}$$



Finally, since $P = (\cos\theta, \sin\theta)$ lies on the unit circle $x^2+y^2 = 1$, we immediately obtain our most well-known identity

$$\sin^2\theta + \cos^2\theta = 1 \tag{0.38}$$

The values of sine and cosine for the angles in (0.35) occur over and over again in calculus; they should be memorized:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$1/2$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

(0.39)

Here is a very useful method for remembering this table:

i. Write down the integers 0 1 2 3 4

ii. Take the square roots: $\sqrt{0}\ \sqrt{1}\ \sqrt{2}\ \sqrt{3}\ \sqrt{4}$

iii. Divide by 2:

$$\frac{\sqrt{0}}{2} \quad \frac{\sqrt{1}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{3}}{2} \quad \frac{\sqrt{4}}{2}$$

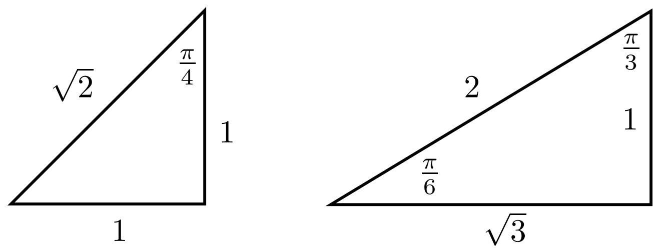
These numbers simplify to

$$0 \quad \frac{1}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{3}}{2} \quad 1$$

which is the row of sine values in the above table! The row of cosine values is obtained simply by writing down the sine values in *reverse order*, i.e.,

$$1 \quad \frac{\sqrt{3}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{1}{2} \quad 0$$

Another way to remember these values is to memorize the following two right triangles:



The sine and cosine of the angles $\pi/6$, $\pi/4$ and $\pi/3$ can be read off from these triangles by

$$\sin\theta = \frac{\text{opposite}}{\text{hypothenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypothenuse}}$$

The other four trigonometric functions are all defined in terms of sine and cosine; these definitions need to be memorized:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \tag{0.40}$$

These definitions apply when the denominators are non-zero, of course.

The secant and cosecant are periodic with period 2π, while the tangent and cotangent are periodic with period π. Notice that, unlike the sine and cosine, these new functions are ***not*** defined for all values of θ since the denominator terms can be zero for some values of θ.

0.7.3 Trigonometric Identities

There are a number of trigonometry formulas which you need to memorize because they are regularly used in calculus and its applications. But, there is a *hierarchy* of importance in the formulas so that you can learn the most important ones now and fill in the others later. Moreover, most of the identities are quickly derived from a small "core" of formulas; memorize the core *very well* and remember the simple derivation tricks, and you'll be in good shape!

You already have most of the "core": identities (0.36), (0.37), (0.38) (quickly derived from "The Basic Picture"), the table of values 0.39, and definitions (0.40). We need only one more pair of formulas;

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\sin\beta + \sin\beta\sin\alpha\end{aligned} \tag{0.41}$$

These are not trivial identities to verify (See Anton's Appendix E) nor are they "intuitive" in any reasonable sense. However, they need to be carefully memorized.

It is *useful* to memorize the rest of the formulas of this section, but all are derivable from our core collection, most in very easy ways. Learning the derivations will free you from the worries of small errors: if unsure of a certain formula, you simply rederive it. We suggest that you try to derive the following formulas from the core formulas yourself before referring to our calculations:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ \sin 2\alpha &= 2\sin\alpha\cos\alpha\end{aligned} \tag{0.42}$$

These two formulas are obtained from (0.41) by setting $\beta = \alpha$. An important feature of the second of these equations is that it allows us to express the product of $\sin\alpha$ and $\cos\alpha$ as one single trigonometric function

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha\psi \tag{0.43}$$

This can come in quite handy in calculus (especially in "integration" of trig functions); we would like to have similar formulas for $\sin^2\alpha$ and $\cos^2\alpha$. This is fortunately provided by the first identity in 0.42 along with 0.38:

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$$

Adding these two equations and dividing by 2 gives a formula for $\cos^2\alpha$; subtracting the two equations and dividing by 2 gives a formula for $\sin^2\alpha$. The results are

$$\cos^2\alpha = 1 + \frac{\cos 2\alpha}{2}$$
$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \tag{0.44}$$

These are very important formulas in "integration theory."

More generally, we can express any product of the form $\sin\alpha\cos\beta$, $\sin\alpha\sin\beta$ or $\cos\alpha\cos\beta$ as a sum of single sine and cosine functions. This is done by first establishing the identities

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \tag{0.45}$$

from 0.41 and 0.37. Then judicious adding or subtracting between 0.41 and 0.45 yields

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{0.46}$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

It is, in our opinion, better to just know that formulas 0.45 and 0.46 exist rather than to memorize them (...but perhaps it's best to check with your calculus instructor on this advice!) There is just one last pair of formulas which we feel is essential to know for calculus (others which are helpful or only occasionally needed can be found in Anton):

$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta \tag{0.47}$$

These are very useful in "integration theory" where it is essential at times to switch between $\tan^2\theta$ and $\sec^2\theta$, or between $\cot^2\theta$ and $\csc^2\theta$. Their derivations are very easy: simply divide 0.38 by $\cos^2\theta$ for the first identity, and by $\sin^2\theta$ for the second.

Here is the first derivation:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

i.e.

$$\tan^2\theta + 1 = \sec^2\theta$$

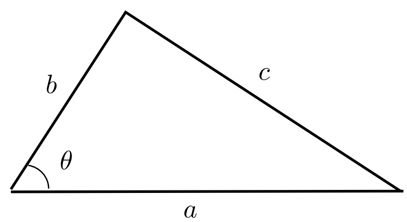
0.7.4 Law of Cosines

Using the cosine function we can obtain an important generalization of the Pythagorean Theorem known as the Law of Cosines:

Law of Cosines

Suppose a triangle has side lengths a, b and c, and θ is the angle between the sides of lengths a and b.
Then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

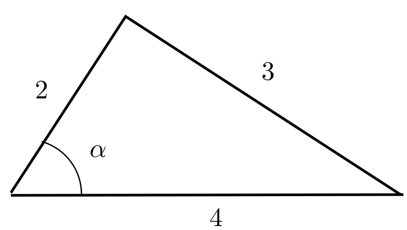


Notice that if θ is a right angle, i.e., $\theta = \pi/2$, then $\cos\theta = 0$ and the Law of Cosines becomes the familiar Pythagorean Theorem

$$c^2 = a^2 + b^2$$

A proof of the Law of Cosines can be found in Anton's Appendix E.

Example 1



Given a triangle with side lengths 2, 3 and 4, determine the angle between the sides of lengths 4 and 2.

