Phasors

Help for Chapter 17 of *Fundamentals of Physics*
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Let’s go through the phasor techniques of Section 17-10 more slowly and with new examples, first with one wave and then two waves. You can then apply the techniques to a few problems (their answers are provided at the end).

One wave

Suppose that a wave given by the function

\[ y(x, t) = (2.00 \text{ mm}) \sin(300x - 700t) \]  

(1)

travels along a string. This function tells us that the wave travels in the positive direction of an \( x \) axis (which is along the string), with the string oscillating parallel to the \( y \) axis (which is perpendicular to the string). In the function, position \( x \) is in meters and time \( t \) is in seconds.

Let’s monitor the displacement of the wave at \( x = 0 \), because that allows us to plug 0 into the term 300\( x \) in Eq. 1. Then the displacement at our monitoring point on the string is given by

\[ y(0, t) = (2.00 \text{ mm}) \sin(-700t). \]  

(2)

At time \( t = 0 \), the displacement is

\[ y(0, 0) = (2.00 \text{ mm}) \sin[-700(0)] = 0. \]

At time \( t = 2.25 \text{ ms} \), the displacement is

\[
\begin{align*}
y(0, 2.25 \text{ ms}) &= (2.00 \text{ mm}) \sin[-700(2.25 \times 10^{-3})] \\
&= -2.00 \text{ mm}.
\end{align*}
\]

(To check this on your calculator, be sure to put the calculator in radian mode.) At time \( t = 4.50 \text{ ms} \), the displacement is

\[
\begin{align*}
y(0, 4.50 \text{ ms}) &= (2.00 \text{ mm}) \sin[-700(4.50 \times 10^{-3})] \\
&= 1.7 \times 10^{-2} \text{ mm} \approx 0.
\end{align*}
\]

In this way, we can construct a table of the displacements at our monitoring point at any desired time, such as

<table>
<thead>
<tr>
<th>Time (ms)</th>
<th>Displacement (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.25</td>
<td>-2.00</td>
</tr>
<tr>
<td>4.50</td>
<td>0</td>
</tr>
<tr>
<td>6.75</td>
<td>+2.00</td>
</tr>
<tr>
<td>9.00</td>
<td>0</td>
</tr>
</tbody>
</table>
One way to depict the oscillation of the string at a monitoring point is to use a phasor in a phasor diagram. In such a diagram, a phasor is a vector that rotates around an origin of two perpendicular axes, with its tail at the origin (Fig. 1).

![Fig. 1](image_url)

The axes are not $x$ and $y$ — let’s just call them the horizontal and vertical axes. The length of the vector represents the amplitude of the wave (2.00 mm in Eqs. 1 and 2). The rate at which the vector rotates around the origin is the angular frequency $\omega$ of the wave (700 rad/s in those equations). The direction of the rotation is clockwise.

As the phasor (the vector) rotates around the origin of the phasor diagram, its projection onto the vertical axis at any time gives the displacement of the wave just then at our monitoring point. (The “projection onto the vertical axis” means the component of the vector on that axis, as shown in Fig. 2.)

![Fig. 2](image_url)

We have already seen that at $x = 0$ the displacement of the wave of Eq. 1 is 0 at time $t = 0$, -2.00 mm at $t = 2.25$ ms, and again 0 at $t = 4.50$ ms. We can represent these results with the phasor drawings of Fig. 3.

![Fig. 3](image_url)

At time $t = 0$, the phasor is directed rightward and has no projection (no component) on the vertical axis (Fig. 3a). Thus this arrangement corresponds to a wave displacement of 0 just then. At $t = 2.25$ ms, the phasor is directed downward and its projection on the vertical axis is the full length of the phasor, 2.00 mm (Fig. 3b). This arrangement corresponds to a wave displacement of $-2.00$ mm just then. At $t = 4.50$ ms, the phasor is
As the wave passes our monitoring point, the displacement of the string is always given by the projection of the phasor on the vertical axis of the phasor diagram, as the phasor rotates. We can stop the rotation with a mental snapshot at any instant, to see what the displacement $y$ is just then. We can either draw the displacement or, given the angle between the phasor and either the horizontal or the vertical axis, calculate the displacement:

$$y = \text{length of phasor} \times \sin(\text{angle relative to horizontal axis})$$

or

$$y = \text{length of phasor} \times \cos(\text{angle relative to vertical axis}).$$

