It's the weekend after midterms, and a hiking trip to Pinnacles National Monument in California is on the agenda. Hiking in Pinnacles will require some advance planning. Some trails go through caves, so a flashlight is needed. Others go to the top of low mountain peaks. No matter where the hiking ends up, some food, such as a trail mix, will be required. Peanuts and raisins sound good. But in what proportions should they be mixed? And what about meeting some minimum calorie requirements? What about carbohydrates and protein? And, of course, fat should be minimized! Fortunately, this chapter was covered before midterms, so these questions can be answered. The Chapter Project at the end of the chapter will guide you.
In Section 1.1, we discussed linear equations (linear equalities) in two variables $x$ and $y$. Recall that these are equations of the form

$$Ax + By = C$$

(1)

where $A$, $B$, and $C$ are real numbers and $A$ and $B$ are not both zero. If in Equation (1) we replace the equal sign by an inequality symbol, namely, one of the symbols $<$, $>$, $\leq$, or $\geq$, we obtain a **linear inequality in two variables** $x$ and $y$. For example, the expressions

$$3x + 2y \geq 4 \quad 2x - 3y < 0 \quad 3x + 5y > -8$$

are each linear inequalities in two variables. The first of these is called a **nonstrict inequality** since the expression is satisfied when $3x + 2y = 4$, as well as when $3x + 2y > 4$. The remaining two linear inequalities are **strict**.

### The Graph of a Linear Inequality

The **graph of a linear inequality** in two variables $x$ and $y$ is the set of all points $(x, y)$ for which the inequality is satisfied.

Let’s look at an example.

**EXAMPLE 1**

**Graphing a Linear Inequality**

Graph the inequality: $2x + 3y \geq 6$
The inequality $2x + 3y \geq 6$ is equivalent to $2x + 3y > 6$ or $2x + 3y = 6$. So we begin by graphing the line $L$: $2x + 3y = 6$, noting that any point on the line must satisfy the inequality $2x + 3y \geq 6$. See Figure 1(a).

Now let’s test a few points, such as $(-1, -1)$, $(5, 5)$, $(4, 0)$, $(-4, 0)$, to see if they satisfy the inequality. We do this by substituting the coordinates of each point into the inequality and determining whether the result is $\geq 6$ or $< 6$.

$$2x + 3y$$

<table>
<thead>
<tr>
<th>Point</th>
<th>Calculation</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-1, -1)$</td>
<td>$2(-1) + 3(-1) = -2 - 3 = -5 &lt; 6$</td>
<td>Not part of graph</td>
</tr>
<tr>
<td>$(5, 5)$</td>
<td>$2(5) + 3(5) = 25 &gt; 6$</td>
<td>Part of graph</td>
</tr>
<tr>
<td>$(4, 0)$</td>
<td>$2(4) + 3(0) = 8 &gt; 6$</td>
<td>Part of graph</td>
</tr>
<tr>
<td>$(-4, 0)$</td>
<td>$2(-4) + 3(0) = -8 &lt; 6$</td>
<td>Not part of graph</td>
</tr>
</tbody>
</table>

Notice that the two points $(4, 0)$ and $(5, 5)$ that are part of the graph both lie on one side of $L$, while the points $(-4, 0)$ and $(-1, -1)$ (not part of the graph) lie on the other side of $L$. This is not an accident. The graph of the inequality consists of all points on the same side of $L$ as $(4, 0)$ and $(5, 5)$. The shaded region of Figure 1(b) illustrates the graph of the inequality.

The inequality in Example 1 was nonstrict, so the corresponding line was part of the graph of the inequality. If the inequality is strict, the corresponding line is not part of the graph of the inequality. We will indicate a strict inequality by using dashes to graph the line.

Let’s outline the procedure for graphing a linear inequality:

**Steps for Graphing a Linear Inequality**

**STEP 1** Graph the corresponding linear equation, a line $L$. If the inequality is non-strict, graph $L$ using a solid line; if the inequality is strict, graph $L$ using dashes.

**STEP 2** Select a test point $P$ not on the line $L$.

**STEP 3** Substitute the coordinates of the test point $P$ into the given inequality. If the coordinates of this point $P$ satisfy the linear inequality, then all points on the same side of $L$ as the point $P$ satisfy the inequality. If the coordinates of the point $P$ do not satisfy the linear inequality, then all points on the opposite side of $L$ from $P$ satisfy the inequality.
EXAMPLE 2  **Graphing a Linear Inequality**

Graph the linear inequality: \(2x - y < -4\)

**SOLUTION**  The corresponding linear equation is the line

\[ L: \quad 2x - y = -4 \]

Since the inequality is strict, points on \(L\) are not part of the graph of the linear inequality. When we graph \(L\), we use a dashed line to indicate this fact. See Figure 2(a).

We select a point not on the line \(L\) to be tested, for example, \((0, 0)\):

\[ 2x - y = 2(0) - 0 = 0 \quad \text{since } 2x - y < -4 \]

Since 0 is not less than \(-4\), the point \((0, 0)\) does not satisfy the inequality. As a result, all points on the opposite side of \(L\) from \((0, 0)\) satisfy the inequality. The graph of \(2x - y < -4\) is the shaded region of Figure 2(b).

NOW WORK PROBLEM 7.

EXAMPLE 3  **Graphing Linear Inequalities**

Graph:  

(a) \(x \leq 3\)  
(b) \(2x \leq y\)

**SOLUTION**  

(a) The corresponding linear equation is \(x = 3\), a vertical line. If we choose \((0, 0)\) as the test point, we find that it satisfies the inequality \([0 \leq 3]\), so all points to the left of, and on, the vertical line also satisfy the inequality. See Figure 3(a).

(b) The corresponding linear equation is \(2x = y\). Since its graph passes through \((0, 0)\), we choose the point \((0, 2)\) as the test point. The inequality is satisfied by the point \((0, 2)[2(0) \leq 2]\), so all points on the same side of the line as \((0, 2)\) also satisfy the inequality. See Figure 3(b).
The set of points belonging to the graph of a linear inequality [for example, the shaded region in Figure 3(b)] is called a **half-plane**.

**NOW WORK PROBLEM 5.**

**COMMENT:** A graphing utility can be used to obtain the graph of a linear inequality. See “Using a graphing utility to graph inequalities” in Appendix C, Section C.4, for a discussion.

**Systems of Linear Inequalities**

A **system of linear inequalities** is a collection of two or more linear inequalities. To graph a system of two inequalities we locate all points whose coordinates satisfy each of the linear inequalities of the system.

**EXAMPLE 4**

**Determining Whether a Point Belongs to the Graph of a System of Two Linear Inequalities**

Determine which of the following points are part of the graph of the system of linear inequalities:

\[
\begin{align*}
2x + y &\leq 6 \\
x - y &\geq 3
\end{align*}
\]

(a) \(P_1 = (6, 0)\)  
(b) \(P_2 = (3, 5)\)  
(c) \(P_3 = (0, 0)\)  
(d) \(P_4 = (3, -2)\)

**SOLUTION**

We check to see if the given point satisfies each of the inequalities of the system.

(a) \(P_1 = (6, 0)\)

\[
\begin{align*}
2x + y &= 2(6) + 0 = 12 \\
x - y &= 6 - 0 = 6
\end{align*}
\]

\(P_1\) satisfies inequality (2) but not inequality (1), so \(P_1\) is not part of the graph of the system.

(b) \(P_2 = (3, 5)\)

\[
\begin{align*}
2x + y &= 2(3) + 5 = 11 \\
x - y &= 3 - 5 = -2
\end{align*}
\]

\(P_2\) satisfies neither inequality (1) nor inequality (2), so \(P_2\) is not part of the graph of the system.

(c) \(P_3 = (0, 0)\)

\[
\begin{align*}
2x + y &= 2(0) + 0 = 0 \\
x - y &= 0 - 0 = 0
\end{align*}
\]

\(P_3\) satisfies inequality (1) but not inequality (2), so \(P_3\) is not part of the graph of the system.