Solution

Let α be the angle between the sides of lengths 4 and 2. Then the Law of Cosines gives

$$3^2 = 4^2 + 2^2 - 2(4)(2)\cos\alpha$$

$$9 = 16 + 4 - 16\cos\alpha$$

so that

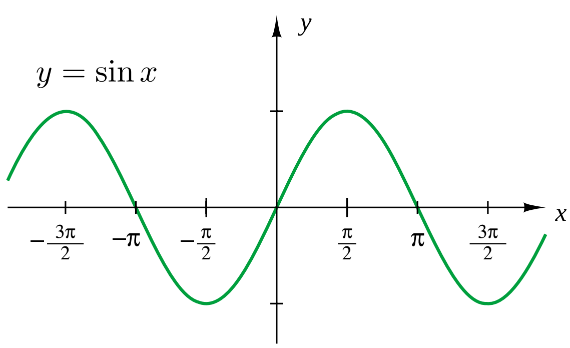
$$\cos\alpha = (16 + 4 - 9)/16 = 11/16 = .6875$$

Using a hand calculator, we see that

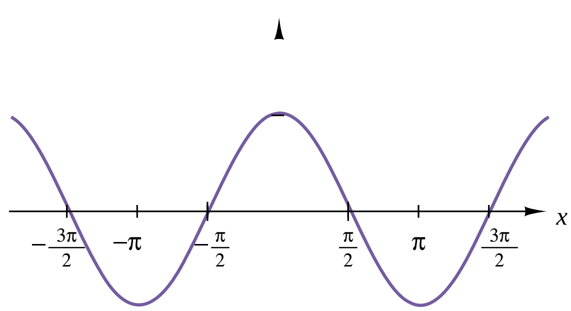
$$\alpha \approx .813 \text{ radians} \approx 46.5^\circ$$

0.7.5 Graphs of Trigonometric Functions

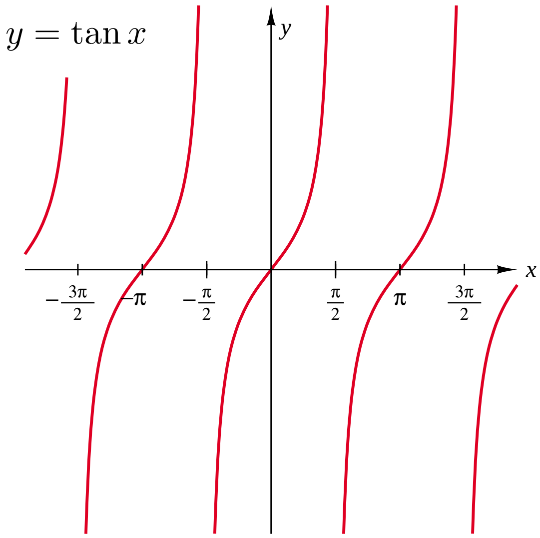
From the graph *The Basic Picture* in Section 0.7.2 it can be seen that the sine function $y = \sin x$ starts at $y = 0$ when $x = 0$, increases to $y = 1$ when $x = \pi/2$, decreases to $y = 0$ when $x = \pi$ and further to $y = -1$ when $x = 3\pi/2$, and then increases back to $y = 0$ when $x = 2\pi$. Since from 0.36 the sine function is periodic with period 2π, the graph just described from $x = 0$ to $x = 2\pi$ is simply repeated over every interval $[2\pi n, 2\pi(n+1)]$ for n any integer. We thus obtain the graph:

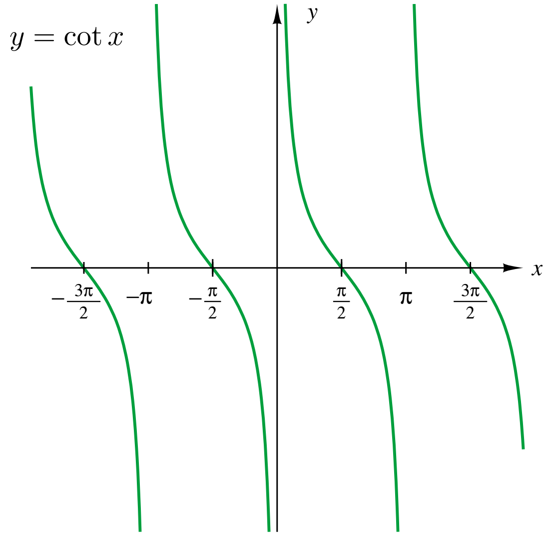


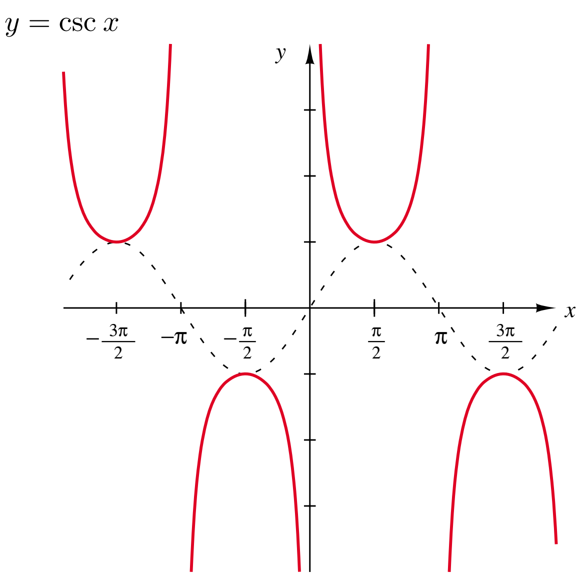
The same analysis for the cosine produces the same type of graph, except that $y = 1$ when $x = 0$. The graph is:

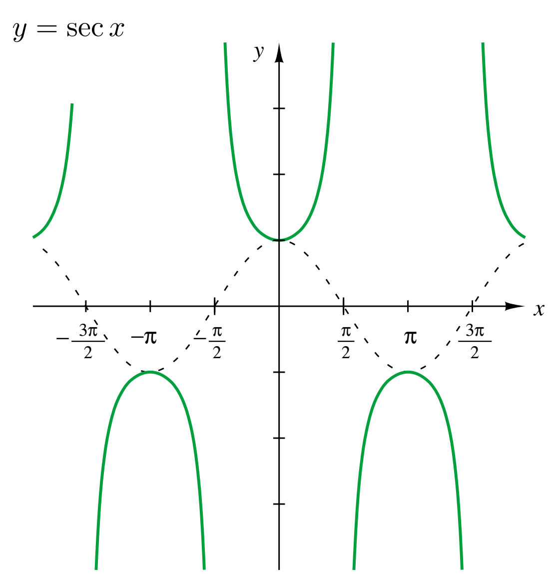


The remaining four graphs are shown in the figures below.









Notice how the graph of $y = \csc x$ "hangs off" of the graph of $y = \sin x$, and similarly for $y = \sec x$ relative to $y = \cos x$. This provides a convenient method for remembering the graphs of the secant and the cosecant.