For example, at time $t = 8.22 \, \text{ms}$, the phasor is oriented at an angle of about $30^\circ$ to the horizontal axis of the phasor diagram. Thus we find that the displacement just then is about

$$y = (2.00 \, \text{mm}) \times \sin(30^\circ) = 1.00 \, \text{mm},$$

which means that the displacement of the string wave at our monitoring point is $1.0 \, \text{mm}$ just then.

The advantage of using a phasor to represent a wave is that it gives us another way to picture how the displacement of the string at our monitoring point varies with time. However, with only one wave, this advantage is small. We shall next examine a more powerful advantage of phasors: When we must combine two (or more) waves with different amplitudes, phasors can save us from a major mathematical headache.

**Two waves**

Suppose we now have two waves traveling along the string. One wave is given by Eq. 1 as before (we include a subscript in the equation to distinguish it):

$$y_1(x, t) = (2.00 \, \text{mm}) \times \sin(300x - 700t)$$

(3)

The other wave is given by

$$y_2(x, t) = (1.00 \, \text{mm}) \times \sin(300x - 700t + \pi/2 \, \text{rad}).$$

(4)

Both waves are traveling in the direction of positive $x$ and have the same wave number ($300 \, \text{m}^{-1}$) and angular frequency ($700 \, \text{rad/s}$). However, they have different amplitudes ($2.00 \, \text{mm}$ for wave 1 and $1.00 \, \text{mm}$ for wave 2). Also, they are not in phase because wave 2 is phase shifted by $+\pi/2 \, \text{rad}$ from wave 1.

If we could actually see both waves come through our monitoring point (still at $x = 0$), that phase shift would make the waves peak (have the greatest displacement) at different times: Wave 1 would peak first and then wave 2 would peak shortly later. However, we cannot see each wave. Rather we see the resultant wave (or net wave) due to the interference of the two waves.
We want to find an equation for that resultant wave and determine how the displacement of the string actually varies with time. Here is a key point:

Because the waves have different amplitudes, we cannot use the simple trig identities of Section 17-9 to end up with the relatively simple result of Eq. 17-38.

In short, trying to algebraically combine the two waves to get the resultant wave leads to a mess. However, phasors come to our rescue. For that rescue, we shall first draw a phasor for each of the two waves on the same phasor diagram. Then we shall add the phasors as vectors to find the resultant phasor, for the resultant wave. From it we can write an equation for the resultant wave and determine how the displacement of the string varies with time.

**Drawing the two phasors**
The phasors for the two waves rotate around the origin of the phasor diagram at the same rate because the two waves have the same angular frequency (ω = 700 rad/s). However, the phasors differ in length because the waves differ in amplitude: Phasor 1 (for wave 1) has a length of 2.00 mm and phasor 2 has a length of 1.00 mm.

The phasors also differ in orientation, because wave 2 is phase shifted by +π/2 rad relative to wave 1. That means that phasor 2 is at an angle of π/2 rad (or 90°) to phasor 1. However, are the phasors oriented as in Fig. 4a or as in Fig. 4b?

Phasor 2 is perpendicular to phasor 1 in both of those figures. To answer, first recall that wave 1 peaks before wave 2. When a wave peaks, its phasor is directed up the vertical axis in a phasor diagram. Thus, phasor 1 should be directed up the vertical axis before phasor 2 is, as the two phasors rotate around the origin. That condition means that Fig. 4a is correct.

**Adding the two phasors**
We can now vectorially add phasors 1 and 2 to get the resultant phasor, for the resultant wave. Although we can add them at any instant during their rotation, let’s add them at t = 0 so that phasor 1 has no component on the vertical axis (Fig. 5a).
Next, to carry out the addition, we can use either the techniques of Chapter 3 or a vector-capable calculator.