(d) \(P_4 = (3, -2)\)

\[
\begin{align*}
2x + y &= 2(3) + (-2) = 4 \\
x - y &= 3 - (-2) = 5
\end{align*}
\]

\(P_4\) satisfies both inequality (1) and inequality (2), so \(P_4\) is part of the graph of the system.

**NOW WORK PROBLEM 13.**
Let’s graph the information from Example 4. Figure 4(a) shows the graph of each of the lines \(2x + y = 6\) and \(x - y = 3\) and the four points \(P_1, P_2, P_3,\) and \(P_4\). Notice that because the two lines of the system intersect, the plane is divided into four regions. Since the graph of each linear inequality of the system is a half-plane, the graph of the system of linear inequalities is the intersection of these two half-planes. As a result, the region containing \(P_4\) is the graph of the system. See Figure 4(b).

We use the method described above in the next example.

**EXAMPLE 5**  
**Graphing a System of Two Linear Inequalities**

Graph the system:  
\[
\begin{align*}
2x - y &\leq -4 \\
2x + y &\geq 1
\end{align*}
\]

**SOLUTION**  
First we graph each inequality separately. See Figures 5(a) and 5(b).

The solution of the system consists of all points common to these two half-planes. The dark blue shaded region in Figure 6 represents the solution of the system.

**NOW WORK PROBLEM 17.**

The lines in the system of linear inequalities given in Example 5 intersect. If the two lines of a system of two linear inequalities are parallel, the system of linear inequalities may or may not have a solution. Examples of such situations follow.
EXAMPLE 6  Graphing a System of Two Linear Inequalities

Graph the system:
\[
\begin{align*}
2x - y &\leq -4 \\
2x - y &\leq -2
\end{align*}
\]

SOLUTION  First we graph each inequality separately. See Figures 7(a) and 7(b). The grey shaded region in Figure 8 represents the solution of the system.

Notice that the solution of this system is the same as that of the single linear inequality \(2x - y \leq -4\).

EXAMPLE 7  Graphing a System of Two Linear Inequalities

The solution of the system
\[
\begin{align*}
2x - y &\geq -4 \\
2x - y &\leq -2
\end{align*}
\]
is the grey shaded region in Figure 9.

EXAMPLE 8  Graphing a System of Two Linear Inequalities

The system
\[
\begin{align*}
2x - y &\leq -4 \\
2x - y &\geq -2
\end{align*}
\]
has no solution, as Figure 10 indicates, because the two half-planes have no points in common.

Until now, we have considered systems of only two linear inequalities. The next example is of a system of four linear inequalities. As we shall see, the technique for graphing such systems is the same as that used for graphing systems of two linear inequalities in two variables.

**EXAMPLE 9 Graphing a System of Four Linear Inequalities**

Graph the system:

\[
\begin{align*}
&x + y \geq 2 \\
&2x + y \geq 3 \\
&x \geq 0 \\
&y \geq 0
\end{align*}
\]

**SOLUTION** Again we first graph the four lines:

\[L_1: \quad x + y = 2\]
\[L_2: \quad 2x + y = 3\]
\[L_3: \quad x = 0 \quad \text{(the y-axis)}\]
\[L_4: \quad y = 0 \quad \text{(the x-axis)}\]

The lines \(L_1\) and \(L_2\) intersect at the point \((1, 1)\). (Do you see why?) The inequalities \(x \geq 0\) and \(y \geq 0\) indicate that the graph of the system lies in quadrant I. The graph of the system consists of that part of the graphs of the inequalities \(x + y \geq 2\) and \(2x + y \geq 3\) that lies in quadrant I. See Figure 11.
Some Terminology

Compare the graphs of the systems of linear inequalities given in Figures 11 and 12. The graph in Figure 11 is said to be **unbounded** in the sense that it extends infinitely far in some direction. The graph in Figure 12 is **bounded** in the sense that it can be enclosed by some circle of sufficiently large radius. See Figure 13.

**EXAMPLE 10** Graphing a System of Four Linear Inequalities

Graph the system:

\[
\begin{align*}
    x + y & \leq 2 \\
    2x + y & \leq 3 \\
    x & \geq 0 \\
    y & \geq 0
\end{align*}
\]

**SOLUTION** The lines associated with these linear inequalities are the same as those of the previous example. Again the inequalities \(x \geq 0, y \geq 0\) indicate that the graph of the system lies in quadrant I. The graph of the system consists of that part of the graphs of the inequalities \(x + y \leq 2\) and \(2x + y \leq 3\) that lie in quadrant I. See Figure 12.

**FIGURE 12**

The boundary of each of the graphs in Figures 11 and 12 consists of line segments. In fact, the graph of any system of linear inequalities will have line segments as boundaries. The point of intersection of two line segments that form the boundary is called a **corner point** of the graph. For example, the graph of the system given in Example 9 has the corner points \((0, 3), (1, 1), (2, 0)\). See Figure 11. The graph of the system given in Example 10 has the corner points \((0, 2), (0, 0), (1, 0), (1, 1)\). See Figure 12.
We shall soon see that the corner points of the graph of a system of linear inequalities play a major role in the procedure for solving linear programming problems.

NOW WORK PROBLEMS 25 AND 29.

Application

EXAMPLE 11 Analyzing a Mixture Problem

Nutt’s Nuts has 75 pounds of cashews and 120 pounds of peanuts. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts.

(a) Use $x$ to denote the number of packages of the low-grade mixture and use $y$ to denote the number of packages of the high-grade mixture to be made and write a system of linear inequalities that describes the possible number of each kind of package.

(b) Graph the system and list its corner points.

SOLUTION  

(a) We begin by naming the variables:

\[
\begin{align*}
x &= \text{Number of packages of low-grade mixture} \\
y &= \text{Number of packages of high-grade mixture}
\end{align*}
\]

First, we note that the only meaningful values for $x$ and $y$ are nonnegative values. We restrict $x$ and $y$ so that

\[
x \geq 0 \quad \text{and} \quad y \geq 0
\]

Next, we note that there is a limit to the number of pounds of cashews and peanuts available. That is, the total number of pounds of cashews cannot exceed 75 pounds (1200 ounces), and the number of pounds of peanuts cannot exceed 120 pounds (1920 ounces). This means that

\[
\begin{align*}
\text{Ounces of cashews required for low-grade mixture} \\
\text{Number of packages of low-grade mixture} + \\
\text{Ounces of cashews required for high-grade mixture} \\
\text{Number of packages of high-grade mixture}
\end{align*}
\]

\[
\left( \text{Ounces of peanuts required for low-grade mixture} \right) + \left( \text{Ounces of peanuts required for high-grade mixture} \right) \right) \text{ cannot exceed 1200}
\]

\[
\left( \text{Number of packages of low-grade mixture} \right) \right) \text{ cannot exceed 1920}
\]

In terms of the data given and the variables introduced, we can write these statements compactly as

\[
\begin{align*}
4x + 8y &\leq 1200 \\
12x + 8y &\leq 1920
\end{align*}
\]
The system of linear inequalities that gives the possible values $x$ and $y$ can take on is

\[
\begin{align*}
4x + 8y &\leq 1200 \\
12x + 8y &\leq 1920 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

(b) The system of linear inequalities given above can be simplified to the equivalent form

\[
\begin{align*}
x + 2y &\leq 300 \quad (1) \\
3x + 2y &\leq 480 \quad (2) \\
x &\geq 0 \quad (3) \\
y &\geq 0 \quad (4)
\end{align*}
\]

The graph of the system is given in Figure 14. Notice that the corner points of the graph are labeled. Three are easy to identify by inspection: $(0, 0)$, $(0, 150)$, and $(160, 0)$. We found the remaining one $(90, 105)$ by solving the system of equations

\[
\begin{align*}
x + 2y &= 300 \\
3x + 2y &= 480
\end{align*}
\]

By subtracting the first equation from the second, we find $2x = 180$ or $x = 90$. Back-substituting in the first equation, we find $y = 105$.