The *translation principles*, discussed in *Companion* Section 0.5.1, tell us how to obtain quickly the graphs of such functions as $y = 2 + \cos(x - 3)$. The graph of this particular example would be the graph of $y = \cos x$ translated up by 2 units and over to the right by 3 units. However, with trigonometric functions it is also common to *expand* or *contract* their graphs, as we now show.

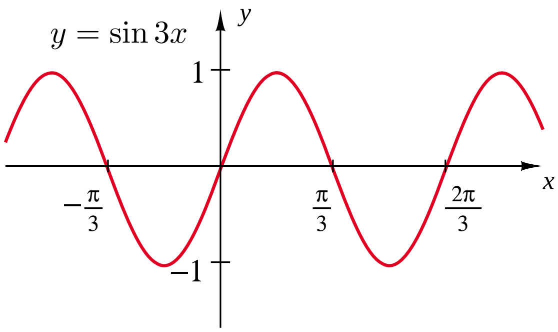
Example 2

Graph the function $y = 2\sin 3x$.

Solution

Let's first examine the function $y = \sin 3x$.

For each x, the corresponding y has the value given by the sine of $3x$. Thus, in the x-interval $\left[0, \frac{2\pi}{3}\right]$ we will have assumed all the sine values that are normally taken on the x-interval $[0, 2\pi]$. Hence the *period*[4] of $y = \sin 3x$ is $2\pi/3$, as opposed to period 2π for $y = \sin x$, and the graph of $\sin x$ is *contracted* by a factor of $1/3$ along the x-axis as shown in the figure.



Return to the function $y = 2\sin 3x$. The 2 merely *expands* the graph of $\sin 3x$ by a factor of 2 along the y-axis as shown in the graph to the right. Thus the *amplitude*[5] of $y = 2\sin 3x$ is 2, as opposed to amplitude 1 for $y = \sin x$.

b) to *expand*/*contract* the graph of an equation along the y-axis by a factor of $a > 0$, replace y with y/a (the graph expands if $a > 1$ and contracts if $1 > a > 0$).

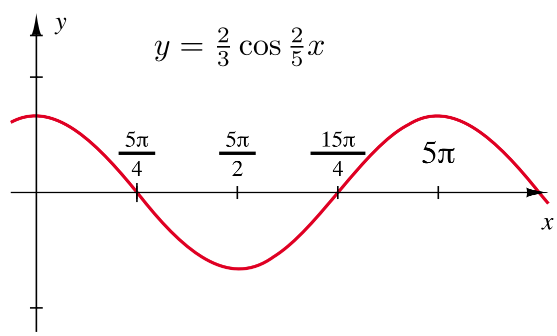
You should compare these with the translation principles as given in Section 0.5.1. In Example 2 we took $y = \sin x$ and replaced x with $3x = \frac{x}{(\frac{1}{3})0.5.1}$ and y with $\frac{y}{2}$; thus we had a *contraction* along the x-direction by a factor of $1/3$ and an *expansion* along the y-direction by a factor of 2, as shown in the final graph of Example 2.

Example 3

Graph the function $y = \frac{2}{3} \cos \frac{2}{5} x$.

Solution

Here we have taken $y = \cos x$ and replaced x with $\frac{2}{5}x = \frac{x}{(5/2)}$ and replaced y with $\frac{y}{(2/3)}$.

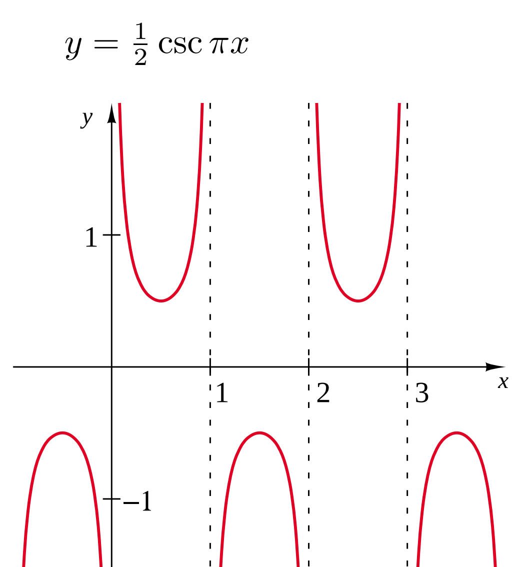


Thus we have expanded the graph $y = \cos x$ by a factor of $5/2$ along the x-axis and contracted the graph by a factor of $2/3$ along the y-axis.

Example 4

Graph the function $y = \frac{1}{2} \csc \pi x$.

Solution



We have taken $y = \csc x$ and replaced x with $\frac{x}{\left(\frac{1}{\pi}\right)}$ and y with $\frac{y}{\left(\frac{1}{2}\right)}$. Thus we need only contract the graph of $\csc x$ by a factor of $1/\pi$ along the x-axis and contract the graph by a factor of $1/2$ along the y-axis.

Numerous exercises for the material of this section can be found at the end of Anton's Appendix E.

0.8 WORD PROBLEMS

Word problems: the very term strikes terror into the hearts of most people, and their appearance on assignments or exams is enough to make a football hero weep. Is all this anguish and gnashing of teeth justified? **No!**

Word problems are difficult only because most people do not approach them in a ***careful***, ***disciplined*** and ***organized*** fashion. In fact, word problems are frequently easier than many calculus problems because the real-word, physical nature of the problem gives you some extra common-sense tools to use in finding its solution. However, if you are careless, lazy, or sloppy, you are going to be in serious trouble. This is especially true with the first step in a word problem solution: *translation* from words into appropriate mathematical equations. This must be done slowly, carefully, one-step-at-a-time. In this section we give a step-by-step procedure which will cover most word problems you'll encounter in calculus. This will give you the *organization*... you must provide the care and discipline.

We should also emphasize how vitally important mastering word problems is for real life applications of calculus. Ninety-nine percent of applications start with a "word problem," a situation described in English sentences which demands translation into, and solution by the methods of, mathematics.

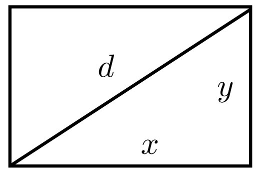
The best way (perhaps the only way) to learn how to solve word problems is to do some. The examples we give are as similar to those in calculus as they can be without actually using calculus. Moreover, to illustrate the thinking that goes into a solution, *our explanations are very long-winded*. So get ready...

Example A

Find the dimensions of a rectangle if the diagonal is 2 more than the longer side, which in turn is 2 more than the shorter side.

Step 1

We must first *translate* our problem into mathematics; this generally takes a number of readings. A quick overview shows that we are looking for the dimensions x and y of a rectangle (let x be the longer of the two).