Chapter 3 techniques:
We shift phasor 2 so that its tail touches the head of phasor 1 and then draw the resultant phasor so that it extends from the origin to the head of phasor 2 (Fig. 5b). In the case here, the three phasors happen to form a right triangle. (Caution: such simplicity is not usually true.) The length of the hypotenuse,

\[
[(2.00 \text{ mm})^2 + (1.00 \text{ mm})^2]^{1/2} = 2.24 \text{ mm},
\]

is the amplitude of the resultant wave. The angle between the hypotenuse and the horizontal phasor 1,

\[
\phi = \tan^{-1} \left(\frac{1.00 \text{ mm}}{2.00 \text{ mm}}\right) = 0.464 \text{ rad},
\]

is the phase constant of the resultant wave relative to wave 1 (Fig. 6).

Vector-capable calculator:
The addition is entered into the calculator in the general form of

\[
[2 \angle 0] + [1 \angle \pi/2]
\]

where vectors are denoted by straight brackets and the calculator is in radian mode. (TI-89 and TI-92 calculators require a comma between the magnitude and the angle symbol. On some other calculators, you may have to substitute 1.571 for \(\pi/2\).) Depending on the mode settings, the calculator gives an answer in the general form of

\[
[2.24 \angle 0.464]
\]

which means that the length and angle of the resultant phasor is 2.24 mm and 0.464 rad. Thus the magnitude and phase constant of the resultant wave is 2.24 mm and 0.464 rad.

If we decide to add the phasors at any other time than \(t = 0\), they would still have the triangular arrangement of Fig. 5b, except that the triangle would be in some other orientation about the origin. The addition would still yield 2.24 mm and 0.464 rad. Because the resultant phasor is always directed from the origin to the head of the second phasor.
phasor in the arrangement, it must rotate about the origin at the same rate as phasors 1 and 2. Thus, the resultant wave must have the same angular frequency ($\omega = 700$ rad/s) as waves 1 and 2.

We can now write the resultant wave as

$$y(x, t) = (2.24 \text{ mm}) \sin(300x - 700t + 0.464 \text{ rad}).$$

As the resultant wave travels through our monitoring point, the displacement of the string varies sinusoidally with an amplitude of 2.24 mm. Because the phase constant of 0.464 rad for the resultant wave is intermediate between the phase constants of 0 and $\pi/2$ rad for waves 1 and 2, the resultant wave peaks after wave 1 and before wave 2. If you want a major headache, try working this problem without phasors.

**Your Turn**

(Answers are at the end.)

1. Wave 1 is again

$$y_1(x, t) = (2.00 \text{ mm}) \sin(300x - 700t)$$

but wave 2 is now

$$y_2(x, t) = (1.00 \text{ mm}) \sin(300x - 700t - \pi/2 \text{ rad}).$$

(Note that the phase constant of wave 2 is now a negative quantity.) (a) For time $t = 0$, draw the phasor diagram for the addition of the phasors, including the resultant phasor. What are (b) the amplitude and (c) the phase constant of the resultant wave?

2. Repeat Problem 1 except with wave 2 as

$$y_2(x, t) = (1.00 \text{ mm}) \sin(300x - 700t + 3\pi/4 \text{ rad}).$$

3. (Now for some real physics courage.) Three waves travel along the same string:

$$y_1(x, t) = (2.00 \text{ mm}) \sin(300x - 700t),$$

$$y_2(x, t) = (1.00 \text{ mm}) \sin(300x - 700t - \pi/2 \text{ rad}),$$

$$y_3(x, t) = (3.00 \text{ mm}) \sin(300x - 700t + 2\pi/3 \text{ rad}).$$

(a) For time $t = 0$, draw the phasor diagram for the addition of the phasors, including the resultant phasor. What are (b) the amplitude and (c) the phase constant of the resultant wave?
**Answers:**

1. (b) 2.24 mm; (c) −0.464 rad (or −26.6°)

2. (b) 1.47 mm; (c) +0.501 rad (or +28.7°)

3. (b) 1.67 mm; (c) +1.27 rad (or +72.7°)