---

**EXERCISE 3.1** Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–12, graph each inequality.

1. $x \geq 0$

5. $y \geq 1$

9. $5x + y \geq 10$

13. Without graphing, determine which of the points $P_1 = (3, 8), P_2 = (12, 9), P_3 = (5, 1)$ are part of the graph of the following system:

\[
\begin{align*}
x + 3y &\geq 0 \\
-3x + 2y &\geq 0
\end{align*}
\]

2. $y \geq 0$

6. $x > 2$

10. $x + 2y > 4$

7. $2x + 3y \leq 6$

11. $x + 5y < 6$

14. Without graphing, determine which of the points $P_1 = (9, -5), P_2 = (12, -4), P_3 = (4, 1)$ are part of the graph of the following system:

\[
\begin{align*}
x + 4y &\leq 0 \\
5x + 2y &\geq 0
\end{align*}
\]
15. Without graphing, determine which of the points $P_1 = (2, 3), P_2 = (10, 10), P_3 = (5, 1)$ are part of the graph of the following system:

\[
\begin{align*}
3x + 2y &\geq 0 \\
x + y &\leq 15
\end{align*}
\]

16. Without graphing, determine which of the points $P_1 = (2, 6), P_2 = (12, 4), P_3 = (4, 2)$ are part of the graph of the following system:

\[
\begin{align*}
2x - 5y &\leq 0 \\
x + 3y &\leq 15
\end{align*}
\]

In Problems 17–24, determine which region $a$, $b$, $c$, or $d$ represents the graph of the given system of linear inequalities. The regions $a$, $b$, $c$, and $d$ are nonoverlapping regions bounded by the indicated lines.

17. \[
\begin{align*}
5x - 4y &\leq 8 \\
2x + 5y &\leq 23
\end{align*}
\]

18. \[
\begin{align*}
4x - 5y &\leq 0 \\
4x + 2y &\leq 28
\end{align*}
\]

19. \[
\begin{align*}
2x - 5y &\geq 0 \\
x + 3y &\leq 15
\end{align*}
\]

20. \[
\begin{align*}
6x - 5y &\leq 5 \\
2x + 4y &\geq 30
\end{align*}
\]

21. \[
\begin{align*}
5x - 3y &\geq 3 \\
2x + 6y &\geq 30
\end{align*}
\]

22. \[
\begin{align*}
5x - 5y &\geq 10 \\
6x + 4y &\geq 48
\end{align*}
\]

23. \[
\begin{align*}
5x - 4y &\leq 0 \\
2x + 4y &\leq 28
\end{align*}
\]

24. \[
\begin{align*}
2x - 5y &\leq -5 \\
3x + 5y &\leq 30
\end{align*}
\]
In Problems 25–36, graph each system of linear inequalities. Tell whether the graph is bounded or unbounded and list each corner point of the graph.

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>System of Linear Inequalities</th>
</tr>
</thead>
</table>
| 25             | \[
\begin{align*}
  x + y &\geq 2 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 26             | \[
\begin{align*}
  2x + 3y &\leq 6 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 27             | \[
\begin{align*}
  x + y &\geq 2 \\
  2x + 3y &\leq 6 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 28             | \[
\begin{align*}
  x + y &\geq 2 \\
  2x + 3y &\leq 6 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 29             | \[
\begin{align*}
  x + y &\geq 2 \\
  x + y &\leq 8 \\
  2x + y &\leq 10 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 30             | \[
\begin{align*}
  x + y &\geq 2 \\
  x + y &\leq 8 \\
  x + 2y &\geq 1 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 31             | \[
\begin{align*}
  x + y &\geq 2 \\
  2x + 3y &\leq 12 \\
  3x + y &\leq 12 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 32             | \[
\begin{align*}
  x + y &\geq 2 \\
  2x + y &\geq 3 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 33             | \[
\begin{align*}
  x + 2y &\geq 1 \\
  y &\leq 4 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 34             | \[
\begin{align*}
  x + 2y &\geq 1 \\
  x + 2y &\leq 10 \\
  x + y &\geq 2 \\
  x + y &\leq 8 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 35             | \[
\begin{align*}
  x + 2y &\geq 2 \\
  x + y &\leq 4 \\
  3x + y &\leq 3 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |
| 36             | \[
\begin{align*}
  x + y &\geq 2 \\
  2x + y &\geq 2 \\
  3x + 2y &\leq 6 \\
  x + y &\geq 2 \\
  x &\geq 0 \\
  y &\geq 0 \\
\end{align*}
\] |

37. Rework Example 11 if 60 pounds of cashews and 90 pounds of peanuts are available.

38. Rework Example 11 if the high-grade mixture contains 10 ounces of cashews and 6 ounces of peanuts.

39. **Manufacturing** Mike’s Famous Toy Trucks company manufactures two kinds of toy trucks—a dumpster and a tanker. In the manufacturing process, each dumpster requires 3 hours of grinding and 4 hours of finishing, while each tanker requires 2 hours of grinding and 3 hours of finishing. The company has two grinders and three finishers, each of whom works at most 40 hours per week.

(a) Using \(x\) to denote the number of dumpsters and \(y\) to denote the number of tankers, write a system of linear inequalities that describes the possible numbers of each truck that can be manufactured.

(b) Graph the system and list its corner points.

40. **Manufacturing** Repeat Problem 39 if one grinder and two finishers, each of whom works at most 40 hours per week, are available.

41. **Financial Planning** A retired couple have up to $25,000 to invest. As their financial adviser, you recommend they place at least $15,000 in Treasury bills yielding 6% and at most $10,000 in corporate bonds yielding 9%.

(a) Use the information supplied in Problem 41, along with the fact that the couple will invest at least $20,000, to answer parts (a), (b), and (c).

(b) Graph the system and list its corner points.

(c) Interpret the meaning of each corner point in relation to the investments it represents.

42. **Financial Planning** Use the information supplied in Problem 41, along with the fact that the couple will invest at least $20,000, to answer parts (a), (b), and (c).

43. **Nutrition** A farmer prepares feed for livestock by combining two types of grain. Each unit of the first grain contains 1 unit of protein and 5 units of iron while each unit of the second grain contains 2 units of protein and 1 unit of iron. Each animal must receive at least 5 units of protein and 16 units of iron each day.

(a) Write a system of linear inequalities that describes the possible amounts of each grain the farmer needs to prepare.

(b) Graph the system and list the corner points.

44. **Investment Strategy** Kathleen wishes to invest up to a total of $40,000 in class AA bonds and stocks. Furthermore, she believes that the amount invested in class AA bonds should be at most one-third of the amount invested in stocks.
(a) Write a system of linear inequalities that describes the possible amount of investments in each security.
(b) Graph the system and list the corner points.

45. Nutrition To maintain an adequate daily diet, nutritionists recommend the following: at least 85 g of carbohydrate, 70 g of fat, and 50 g of protein. An ounce of food A contains 5 g of carbohydrate, 3 g of fat, and 2 g of protein, while an ounce of food B contains 4 g of carbohydrate, 3 g of fat, and 3 g of protein.

(a) Write a system of linear inequalities that describes the possible quantities of each food.
(b) Graph the system and list the corner points.

46. Transportation A microwave company has two plants, one on the East Coast and one in the Midwest. It takes 25 hours (packing, transportation, and so on) to transport an order of microwaves from the eastern plant to its central warehouse and 20 hours from the Midwest plant to its central warehouse. It costs $80 to transport an order from the eastern plant to the central warehouse and $40 from the midwestern plant to its central warehouse. There are 1000 work-hours available for packing, transportation, and so on, and $3000 for transportation cost.

(a) Write a system of linear inequalities that describes the transportation system.
(b) Graph the system and list the corner points.

47. Make up a system of linear inequalities that has no solution.

48. Make up a system of linear inequalities that has a single point as solution.

### 3.2 A Geometric Approach to Linear Programming Problems

**OBJECTIVES**

1. Identify a linear programming problem
2. Solve a linear programming problem

**Identify a linear programming problem**

To help see the characteristics of a linear programming problem, we look again at Example 11 of the previous section.

Nutt’s Nuts has 75 pounds of cashews and 120 pounds of peanuts. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts.

Suppose that in addition to the information given above, we also know what the profit will be on each type of mixture. For example, suppose the profit is $0.25 on each package of the low-grade mixture and is $0.45 on each package of the high-grade mixture. The question of importance to the manager is “How many packages of each type of mixture should be prepared to maximize the profit?”