The length of the diagonal also plays an important role in the problem, and so we assign it the label d. We now reread the word problem very slowly and carefully, translating every detail into an equation:

Find the dimensions of a rectangle

$$\to x = ?\ \ y + ?$$

if the *diagonal is* 2 *more than the longer side*

$$\to d = x + 2$$

which in turn is 2 *more than the shorter side*

$$\to x = y + 2.$$

Read the problem another time: did we miss anything? It appears that we caught everything in either our picture or our equations, and thus the translation phase is complete. You should, from this point on, have little need to refer to the actual word problem again; you have only to use your picture and equations.

Step 2

Think about your method of solution, *game plan*.

Question: What are you looking for?

Answer: x and y.

Question: What equations do you have to work with?

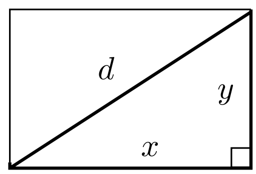
Answer:

$$d = x + 2$$

$$x = y + 2$$

Oops! Two equations in three unknowns!

This should immediately cause a bell to ring: in general you need *three* equations to solve for three unknowns—two equations are *not enough*! So we go back and look at our word problem again (we shouldn't have to do this—but we're worried!): did we miss an equation relating x, y and d? Nope... well, wait a minute. We certainly did not miss any *explicit* equations, i.e., relationships which are explicitly stated in the problem. However, are there some *implicit* equations that we could find, i.e., relationships which, although not directly stated in the problem, are implied by the relationships which are given? Ah-ha! Yes! [Before reading on... can YOU find this relationship? Try it!] Look at our picture:



There is a right triangle crying out to be "Pythagorized":

$$d^2 = x^2 + y^2$$

So... now we do have three equations in three unknowns. Our *game plan* is thus to solve the following system of simultaneous equations:

$$\left\{ \begin{array}{l} d = x + 2 \\ x = y + 2 \\ d^2 = x^2 + y^2 \end{array} \right\}$$

Step 3

It's time to execute our game plan. (This is the easy part, we hope !) Our first two equations give d and y in terms of x:

$$\left\{ \begin{array}{l} d = x + 2 \\ y = x - 2 \end{array} \right\}$$

Plug these into the third equation and we get

$$(x + 2)^2 = x^2 + (x - 2)^2$$

which simplifies to

$$0 = x^2 - 8x = x(x - 8)$$

Thus either $x = 0$ (impossible since a rectangle has non-zero side lengths) or $x = 8$. But if $x = 8$, then $y = 8 - 2 = 6$ or $d = 8 + 2 = 10$. Thus $(x, y, z) = (8, 6, 10)$ is the only possible solution, and sure enough, it does check out in all of our equations.

We summarize our method as follows:

How to tackle a word problem (. . . and live to tell about it).

Step 1. Translate into mathematics.

This major step takes several readings of the problem. The procedure to use (more-or-less in the following order) are

(a) obtain a general overview of the problem, sketching a picture or constructing a table if appropriate,

(b) determine the quantities which you desire to compute, and assign labels to them (e.g., x, y, t, etc.). These are the "unknowns"

(c) label other quantities which you believe will be important,

(d) write down equations for relationships between your labelled quantities: use as few variables as possible, preferably only your "unknowns" from (b),

(e) check that every place of information has been translated into an equation and/or has been placed in your picture.

Step 2. Devise game plan.

(a) determine if you have enough equations to solve for the unknown quantities,

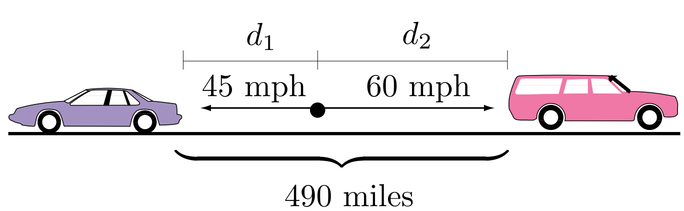
(b) if necessary, look for relationships between variables which are *implicit* in the problem; use these to eliminate variables if possible,

(c) decide how you will solve for your unknowns.

Step 3. Execute game plan . . .

. . . and then check that your answers are reasonable. Step 1, the translation from words to mathematics, is surely the step to concentrate on. Most people do reasonably well once a word problem has been accurately translated; however, the translation is often done incompletely or inaccurately, and dooms the solver to failure. Use our method faithfully and your success rate with word problems is guaranteed to improve. We give a few more examples to illustrate our method.

Example B



Two cars start from the same point and travel in opposite directions with speeds of 45 and 60 miles per hour respectively. In how many hours will they be 490 miles apart?

Step 1. Translation.

This is a problem involving (constant) *speed*, hence (from the familiar formula $d = rt$, i.e., distance equals speed times time) we need to consider *distance* and *time* quantities. We are looking for the (unknown) time t which it takes for two autos to be a certain distance apart; we thus label the distances which each car travels in time t:

$$d_1 = \text{distance traveled by 45 mph car}$$
$$d_2 = \text{distance traveled by 60 mph car}$$

Each of these quantities can be related to our unknown time t (as we always desire to do, when possible) by using the rate equation $d = rt$:

$$d_1 = 45t \text{ and } d_2 = 60t$$

Now we reread our problem very carefully for every detail.

Two cars start from the same point and travel in opposite directions with *speeds of* 45 *and* 60 *miles per hour* respectively. In *how many hours* will they be 490 miles apart?

$$\left.\begin{aligned} \frac{d_1}{t} &= 45 \\ \frac{d_2}{t} &= 60 \end{aligned}\right\} \text{As noted above}$$

$t =$? when

$$d_1 + d_2 = 490$$

Our total collection of equations is therefore:

$$d_1 = 45t, \; d_2 = 60t$$
$$t = ? \text{ when } d_1 + d_2 = 490$$

Step 2. Devise game plan.

We wish to solve for t, and we have three equations in the three unknowns d_1, d_2 and t. So we solve the equations. . .

Step 3. Execute game plan.

Since $d_1 = 45t$ and $d_2 = 60t$, then

$$490 = d_1 + d_2 = 45t + 60t = 105t$$

Thus

$$t = \frac{490}{105} = 4\frac{2}{3} \text{ hours}$$

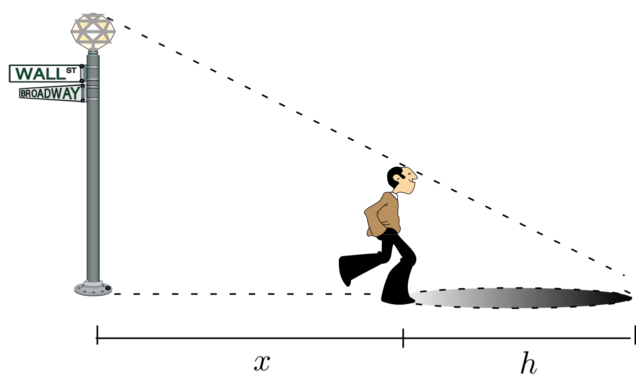
This answer is easily checked:

$$d_1 = 45\left(4\tfrac{2}{3}\right) = 210$$
$$d_2 = 60\left(4\tfrac{2}{3}\right) = 280$$

Thus $d_1 + d_2 = 490$, as desired.

Example C

A 6-foot man, walking at a rate of 5 feet/sec, passes under an 18-foot lamp post. How long is his shadow 10 seconds after passing the lamp post?



Step 1. Translate.

We have another *speed* ("rate") problem, and thus *time* and *distance* variables must be considered. We are looking for the length of our walker's shadow at a certain time; label this desired unknown as h. The picture then screams out for us to label the distance from the walker to the lamp post; label this as x. Since his speed is 5 feet/sec and he covers the distance x in 10 seconds, the rate equation $d = rt$ yields

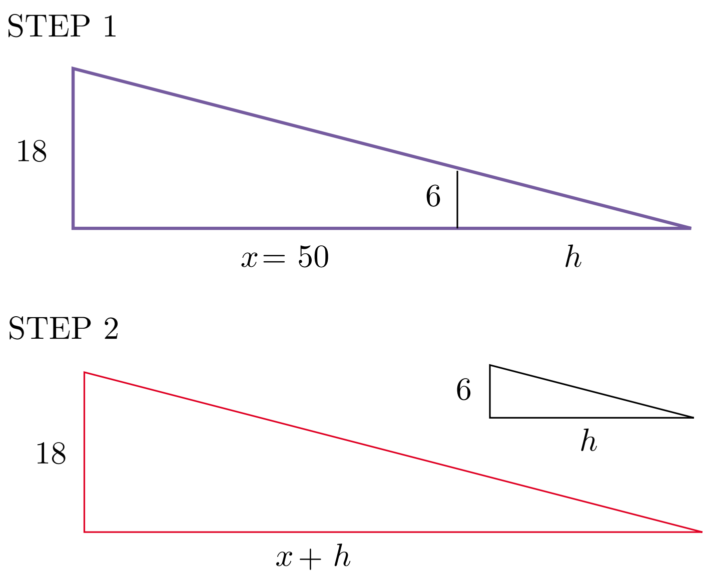
$$x = 5(10) = 50 \text{ feet.}$$

We have one labelled unknown, h, and a basic picture; we reread the problem for other relationships. A 6-*foot man*, walking at a *rate of* 5 feet/sec, passes under an 18-*foot* *lamp post. How long is his shadow* 10 *seconds* after passing the lamp post?

Place 6 in the picture,
$5 = x/t = x/10$,
as noted above,
Place 18 in the picture
h =? when
$t = 10$ *seconds have elapsed.*

Our picture is thus as shown in Step 1 (figure below) and our equations are

$$h = ? \text{ when } t = 10, x = 50$$



Step 2. Devise game plan

We want h, but we have not written down any equation relating h with x or t. Clearly there must be some *implicit* relationship between these variables. (Before reading on—can YOU find this relationship? Try it!) Let's turn to the picture (Figure 2 above.). Ah-ha!, similar triangles (*Companion* Section 0.1.4) are staring us in the face:

$$\frac{x+h}{h} = \frac{18}{6}$$

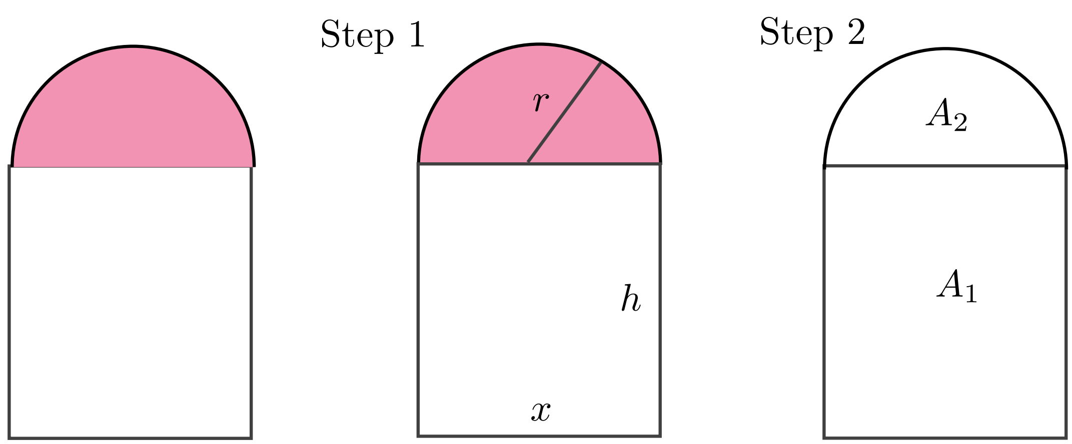
This is how we relate x and h. The game plan is simply to solve for h in this equation.

Step 3. Execute game plan

$$\begin{aligned} 6(x+h) &= 18h \\ 6x+6h &= 18h \\ 6x &= 12h \\ h &= \frac{1}{2}x \\ &= \frac{1}{2}(50) \\ &= 25 \text{ feet} \end{aligned}$$

Example D

A builder has constructed a large window which is a rectangle of clear glass with a semicircle of red-tinted glass above it.



From his records you see that he used 24 feet of molding around the outer perimeter of the window, that the cost of clear glass is \$1/ft^2, the cost of red-tinted glass is \$2/ft^2, and that the builder's total glass cost was \$54. What are the dimensions of the window?

Step 1. Translate

It's pretty clear that we will need to label the dimensions of the window:

$$\begin{aligned} x &= \text{length of rectangle} \\ h &= \text{height of rectangle} \\ r &= \text{radius of semicircle} \\ 24 &= \text{perimeter of the window} \end{aligned}$$

Two relationships which are immediately seen are

$$x = 2r \tag{0.48}$$

$$24 = x + 2h + \pi r = 2h + (2 + \pi)r \tag{0.49}$$

Thus x has been effectively eliminated and we are left with finding h and r. We now fill in details with a rereading of the problem:

...24 feet of molding around the outer perimeter...	perimeter of 24, shown above to give the equation $24 = 2h + (2 + \pi)r$
...cost of clear glass is \$1/ft^2...	$C_1 = \$1/\text{ft}^2$
...cost of red-tinted glass is \$2/ft^2...	$C_2 = \$2/\text{ft}^2$
...total glass cost was \$54.	$C = \$54$
What are the dimensions of the window?	$x = ?$, $h = ?$, $r = ?$ ($x = 2r$, as shown above)

Step 2. Devise game plan

We have 2 unknowns (h and r) and desire to compute both of them. But we have only one equation, (0.49), which relates h and r, and our general knowledge of solving for unknowns should tell us that we'll need at least 2 equations. (Before going on ... can YOU find the 2nd relationship? Try it!) We look to the cost for our second equation: we know that the total cost C of the glass must depend on the cost per square foot of clear and red-tinted glass, along with the dimensions of the window panes. Thus

$$C = A_1C_1 + A_2C_2$$

where

$$A_1 = \text{area of rectangular pane} = xh = 2rh$$

and

$$A_2 = \text{area of semi-circular pane} = \frac{1}{2}\pi r^2$$

See Step 2 in the figure.