If \( P \) symbolizes the profit, \( x \) the number of packages of low-grade mixture, and \( y \) the number of high-grade packages, then the question can be restated as “What are the values of \( x \) and \( y \) so that the expression

\[
P = 0.25x + 0.45y
\]

is a maximum?”

This problem is typical of a linear programming problem. It requires that a certain linear expression, in this case the profit, be maximized. This linear expression is called the **objective function**. Furthermore, the problem requires that the maximum profit be achieved under certain restrictions or **constraints**, each of which are linear inequalities involving the variables. The linear programming problem may be restated as

Maximize

\[
P = 0.25x + 0.45y \quad \text{Objective function}
\]
subject to the conditions that
\[
\begin{align*}
    x + 2y &\leq 300 & \text{Cashew constraint} \\
    3x + 2y &\leq 480 & \text{Peanut constraint} \\
    x &\geq 0 & \text{Nonnegativity constraint} \\
    y &\geq 0 & \text{Nonnegativity constraint}
\end{align*}
\]

In general, every linear programming problem has two components:

1. A linear objective function to be maximized or minimized.
2. A collection of linear inequalities that must be satisfied simultaneously.

**Linear Programming Problem**

A linear programming problem in two variables, \( x \) and \( y \), consists of maximizing or minimizing an objective function

\[
z = Ax + By
\]

where \( A \) and \( B \) are given real numbers, not both zero, subject to certain conditions or constraints expressible as a system of linear inequalities in \( x \) and \( y \).

Let's look at this definition more closely. To maximize (or minimize) the quantity \( z = Ax + By \) means to locate the points \((x, y)\) that result in the largest (or smallest) value of \( z \). But not all points \((x, y)\) are eligible. Only the points that obey all the constraints are potential solutions. We refer to such points as feasible points.

In a linear programming problem we want to find the feasible point that maximizes (or minimizes) the objective function.

By a solution to a linear programming problem we mean a feasible point \((x, y)\), together with the value of the objective function at that point, which maximizes (or minimizes) the objective function. If none of the feasible points maximizes (or minimizes) the objective function, or if there are no feasible points, then the linear programming problem has no solution.

**EXAMPLE 1 Solving a Linear Programming Problem**

Minimize the quantity

\[
z = x + 2y
\]

subject to the constraints

\[
\begin{align*}
    x + y &\geq 1 \\
    x &\geq 0 \\
    y &\geq 0
\end{align*}
\]
The objective function to be minimized is $z = x + 2y$. The constraints are the linear inequalities

$$\begin{align*}
x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}$$

The shaded portion of Figure 15 illustrates the set of feasible points.

To see if there is a smallest $z$, we graph $z = x + 2y$ for some choice of $z$, say, $z = 3$. See Figure 16. By moving the line $x + 2y = 3$ parallel to itself, we can observe what happens for different values of $z$. Since we want a minimum value for $z$, we try to move $z = x + 2y$ down as far as possible while keeping some part of the line within the set of feasible points. The “best” solution is obtained when the line just touches a corner point of the set of feasible points. If you refer to Figure 16, you will see that the best solution is $x = 1, y = 0$, which yields $z = 1$. There is no other feasible point for which $z$ is smaller.

The next example illustrates a linear programming problem that has no solution.

**EXAMPLE 2 A Linear Programming Problem without a Solution**

Maximize the quantity $z = x + 2y$

subject to the constraints

$$\begin{align*}
x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}$$

**SOLUTION**

First, we graph the constraints. The shaded portion of Figure 17 illustrates the set of feasible points.

The graphs of the objective function $z = x + 2y$ for $z = 2, z = 8$, and $z = 12$ are also shown in Figure 17. Observe that we continue to get larger values for $z$ by moving the graph of the objective function upward. But there is no feasible point that will make $z$ largest. No matter how large a value is assigned to $z$, there is a feasible point that
will give a larger value. Since there is no feasible point that makes \( z \) largest, we conclude that this linear programming problem has no solution.

Examples 1 and 2 demonstrate that sometimes a linear programming problem has a solution and sometimes it does not. The next result gives conditions on the set of feasible points that determine when a solution to a linear programming problem exists.

**Existence of a Solution**

Consider a linear programming problem with the set \( R \) of feasible points and objective function \( z = Ax + By \).

1. If \( R \) is bounded, then \( z \) has both a maximum and a minimum value on \( R \).
2. If \( R \) is unbounded and \( A \geq 0, B \geq 0 \), and the constraints include \( x \geq 0 \) and \( y \geq 0 \), then \( z \) has a minimum value on \( R \) but not a maximum value (see Example 2).
3. If \( R \) is the empty set, then the linear programming problem has no solution and \( z \) has neither a maximum nor a minimum value.

In Example 1 we found that the feasible point that minimizes \( z \) occurs at a corner point. This is not an unusual situation. If there are feasible points minimizing (or maximizing) the objective function, at least one will be at a corner point of the set of feasible points.

**Fundamental Theorem of Linear Programming**

If a linear programming problem has a solution, it is located at a corner point of the set of feasible points; if a linear programming problem has multiple solutions, at least one of them is located at a corner point of the set of feasible points. In either case the corresponding value of the objective function is unique.

The result just stated indicates that it is possible for a feasible point that is not a corner point to minimize (or maximize) the objective function. For example, if the slope of the objective function is the same as the slope of one of the boundaries of the set of feasible points and if the two adjacent corner points are solutions, then so are all the points on the line segment joining them. The following example illustrates this situation.

**EXAMPLE 3**  
**A Linear Programming Problem with Multiple Solutions**

Minimize the quantity

\[ z = x + 2y \]

subject to the constraints
Again we first graph the constraints. The shaded portion of Figure 18 illustrates the set of feasible points.

If we graph the objective equation \( z = x + 2y \) for some choice of \( z \) and move it down, we find that a minimum is reached when \( z = \frac{3}{2} \). In fact, any point on the line \( 2x + 4y = 3 \) between the adjacent corner points \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( \left( \frac{3}{2}, 0 \right) \) and including these corner points will minimize the objective function. Of course, the reason any feasible point on \( 2x + 4y = 3 \) minimizes the objective equation \( z = x + 2y \) is that these two lines each have slope \(-\frac{1}{2}\). This linear programming problem has infinitely many solutions.

NOW WORK PROBLEM 1.

Since the objective function attains its maximum or minimum value at the corner points of the set of feasible points, we can outline a procedure for solving a linear programming problem provided that it has a solution.

**Steps for Solving a Linear Programming Problem**

If a linear programming problem has a solution, follow these steps to find it:

**STEP 1** Write an expression for the quantity that is to be maximized or minimized (the objective function).

**STEP 2** Determine all the constraints and graph the set of feasible points.

**STEP 3** List the corner points of the set of feasible points.

**STEP 4** Determine the value of the objective function at each corner point.

**STEP 5** Select the maximum or minimum value of the objective function.

Let’s look at some examples.
EXAMPLE 4  Solving a Linear Programming Problem

Maximize and minimize the objective function

\[ z = x + 5y \]

subject to the constraints

\[
\begin{align*}
  x + 4y &\leq 12 \quad (1) \\
  x &\leq 8 \quad (2) \\
  x + y &\geq 2 \quad (3) \\
  x &\geq 0 \quad (4) \\
  y &\geq 0 \quad (5)
\end{align*}
\]

SOLUTION  The objective function is \( z = x + 5y \) and the constraints consist of a system of five linear inequalities. We proceed to graph the system of five linear inequalities. The shaded portion of Figure 19 illustrates the graph, the set of feasible points. Since this set is bounded, we know a solution to the linear programming problem exists. Notice in Figure 19 that we have labeled each line from the system of linear inequalities. We have also labeled the corner points.

**FIGURE 19**

The corner points of the set of feasible points are 

\((0, 3), (8, 1), (8, 0), (2, 0), (0, 2)\)

To find the maximum and minimum value of the objective function \( z = x + 5y \), we construct Table 1:

<table>
<thead>
<tr>
<th>Corner Point ((x, y))</th>
<th>Value of Objective Function ( z = x + 5y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3)</td>
<td>( z = 0 + 5(3) = 15 )</td>
</tr>
<tr>
<td>(8, 1)</td>
<td>( z = 8 + 5(1) = 13 )</td>
</tr>
<tr>
<td>(8, 0)</td>
<td>( z = 8 + 5(0) = 8 )</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>( z = 2 + 5(0) = 2 )</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>( z = 0 + 5(2) = 10 )</td>
</tr>
</tbody>
</table>

The maximum value of \( z \) is 15, and it occurs at the point \((0, 3)\). The minimum value of \( z \) is 2, and it occurs at the point \((2, 0)\).

NOW WORK PROBLEMS 17 AND 29.

Now let’s solve the problem of the cashews and peanuts that we discussed at the start of this section.
EXAMPLE 5  **Maximizing Profit**

Maximize

\[ P = 0.25x + 0.45y \]

subject to the constraints

\[
\begin{align*}
  x + 2y & \leq 300 \quad (1) \\
  3x + 2y & \leq 480 \quad (2) \\
  x & \geq 0 \quad (3) \\
  y & \geq 0 \quad (4)
\end{align*}
\]

**SOLUTION**  Before applying the method of this chapter to solve this problem, let’s discuss a solution that might be suggested by intuition. Namely, since the profit is higher for the high-grade mixture, you might think that Nutt’s Nuts should prepare as many packages of the high-grade mixture as possible. If this were done, then there would be a total of 150 packages (8 ounces divides into 75 pounds of cashews exactly 150 times) and the total profit would be

\[ 150(0.45) = 67.50 \]

As we shall see, this is not the best solution to the problem.

To obtain the maximum profit, we use linear programming. We reproduce here in Figure 20 the graph of the set of feasible points we obtained earlier (see Figure 14, page 168)

**FIGURE 20**

Notice that this set is bounded. The corner points of the set of feasible points are

\( (0, 0) \quad (0, 150) \quad (160, 0) \quad (90, 105) \)

It remains only to evaluate the objective function at each corner point: see Table 2.

**TABLE 2**

<table>
<thead>
<tr>
<th>Corner Point ((x, y))</th>
<th>Value of Objective Function (P = (0.25)x + (0.45)y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>(P = (0.25)(0) + (0.45)(0) = 0)</td>
</tr>
<tr>
<td>((0, 150))</td>
<td>(P = (0.25)(0) + (0.45)(150) = 67.50)</td>
</tr>
<tr>
<td>((160, 0))</td>
<td>(P = (0.25)(160) + (0.45)(0) = 40.00)</td>
</tr>
<tr>
<td>((90, 105))</td>
<td>(P = (0.25)(90) + (0.45)(105) = 69.75)</td>
</tr>
</tbody>
</table>
A maximum profit is obtained if 90 packages of low-grade mixture and 105 packages of high-grade mixture are made. The maximum profit obtainable under the conditions described is $69.75.

**NOW WORK PROBLEM 49.**

**EXERCISE 3.2** Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–10, the given figure illustrates the graph of the set of feasible points of a linear programming problem. Find the maximum and minimum values of each objective function.

1. $z = 2x + 3y$
2. $z = 3x + 2y$
3. $z = x + y$
4. $z = 3x + 3y$
5. $z = x + 6y$
6. $z = 6x + y$
7. $z = 3x + 4y$
8. $z = 4x + 3y$
9. $z = 10x + y$
10. $z = x + 10y$

In Problems 11–16, list the corner points for each collection of constraints of a linear programming problem.

11. $\begin{cases} x \leq 13 \\ 4x + 3y \geq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$
12. $\begin{cases} x \leq 8 \\ 2x + 3y \geq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$
13. $\begin{cases} y \leq 10 \\ x + y \leq 15 \\ x \geq 0 \\ y \geq 0 \end{cases}$
14. $\begin{cases} y \leq 8 \\ 2x + y \geq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$
15. $\begin{cases} x \leq 10 \\ y \leq 8 \\ 4x + 3y \geq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$
16. $\begin{cases} x \leq 9 \\ y \leq 12 \\ 2x + 3y \geq 24 \\ x \geq 0 \\ y \geq 0 \end{cases}$

In Problems 17–24, maximize (if possible) the quantity $z = 5x + 7y$ subject to the given constraints.

17. $\begin{cases} x + y \leq 2 \\ y \geq 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$
18. $\begin{cases} 2x + 3y \leq 6 \\ x \geq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}$
19. $\begin{cases} x + y \geq 2 \\ 2x + 3y \geq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$
20. $\begin{cases} x + y \geq 2 \\ 2x + 3y \leq 12 \\ 3x + 2y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$
21. $\begin{cases} x + y \geq 2 \\ x + y \leq 8 \\ 2x + y \leq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$
22. $\begin{cases} x + y \geq 2 \\ x + y \leq 8 \\ x + 2y \geq 1 \\ x + 2y \leq 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$
23. $\begin{cases} x + y \leq 10 \\ x \geq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$
24. $\begin{cases} x + y \leq 8 \\ x \geq 2 \\ x \geq 0 \\ y \geq 0 \\ y \geq 0 \end{cases}$
In Problems 25–32, minimize (if possible) the quantity \( z = 2x + 3y \) subject to the given constraints.

25. \[
\begin{align*}
    x + y &\leq 2 \\
    y &\leq x \\
    x &\geq 0 \\
    y &\geq 0
\end{align*}
\]

26. \[
\begin{align*}
    3x + y &\leq 3 \\
    y &\geq x \\
    x &\geq 0 \\
    y &\geq 0
\end{align*}
\]

In Problems 33–40, find the maximum and minimum values (if possible) of the given objective function subject to the constraints

\[
\begin{align*}
    x + y &\leq 10 \\
    2x + y &\geq 10 \\
    x + 2y &\geq 10 \\
    x &\geq 0 \\
    y &\geq 0
\end{align*}
\]

33. \( z = x + y \)

34. \( z = 2x + 3y \)

37. \( z = 3x + 4y \)

38. \( z = 3x + 6y \)

41. Find the maximum and minimum values of \( z = 18x + 30y \) subject to the constraints \( 3x + 3y \geq 9, -x + 4y \leq 12, \) and \( 4x - y \leq 12, \) where \( x \geq 0 \) and \( y \geq 0. \)

42. Find the maximum and minimum values of \( z = 20x + 16y \) subject to the constraints \( 4x + 3y \geq 12, -2x + 4y \geq 16, \) and \( 6x - y \leq 18, \) where \( x \geq 0 \) and \( y \geq 0. \)

43. Find the maximum and minimum values of \( z = 7x + 6y \) subject to the constraints \( 2x + 3y \geq 6, -3x + 4y \leq 8, \) and \( 5x - y \leq 15, \) where \( x \geq 0 \) and \( y \geq 0. \)

44. Find the maximum and minimum values of \( z = 6x + 3y \) subject to the constraints \( 2x + 2y \geq 4, -x + 5y \leq 10, \) and \( 3x - 3y \leq 6, \) where \( x \geq 0 \) and \( y \geq 0. \)

45. Maximize \( z = -20x + 30y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 5x + 3y \geq 15, \) and \( -3x + 3y \leq 21, \) where \( x \geq 0 \) and \( y \geq 0. \)

46. Maximize \( z = -10x + 10y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 6x + y \geq 6, \) and \( -3x + y \leq 7, \) where \( x \geq 0 \) and \( y \geq 0. \)

47. Maximize \( z = -12x + 24y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 3x + 3y \geq 9, \) and \( -3x + 2y \leq 14, \) where \( x \geq 0 \) and \( y \geq 0. \)

48. Maximize \( z = -20x + 10y \) subject to the constraints \( 0 \leq x \leq 15, 0 \leq y \leq 10, 4x + 3y \geq 12, \) and \( -3x + y \leq 7, \) where \( x \geq 0 \) and \( y \geq 0. \)

49. In Example 5, if the profit on the low-grade mixture is $0.30 per package and the profit on the high-grade mixture is $0.40 per package, how many packages of each mixture should be made for a maximum profit?