Thus

$$C = 2rhC_1 + \frac{1}{2}\pi r^2 C_2.$$

Using the values of C, C_1 and C_2, yields

$$54 = 2rh + \pi r^2 \tag{0.50}$$

The game plan is thus to solve Equations (0.49) and 0.50 for h and r, and then compute x from Equation (0.48).

Step 3. Execute game plan

Listing our equations together gives

$$\begin{aligned} 24 &= 2h + (\pi + 2)r \\ 54 &= 2rh + \pi r^2 \end{aligned}$$

Solving the first equation for h in terms of r yields

$$h = 12 - \tfrac{1}{2}(\pi + 2)r \tag{0.51}$$

Plugging this into the second equation yields

$$54 = r(24 - (\pi + 2)r + \pi r^2$$

which simplifies to

$$0 = r^2 - 12r + 27 = (r - 3)(r - 9)$$

Thus $r = 3$ or 9. However, using Equation (0.51) for h we find that $r = 9$ gives a *negative* value for h:

$$\begin{aligned} h &= 12 - \frac{1}{2}(\pi + 2)9 \\ &\cong 12 - \frac{9}{2}(5.14) \\ &= -11.13 \end{aligned}$$

Since h, as the height of the rectangle, cannot be negative, this rules out $r = 9$ as a possible value. Thus turn to $r = 3$. With this value, $x = 6$ from (0.48) and $h = 9 - 3/2\pi \approx 4.29$ from (0.51). These are physically allowable dimensions, and the numbers do check out in Equations 0.48, 0.49 and 0.50. Thus our solution is

$$\boxed{r = 3, \ \ x = 6, \ \ h \approx 4.29}$$

Example E

A doctor has a 2-liter solution of 3% boric acid. How much of a 10% solution of the acid must she add to have a 4% solution?

Step 1. Translate

Percentages are proportions, and as such, are divisions of one quantity by another. In this case we have percentages of boric acid in a solution, i.e.,

$$[\text{\% of boric acid}] = \left[\frac{\begin{array}{c}\text{volume of}\\ \text{boric acid}\end{array}}{\begin{array}{c}\text{total volume}\\ \text{of solution}\end{array}}\right] \tag{0.52}$$

The quantity we desire to compute is the *total volume* of added 10% solution; label this as h. From (0.52) we then can compute the *volume of boric acid* which we will be adding. There are, of course, two other solutions (the initial 3% solution and the final 4% solution), and each has a *total volume* and a *volume of boric acid*. We could give labels to each of these, but introducing so many new variable designations seems unwise if it can be avoided. Instead we set up a table for all these quantities which "incorporates" Equation (0.52) in a convenient way, and which contains all the specific information given in the word problem itself:

% of boric acid in solution	×	total volume of solution (liters)	=	volume of boric acid (liters)
3%		2		?
10%		h		?
4%		$2+h$		?

The "2-liter" entry is a given piece of information, and the $2+h$ reflects the fact that the 4% solution is the sum of the 3% and 10% solutions. We now can use Equation (0.52) to fill in the last column:

% of acid	×	total volume	=	volume of acid
3%		2		.06
10%		h		$(.10)h$
4%		$2+h$		$(.04)\,(2+h)$

A rereading of the problem will show that we have not missed any information.

Step 2. Devise a game plan

We have one unknown quantity, h, one table, and no equations; clearly there must be at least one implicit equation for h which can be drawn out of our table. The game plan is therefore to analyze our table to find this relationship. (Before going on! can YOU find the relationship? Try it!)

Step 3. Execute game plan

The trick lies in checking the consistency of the table. *The volume of acid in the final mixture must equal the sum of volumes of acid in the solutions being mixed;* after all, every drop of acid in the final mixture came from one of the two! That is, the sum of the first two entries in the 3rd column must equal the third entry in the 3rd column:

$$.06 + (.10)h = (.04)(2 + h)$$

Ah ha! This equation will be true only for one value of h:

$$\begin{aligned} .06 + (.10)h &= .08 + (.04)h \\ .06h &= .02 \end{aligned}$$

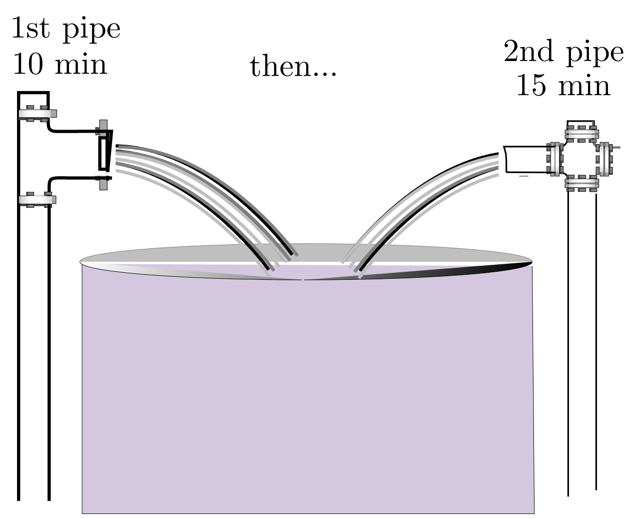
$$\boxed{h = 1/3 \text{ liter}}$$

Example F

One pipe takes 30 minutes to fill a tank. After it has been running for 10 minutes, it is shut off. A second pipe is then opened and it finishes filling the tank in 15 minutes. How long would it take the second pipe alone to fill the tank?

Step 1. Translate

This is a *rate* problem, since clearly the difference in the two pipes is that they allow fluid to pass through at differing speeds.



The basic relationship which governs our situation is

$$\begin{bmatrix} \text{volume of} \\ \text{fluid passing} \\ \text{through pipe} \end{bmatrix} = [\text{rate of flow}] \times [\text{time}]$$

i.e.,

$$[\text{rate of flow}] = [\text{volume}/\text{time}]$$

The unknown which we desire to find is

$t =$ the amount of time which the 2nd pipe would need to fill the tank

Other quantities which should play a role in this problem are:

$V =$	volume of tank
$r_1 =$	rate of flow for 1st pipe
$r_2 =$	rate of flow for 2nd pipe

As in the previous problem, we have one relationship which applies a number of times. This suggests the use of a table again:

rate × of flow	time =	volume		
r_1	30	V		1st pipe filling tank alone
r_2	t	V		2nd pipe filling tank alone
r_1	10	V_1	sum is V	Both pipes filling tank
r_2	15	V_2		

Here, of course, V_1 is the part of the volume of the tank filled by the 1st pipe in 10 minutes, and V_2 is the part left for the 2nd pipe to fill. Notice how all the given information is nicely recorded and organized in such a table.