3.3 Applications

OBJECTIVES 1 Solve applied problems

Solve applied problems 1 In this section, several situations that lead to linear programming problems are presented.
EXAMPLE 1 Maximizing Profit

Mike’s Famous Toy Trucks manufactures two kinds of toy trucks—a standard model and a deluxe model. In the manufacturing process each standard model requires 2 hours of grinding and 2 hours of finishing, and each deluxe model needs 2 hours of grinding and 4 hours of finishing. The company has two grinders and three finishers, each of whom works at most 40 hours per week. Each standard model toy truck brings a profit of $3 and each deluxe model a profit of $4. Assuming that every truck made will be sold, how many of each should be made to maximize profits?

SOLUTION First, we name the variables:

\[ x = \text{Number of standard models made} \]
\[ y = \text{Number of deluxe models made} \]

The quantity to be maximized is the profit, which we denote by \( P \):

\[ P = 3x + 4y \]

This is the objective function. To manufacture one standard model requires 2 grinding hours and to make one deluxe model requires 2 grinding hours. The number of grinding hours needed to manufacture \( x \) standard and \( y \) deluxe models is

\[ 2x + 2y \]

But the total amount of grinding time available is only 80 hours per week. This means we have the constraint

\[ 2x + 2y \leq 80 \quad \text{Grinding time constraint} \]

Similarly, for the finishing time we have the constraint

\[ 2x + 4y \leq 120 \quad \text{Finishing time constraint} \]

By simplifying each of these constraints and adding the nonnegativity constraints \( x \geq 0 \) and \( y \geq 0 \), we may list all the constraints for this problem:

\[
\begin{align*}
    x + y &\leq 40 & (1) \\
    x + 2y &\leq 60 & (2) \\
    x &\geq 0 & (3) \\
    y &\geq 0 & (4)
\end{align*}
\]

Figure 21 illustrates the set of feasible points, which is bounded.

The corner points of the set of feasible points are

\( (0, 0) \quad (0, 30) \quad (40, 0) \quad (20, 20) \)
Table 3 lists the corresponding values of the objective equation:

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Value of Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, y))</td>
<td>(P = 3x + 4y)</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>(P = 0)</td>
</tr>
<tr>
<td>(0, 30)</td>
<td>(P = 120)</td>
</tr>
<tr>
<td>(40, 0)</td>
<td>(P = 120)</td>
</tr>
<tr>
<td>(20, 20)</td>
<td>(P = 3(20) + 4(20) = 140)</td>
</tr>
</tbody>
</table>

A maximum profit is obtained if 20 standard trucks and 20 deluxe trucks are manufactured. The maximum profit is $140.

**NOW WORK PROBLEM 1.**

### EXAMPLE 2  Financial Planning

A retired couple have up to $30,000 to invest in fixed-income securities. Their broker recommends investing in two bonds: one a AAA bond yielding 8%; the other a B\(^+\) bond paying 12%. After some consideration, the couple decide to invest at most $12,000 in the B\(^+\)-rated bond and at least $6000 in the AAA bond. They also want the amount invested in the AAA bond to exceed or equal the amount invested in the B\(^+\) bond. What should the broker recommend if the couple (quite naturally) want to maximize the return on their investment?

**SOLUTION** First, we name the variables:

\[
x = \text{Amount invested in the AAA bond} \\
y = \text{Amount invested in the } B^+ \text{ bond}
\]

The quantity to be maximized, the couple’s return on investment, which we denote by \(P\), is

\[
P = 0.08x + 0.12y
\]

This is the objective function. The conditions specified by the problem are:

- Up to $30,000 available to invest: \(x + y \leq 30,000\)  
- Invest at most $12,000 in the B\(^+\) bond: \(y \leq 12,000\)  
- Invest at least $6000 in the AAA bond: \(x \geq 6000\)  
- Amount in the AAA bond must exceed or equal the amount in the B\(^+\) bond: \(x \geq y\)

In addition, we must have the conditions \(x \geq 0\) and \(y \geq 0\). The total list of constraints is

\[
\begin{align*}
x + y & \leq 30,000 \quad (1) \\
y & \leq 12,000 \quad (2) \\
x & \geq 6000 \quad (3) \\
x & \geq y \quad (4) \\
x & \geq 0 \quad (5) \\
y & \geq 0 \quad (6)
\end{align*}
\]
Figure 22 illustrates the set of feasible points, which is bounded. The corner points of the set of feasible points are

\[(6000, 0) \quad (6000, 6000) \quad (12000, 12000) \quad (18000, 12000) \quad (30000, 0)\]

The corresponding return on investment at each corner point is

\[P = 0.08(6000) + 0.12(0) = 480\]
\[P = 0.08(6000) + 0.12(6000) = 480 + 720 = 1200\]
\[P = 0.08(12,000) + 0.12(12,000) = 960 + 1440 = 2400\]
\[P = 0.08(18,000) + 0.12(12,000) = 1440 + 1440 = 2880\]
\[P = 0.08(30,000) + 0.12(0) = 2400\]

The maximum return on investment is $2880, obtained by placing $18,000 in the AAA bond and $12,000 in the B+ bond.

\[\text{NOW WORK PROBLEM 3.}\]

\[\text{EXAMPLE 3 Manufacturing Vitamin Pills—Maximizing Profit}\]

A pharmaceutical company makes two types of vitamins at its New Jersey plant—a high-potency, antioxidant vitamin and a vitamin enriched with added calcium. Each high-potency vitamin contains, among other things, 500 mg of vitamin C and 40 mg of calcium and generates a profit of $0.10 per tablet. A calcium-enriched vitamin tablet contains 100 mg of vitamin C and 400 mg of calcium and generates a profit of $0.05 per tablet. Each day the company has available 235 kg of vitamin C and 156 kg of calcium for use. Assuming all vitamins made are sold, how many of each type of vitamin should be manufactured to maximize profit?

\[\text{Source: Centrum Vitamin Supplements.}\]
First we name the variables:

\[ x = \text{Number (in thousands) of high-potency vitamins to be produced} \]
\[ y = \text{Number (in thousands) of calcium-enriched vitamins to be produced} \]

We want to maximize the profit, \( P \), which is given by:

\[ P = 0.10x + 0.05y \quad x \text{ and } y \text{ in thousands} \]

Since 1 kg = 1,000,000 mg, the constraints, in mg, take the form

\[
\begin{align*}
500x + 100y & \leq 235,000 & \text{vitamin C constraint (in thousands of mg)} \\
40x + 400y & \leq 156,000 & \text{calcium constraint (in thousands of mg)} \\
x & \geq 0 & \text{non-negativity constraints (in thousands)} \\
y & \geq 0
\end{align*}
\]

Figure 23 illustrates the set of feasible points, which is bounded. The corner points of the set of feasible points are

\[(0, 0) \quad (0, 390) \quad (400, 350) \quad (470, 0)\]
1. **Optimal Land Use** A farmer has 70 acres of land available on which to grow some soybeans and some corn. The cost of cultivation per acre, the workdays needed per acre, and the profit per acre are indicated in the table:

<table>
<thead>
<tr>
<th>Cultivation Cost per Acre</th>
<th>Soybeans</th>
<th>Corn</th>
<th>Total Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soybeans</td>
<td>$60</td>
<td>$30</td>
<td>$1800</td>
</tr>
<tr>
<td>Days of Work per Acre</td>
<td>3 days</td>
<td>4 days</td>
<td>120 days</td>
</tr>
<tr>
<td>Profit per Acre</td>
<td>$300</td>
<td>$150</td>
<td></td>
</tr>
</tbody>
</table>

As indicated in the last column, the acreage to be cultivated is limited by the amount of money available for cultivation costs and by the number of working days that can be put into this part of the business. Find the number of acres of each crop that should be planted in order to maximize the profit.

2. **Manufacturing** A factory manufactures two products, each requiring the use of three machines. The first machine can be used at most 70 hours; the second machine at most 40 hours; and the third machine at most 90 hours. The first product requires 2 hours on machine 1, 1 hour on machine 2, and 1 hour on machine 3; the second product requires 1 hour each on machines 1 and 2, and 3 hours on machine 3. If the profit is $40 per unit for the first product and $60 per unit for the second product, how many units of each product should be manufactured to maximize profit?