Step 2. Devise game plan

Somehow we need to obtain equations from our table to solve for t. Well, we have a lot of variables in our table; let's cut down the number until perhaps we'll end up with an equation just involving t. That's our game plan.

Step 3. Execute game plan

The first two lines of our table will eliminate r_1 and r_2:

$$r_1 = V/30 \text{ and } r_2 = V/t$$

The third and fourth lines together will then yield

$$V = V_1 + V_2 = 10r_1 + 15r_2$$

Substituting from above for r_1 and r_2 yields

$$\begin{aligned} V &= 10(V/30) + 15(V/t) \\ 1 &= 1/3 + 15/t \end{aligned}$$

(V just cancels out of the problem)

$$\boxed{t = 22.5 \text{ minutes}}$$

Disclaimer:

We hope that our method and examples have shown that word problems can be effectively tackled in a coherent, step-by-step fashion. The main ingredients are complete and accurate *translation* of the problem into mathematics, and a systematic *analysis* of what is necessary to solve it. The method we have described is one way to organize these ingredients; however, it is not the only way, nor will it in all cases provide the most efficient means of solution. To begin with, no one method can be expected to cover so vast and varied a collection as "word problems"—that's almost equivalent to devising a method of solving *any type of mathematical problem*! Any such list of procedures is going to have inherent shortfalls; our list is probably a bit too rigid and detailed, and in some instances it might lead you to introduce more variables than are actually needed (and, in so doing, will make the problem solution more complicated than it need be). As you gain in experience and confidence, you will be better able to tailor the method to individual problems, and thus achieve more efficient solutions.

Nonetheless, the solution of any word problem requires the basic procedures of our method, and you are thus encouraged to follow our line pretty closely.

EXERCISES

1. Howard Anton takes $7\frac{1}{2}$ hours to make a trip overseas in a prop plane. Later he discovers that he could have taken a jet and saved $2\frac{1}{2}$ hours of flying time. Find the speed of the jet if the jet travels 225 mph faster than the prop plane.

2. On another trip, this one an auto trip of 126 miles, Howard Anton calculated that had he decreased his average speed by 8 m.p.h., his trip would have taken one hour longer. What was his original rate?

3. One solution is 20% sulfuric acid while another is 12% sulfuric acid. How much of each solution must be mixed together to produce 60 milliliters of solution containing 9 milliliters of sulfuric acid?

4. Flying east between two cities, a plane's speed is 380 mph. On the return trip, it flies at 420 mph. Find the average speed over the whole round trip. (No, the answer is not 400 mph.)

5. A rectangle is said to be in the "Divine Proportion" if the ratio of its width to its length is equal to the ratio of its length to the sum of its length and width. What are the dimensions of a rectangle in the Divine Proportion if its perimeter is 10 meters?

6. An offshore oil well is located in the ocean at a point W, which is 3 miles from the closest shorepoint A on a straight shoreline. The oil is to be piped to a shorepoint B that is 9 miles from A by piping it on a straight line underwater from W to some shorepoint P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is $500,000 per mile underwater, $300,000 per mile over land, and the total cost of the pipe installation is $4,000,000, then how far is point P from point A if this distance is known to be at least one mile? (For convenience, calculate with money in units of $100,000.)

7. In a long-distance race around a 400-meter track, the winner finished the race one lap ahead of the loser. If the average speed of the winner was 6 meters/sec and the average speed of the loser was 5.75 meters/sec, how soon after the start did the winner complete the race?

8. At 8 a.m., a bus traveling 100 kilometers/hr leaves Philadelphia for Boston, a distance of 500 kilometers. At 10 a.m., a bus traveling 80 kilometers/hr leaves Boston for Philadelphia. At what time do the two buses pass each other?

9. A confectioner has 15 pounds of chocolate worth $3.20 per pound and 12 pounds of caramels worth $2.20 per pound. How many pounds of nougats worth $2.40 per pound should be added to these candies to obtain a mixture that is to sell for $2.60 per pound?

10. A radiator contains 10 liters of a water and antifreeze solution of which 60% is anti-freeze. How much of this solution should be drained and replaced with water in order for the new solution to be 40% antifreeze?

11. One painter can paint a house in 40 hours and another painter can paint the same house in 35 hours. How long will it take to paint the house if they work together?

12. A pipe can fill a swimming pool in 10 hours. If a second pipe is opened, the two pipes together can fill the pool in 4 hours. How long would it take the second pipe alone to fill the pool?

ANSWERS

1. d = distance of trip

   s_1 = speed of prop plane
   s_2 = speed of jet
   t_1 = time of prop plane trip
   t_2 = time of jettrip
   $t_1 = 7\frac{1}{2}$
   $t_2 = 5$
   $s_1 = d/t_1$
   $s_2 = d/t_2$
   $s_2 = s_1 + 225$
   Thus $s_2 = 675$

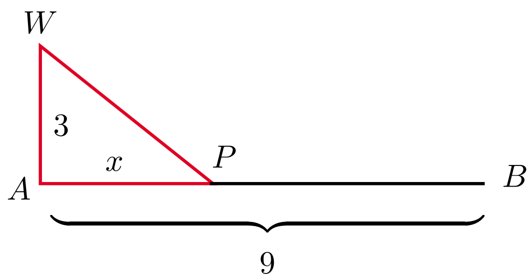
2. 36 mph

3. 22.5 and 37.5 milliliters

4. 339 mph

5. length $= 5\sqrt{5} - 5$, width $= 15 - 5\sqrt{5}$

6. x = distance from A to P = 4 miles



7. 26 minutes and 40 seconds

8. 11:40 a.m.

9. 21 pounds

10. $3\frac{1}{3}$ liters

11. $18\frac{2}{3}$ hours

12. $6\frac{2}{3}$ hours

0.9 MATHEMATICAL INDUCTION

Mathematical induction is the sophisticated name given to a simple logical principle that can be used to prove certain types of mathematical statements.

Suppose we have a series of statements, one about each positive integer. For example, the statement

"the sum of the first n positive integers is $\frac{n(n+1)}{2}$"

is, in fact, a series of statements, one for each positive integer n

"the sum of the first 6 positive integers is $\frac{6(6+1)}{2} = 21$,"

"the sum of the first 7 positive integers is $\frac{7(7+1)}{2} = 28$," etc.