3. **Investment Strategy** An investment broker wants to invest up to $20,000. She can purchase a type A bond yielding a 10% return on the amount invested, and she can purchase a type B bond yielding a 15% return on the amount invested. She wants to invest at least as much in the type A bond as in the type B bond. She will also invest at least $5000 in the type A bond and no more than $8000 in the type B bond. How much should she invest in each type of bond to maximize her return?

4. **Diet** A diet is to contain at least 400 units of vitamins, 500 units of minerals, and 1400 calories. Two foods are available: $F_1$, which costs $0.05 per unit, and $F_2$, which costs $0.03 per unit. A unit of food $F_1$ contains 2 units of vitamins, 1 unit of minerals, and 4 calories; a unit of food $F_2$ contains 1 unit of vitamins, 2 units of minerals, and 4 calories. Find the minimum cost for a diet that consists of a mixture of these two foods and also meets the minimal nutrition requirements.

5. **Manufacturing Vitamin Pills** After changing suppliers, the pharmaceutical company in Example 3 has 300 kg of vitamin C and 220 kg of calcium available each day for the manufacture of the high-potency, antioxidant vitamins and vitamins enriched with added calcium. If each high-potency vitamin contains, among other things, 500 mg of vitamin C and 40 mg of calcium and generates a profit of $0.10 per tablet, and each calcium-enriched vitamin tablet contains 100 mg of vitamin C and 400 mg of calcium and generates a profit of $0.05 per tablet, how many of each type of vitamin should be manufactured to maximize profit?

Source: Centrum Vitamin Supplements.

6. **Investment Strategy** A financial consultant wishes to invest up to a total of $30,000 in two types of securities, one that yields 10% per year and another that yields 8% per year. Furthermore, she believes that the amount invested in the first security should be at least one-third of the amount invested in the second security. What investment program should the consultant pursue in order to maximize income?
7. Scheduling Blink Appliances has a sale on microwaves and stoves. Each microwave requires 2 hours to unpack and set up, and each stove requires 1 hour. The storeroom space is limited to 50 items. The budget of the store allows only 80 hours of employee time for unpacking and setup. Microwaves sell for $300 each, and stoves sell for $200 each. How many of each should the store order to maximize revenue?

8. Transportation An appliance company has a warehouse and two terminals. To minimize shipping costs, the manager must decide how many appliances should be shipped to each terminal. There is a total supply of 1200 units in the warehouse and a demand for 400 units in terminal A and 500 units in terminal B. It costs $12 to ship each unit to terminal A and $16 to ship to terminal B. How many units should be shipped to each terminal in order to minimize cost?

9. Pension Fund Investments A pension fund has decided to invest $45,000 in the two high-yield stocks listed in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Price per Share 2/21/03</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duke Energy Corp.</td>
<td>$14</td>
<td>8%</td>
</tr>
<tr>
<td>Eastman Kodak</td>
<td>$30</td>
<td>6%</td>
</tr>
</tbody>
</table>

This pension fund has decided to invest at least 25% of the $45,000 in each of the two stocks. Further, it has been decided that at most 63% of the $45,000 can be invested in either one of the stocks. How many shares of each stock should be purchased in order to maximize the annual yield, while meeting the stipulated requirements? What is the annual yield in dollars for the optimal investment plan?

Source: Yahoo! Finance. Prices and yields have been rounded.

10. Pollution Control A chemical plant produces two items A and B. For each item A produced, 2 cubic feet of carbon monoxide and 6 cubic feet of sulfur dioxide are emitted into the atmosphere; to produce item B, 4 cubic feet of carbon monoxide and 3 cubic feet of sulfur dioxide are emitted into the atmosphere. Government pollution standards permit the manufacturer to emit a maximum of 3000 cubic feet of carbon monoxide and 5400 cubic feet of sulfur dioxide per week. The manufacturer can sell all of the items that it produces and makes a profit of $1.50 per unit for item A and $1.00 per unit for item B. Determine the number of units of each item to be produced each week to maximize profit without exceeding government standards.

11. Baby Food Servings Gerber Banana Plum Granola costs $0.89 per 5.5-oz serving; each serving contains 140 calories, 31 g of carbohydrates, and 0% of the recommended daily allowance of vitamin C. Gerber Mixed Fruit Carrot Juice costs $0.79 per 4-oz serving; each serving contains 60 calories, 13 g of carbohydrates, and 100% of the recommended daily allowance of vitamin C. Determine how many servings of each of the above foods would be needed to provide a child at least 160 calories, 40 g of carbohydrates, and 70% of the recommended daily allowance of vitamin C at minimum cost. Fractions of servings are permitted.

Source: Gerber website and Safeway Stores, Inc.

12. Production Scheduling A company produces two types of steel. Type 1 requires 2 hours of melting, 4 hours of cutting, and 10 hours of rolling per ton. Type 2 requires 5 hours of melting, 1 hour of cutting, and 5 hours of rolling per ton. Forty hours are available for melting, 20 for cutting, and 60 for rolling. Each ton of Type 1 produces $240 profit, and each ton of Type 2 yields $80 profit. Find the maximum profit and the production schedule that will produce this profit.

13. Website Ads Nielson’s Net Ratings for the month of December 2002 indicated that in the United States, AOL had a unique audience of about 76.4 million people and Yahoo! had a unique audience of about 66.2 million people. An advertising company wants to purchase website ads to promote a new product. Suppose that the monthly cost of an ad on the AOL website is $1200 and the monthly cost of an ad on the Yahoo! website is $1100. Determine how many months an ad should run on each website to maximize the number of people who would be exposed to it. Assume that, for future months, the monthly website audience remains the same as given for December 2002. Also assume that the advertising budget is $35,000 and that it has been decided to advertise on Yahoo! for at least ten months.

Source: Nielson’s Net Ratings.

14. Diet Danny’s Chicken Farm is a producer of frying chickens. In order to produce the best fryers possible, the regular chicken feed is supplemented by four vitamins. The minimum amount of each vitamin required per 100 ounces of feed is: vitamin 1, 50 units; vitamin 2, 100 units; vitamin 3, 60 units; vitamin 4, 180 units. Two supplements are available: supplement I costs $0.03 per ounce and contains 5 units of vitamin 1 per ounce, 25 units of vitamin 2 per ounce, 10 units of vitamin 3 per ounce, and 35 units of vitamin 4 per ounce. Supplement II costs $0.04 per ounce and contains 25 units of vitamin 1 per ounce, 25 units of vitamin 2 per ounce, 10 units of vitamin 3 per ounce, and 20 units of vitamin 4 per ounce. How much of each supplement should Danny buy to add to each 100 ounces of feed in order to minimize his cost, but still have the desired vitamin amounts present?

15. Maximizing Income J. B. Rug Manufacturers has available 1200 square yards of wool and 1000 square yards of nylon for
the manufacture of two grades of carpeting: high-grade, which sells for $500 per roll, and low-grade, which sells for $300 per roll. Twenty square yards of wool and 40 square yards of nylon are used in a roll of high-grade carpet, and 40 square yards of nylon are used in a roll of low-grade carpet. Forty work-hours are required to manufacture each roll of the high-grade carpet, and 20 work-hours are required for each roll of the low-grade carpet, at an average cost of $6 per work-hour. A maximum of 800 work-hours are available. The cost of wool is $5 per square yard and the cost of nylon is $2 per square yard. How many rolls of each type of carpet should be manufactured to maximize income? [Hint: Income = Revenue from sale − (Production cost for material + labor)]

16. The rug manufacturer in Problem 15 finds that maximum income occurs when no high-grade carpet is produced. If the price of the low-grade carpet is kept at $300 per roll, in what price range should the high-grade carpet be sold so that income is maximized by selling some rolls of each type of carpet? Assume all other data remain the same.