In proving statements of this type, we have a fundamental problem: there are an infinite number of the statements and so it is impossible to check them all, one by one! However, there is a pattern to the statements and, by taking advantage of it, we can derive a simple, do-able procedure for proving them. To prove statements of this type by mathematical induction, we reason as follows

The Principle of Mathematical Induction

To verify a statement for every positive integer n such that $n \geq n_0$, where n_0 is some fixed starting integer (usually $n_0 = 1$):

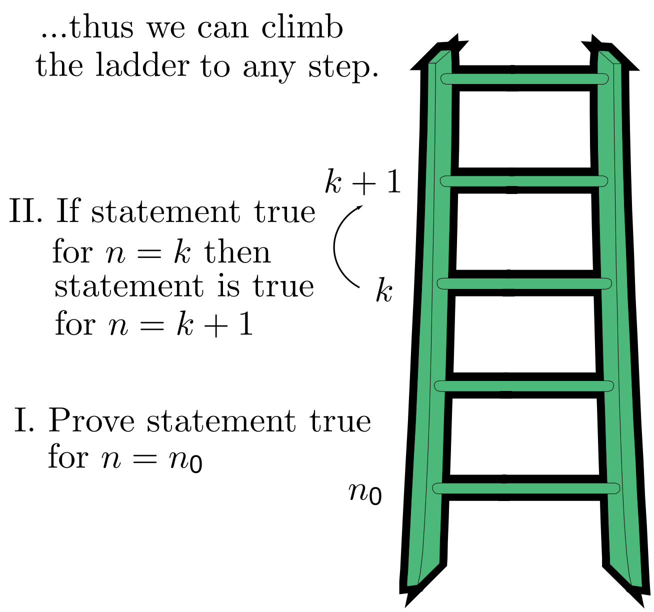
I. Prove that the statement is true for $n = n_0$

II. (a) (The Induction Hypothesis) Assume that the statement is true for $n = k$ (where k is an arbitrary value of n), and then

(b) Prove that the statement is true for $n = k + 1$. Then the statement is true for all values of $n \geq n_0$.

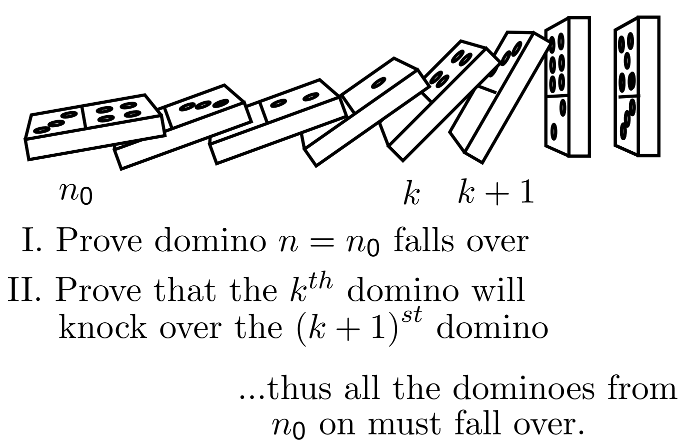
It is really *very* easy to see why this principle works (and, after all, a *principle* in mathematics is something that is not proved but is accepted as being obvious, so it had *better* be easy to see why it works). In Part 1, it has been verified that the statement is true for the first value of n. Say that first value is $n = 1$. Then, because it is true for $n = 1$, Part 2 says that it must be true for $n = 2$ (by using $k = 1$ and $k + 1 = 2$).

And then, because it is true for $n = 2$, Part 2 says that it must be true for $n = 3$ (by using $k = 2$ and $k + 1 = 3$). And then it must be true for $n = 4$; and so on. The inescapable conclusion is that the statement must be true for *all* the positive integers. The principle of mathematical induction has been described as the "ladder climbing" principle. We start at the bottom . . .



It says that if you can get on the ladder (at Step 1, usually) and if you can go from any step (Step k) to the *next* step (Step $k + 1$), then you can climb the ladder to any step.

The principle of mathematical induction can also be thought of in terms of falling dominoes. If you can knock over one domino (usually the first) and if the dominoes are arranged so that each domino (the k-th) knocks over the next one in line, the $(k + 1)$-th, then all the dominoes will fall.



The use of the principle of mathematical induction is illustrated in the proof of the following theorem.

THEOREM (Anton's Theorem 7.4.2)

a) The sum of the first n positive integers is $\dfrac{n(n+1)}{2}$.

b) The sum of the squares of the first n positive integers is

$$\frac{n(n+1)(2n+1)}{6}$$

PROOF:

a) Let us define $S_n = 1+2+3+\ldots+n$ where n is any positive integer. Then the statement we want to prove is

$$\text{“}S_n = \frac{n(n+1)}{2}\text{”}$$

I. First observe that for $n=1$, the statement is true since

$$S_1 = 1 \text{ and } \frac{1(1+1)}{2} = 1$$

II. a) Assume that the statement is true for $n=k$. That is, assume that

$$S_k = \frac{k(k+1)}{2}$$

(This is the induction hypothesis.)

b) We want to use the induction hypothesis to show that the statement is true for $n=k+1$, i.e., that

$$\begin{aligned} S_{k+1} &= \frac{(k+1)((k+1)+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

This is accomplished through the following sequence of equations:

$$\begin{aligned} S_{k+1} &= S_k + (k+1) \overset{*}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

*(the induction hypothesis)

Therefore S_{k+1} has the required form. This completes the induction argument and shows, by the principle of mathematical induction, that $S_n = \frac{n(n+1)}{2}$ for all positive integers n.

b) Define

$$S_n^{(2)} = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

where n is any positive integer. Then the statement we want to prove is

$$\text{“}S_n^{(2)} = \frac{n(n+1)(2n+1)}{6}.\text{”}$$

I. When

$$n = 1,\ S_1^{(2)} = 1^2 = 1$$

and

$$\frac{1(1+1)(2(1)+1)}{6} = 1$$

so the statement is true.

II. a) Assume that the statement is true for $n = k$, i.e., that

$$S_k^{(2)} = \frac{k(k+1)(2k+1)}{6}$$

(This is the induction hypothesis.)

b) Under this assumption, we want to show that the statement is true for $n = k+1$, i.e., that

$$\begin{aligned} S_{k+1}^{(2)} &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

This is done as follows:

$$\begin{aligned} S_{k+1}^{(2)} &= S_k^{(2)} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \end{aligned}$$

(The induction hypothesis.)

$$\begin{aligned}
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right] \\
&= (k+1)\left[\frac{2k^2+7k+6}{6}\right] \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Therefore $S_{k+1}^{(2)}$ has the required form, proving by the principle of mathematical induction that

$$S_n^{(2)} = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n. ■

NOTE: Very often, a proof by mathematical induction is not the only way to prove that statements are true for all positive integers. For example, in Section 7.4 Anton proves the two results above by different methods. However, in general you will find that a proof by mathematical induction is more straightforward and involves fewer tricks.

EXERCISES

1. Prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (n^2(n+1)^2)/4.$$

(This is Anton's Theorem 7.4.2(c).)

2. Prove that if $x \neq 1$, then

$$1 + x + x^2 + \ldots + x^n = \frac{(1-x^{n+1})}{(1-x)}.$$

3. If a set S contains n elements, show that S has 2^n subsets (counting S itself and the empty set ϕ as subsets).