### Chapter 3 Review

**OBJECTIVES**

<table>
<thead>
<tr>
<th>Section</th>
<th>You should be able to</th>
<th>Review Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>1 Graph linear inequalities</td>
<td>1–4</td>
</tr>
<tr>
<td></td>
<td>2 Graph systems of linear inequalities</td>
<td>9–14</td>
</tr>
<tr>
<td>3.2</td>
<td>1 Identify a linear programming problem</td>
<td>33–38</td>
</tr>
<tr>
<td></td>
<td>2 Solve a linear programming problem</td>
<td>15–32</td>
</tr>
<tr>
<td>3.3</td>
<td>1 Solve applied problems</td>
<td>33–38</td>
</tr>
</tbody>
</table>

**THINGS TO KNOW**

**Graphs of Inequalities (p. 160)**

The graph of a strict inequality is represented by a dashed line and the half-plane satisfying the inequality.

The graph of a nonstrict inequality is represented by a solid line and the half-plane satisfying the inequality.

**Graphs of Systems of Linear Inequalities (p. 162)**

The graph of a system of linear inequalities is the set of all points that satisfy each inequality in the system.

**Bounded (p. 166)**

The graph is called bounded if some circle can be drawn around it.

The graph is called unbounded if it extends infinitely far in at least one direction.

**Corner Point (p. 166)**

A corner point is the intersection of two line segments that form the boundary of the graph of a system of linear inequalities.

**Linear Programming (p. 172)**

Maximize (or minimize) a linear objective function, $z = Ax + By$, subject to certain conditions, or constraints, expressible as linear inequalities in $x$ and $y$. A feasible point $(x, y)$ is a point that satisfies the constraints of a linear programming problem.

**Solution to a Linear Programming Problem (p. 172)**

A solution to a linear programming problem is a feasible point that maximizes (or minimizes) the objective function together with the value of the objective function at that point.

**Location of Solution (p. 174)**

If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points.

If a linear programming problem has multiple solutions, at least one of them is located at a corner point of the graph of the feasible points.

In either case, the corresponding value of the objective function is unique.
**Chapter Review 187**

**TRUE–FALSE ITEMS** Answers are on page AN-00.

T  F  1. The graph of a system of linear inequalities may be bounded or unbounded.

T  F  2. The graph of the set of constraints of a linear programming problem, under certain conditions, could have a circle for a boundary.

T  F  3. The objective function of a linear programming problem is always a linear equation involving the variables.

T  F  4. In a linear programming problem, there may be more than one point that maximizes or minimizes the objective function.

T  F  5. Some linear programming problems will have no solution.

T  F  6. If a linear programming problem has a solution, it is located at the center of the set of feasible points.

**FILL IN THE BLANKS** Answers are on page AN-00.

1. The graph of a linear inequality in two variables is called a _________.

2. In a linear programming problem the quantity to be maximized or minimized is referred to as the _________ function.

3. The points that obey the collection of constraints of a linear programming problem are called _________ points.

4. A linear programming problem will always have a solution if the set of feasible points is _________.

5. If a linear programming problem has a solution, it is located at a _________ of the set of feasible points.

**REVIEW EXERCISES** Answers to odd-numbered problems begin on page AN-00.

**Blue problem numbers indicate the author’s suggestions for use in a practice test.**

In Problems 1–4, graph each linear inequality.

1. \( x + 3y \leq 0 \)

2. \( 4x + y \geq 0 \)

3. \( 5x + y > 10 \)

4. \( 2x + 3y < 6 \)

5. Without graphing, determine which of the points \( P_1 = (4, -3) \), \( P_2 = (2, -6) \), \( P_3 = (8, -3) \) are part of the graph of the following system:

\[
\begin{align*}
x + 2y & \leq 8 \\
2x - y & \geq 4
\end{align*}
\]

6. Without graphing, determine which of the points \( P_1 = (8, 6) \), \( P_2 = (2, -5) \), \( P_3 = (4, 1) \) are part of the graph of the following system:

\[
\begin{align*}
5x - y & \geq 2 \\
x - 4y & \leq -2
\end{align*}
\]

In Problems 7–8, determine which region — a, b, c, or d — represents the graph of the given system of linear inequalities.

7. \[
\begin{align*}
6x - 4y & \leq 12 \\
3x + 2y & \leq 18
\end{align*}
\]

8. \[
\begin{align*}
6x - 5y & \geq 5 \\
6x + 6y & \leq 60
\end{align*}
\]

In Problems 9–14, graph each system of linear inequalities. Locate the corner points and tell whether the graph is bounded or unbounded.

9. \[
\begin{align*}
3x + 2y & \leq 12 \\
x + y & \geq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

10. \[
\begin{align*}
x + y & \leq 8 \\
2x + y & \geq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

11. \[
\begin{align*}
x + 2y & \geq 4 \\
3x + y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
In Problems 15–22, use the constraints below to solve each linear programming problem.

\[
\begin{align*}
15. & \quad \text{Maximize } z = x + y \\
16. & \quad \text{Maximize } z = 2x + 3y \\
17. & \quad \text{Minimize } z = 5x + 2y \\
18. & \quad \text{Minimize } z = 3x + 2y \\
19. & \quad \text{Maximize } z = 2x + y \\
20. & \quad \text{Maximize } z = x + 2y \\
21. & \quad \text{Minimize } z = 2x + 5y \\
22. & \quad \text{Minimize } z = x + y
\end{align*}
\]

In Problems 23–26, maximize and minimize (if possible) the quantity \( z = 15x + 20y \) subject to the given constraints.

\[
\begin{align*}
23. & \quad \begin{align*} 
 x & \leq 5 \\
 y & \leq 8 \\
3x + 4y & \geq 12 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \\
24. & \quad \begin{align*} 
 x + 2y & \leq 4 \\
3x + 2y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \\
25. & \quad \begin{align*} 
 2x + 3y & \leq 22 \\
x & \leq 7 \\
y & \leq 8 \\
x + y & \geq 2 \\
x & \geq 0 \\
y & \geq 0
\end{align*} \\
26. & \quad \begin{align*} 
 x + 2y & \leq 20 \\
x + 10y & \geq 36 \\
x & \leq 6 \\
x + y & \geq 1 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

In Problems 27–32, solve each linear programming problem.

\[
\begin{align*}
27. & \quad \text{Maximize } z = 2x + 3y \\
28. & \quad \text{Maximize } z = 4x + y \\
29. & \quad \text{Maximize } z = x + 2y \\
30. & \quad \text{Maximize } z = 3x + 4y \\
31. & \quad \text{Minimize } z = 3x + 2y \\
32. & \quad \text{Minimize } z = 2x + 5y
\end{align*}
\]
33. **Maximizing Profit** A ski manufacturer makes two types of skis: downhill and cross-country. Using the information given in the table below, how many of each type of ski should be made for a maximum profit to be achieved? What is the maximum profit?

<table>
<thead>
<tr>
<th></th>
<th>Cross-Country</th>
<th>Maximum Time Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturing Time per Ski</td>
<td>2 hours</td>
<td>40 hours</td>
</tr>
<tr>
<td>Finishing Time per Ski</td>
<td>1 hour</td>
<td>32 hours</td>
</tr>
<tr>
<td>Profit per Ski</td>
<td>$70</td>
<td>$50</td>
</tr>
</tbody>
</table>

34. Rework Problem 33 if the manufacturing unit has a maximum of 48 hours available.

35. **Nutrition** Katy needs at least 60 units of carbohydrates, 45 units of protein, and 30 units of fat each month. From each pound of food A, she receives 5 units of carbohydrates, 3 of protein, and 4 of fat. Food B contains 2 units of carbohydrates, 2 units of protein, and 1 unit of fat per pound. If food A costs $1.30 per pound and food B costs $0.80 per pound, how many pounds of each food should Katy buy each month to keep costs at a minimum?

36. **Production Scheduling** A company sells two types of shoes. The first uses 2 units of leather and 2 units of synthetic material and yields a profit of $8 per pair. The second type requires 5 units of leather and 1 unit of synthetic material and gives a profit of $10 per pair. If there are 40 units of leather and 16 units of synthetic material available, how many pairs of each type of shoe should be manufactured to maximize profit? What is the maximum profit?

37. **Baby Food Servings** Gerber Banana Oatmeal and Peach costs $0.79 per 4-oz serving; each serving contains 90 calories, 19 g of carbohydrates, and 45% of the recommended daily allowance of vitamin C. Gerber Mixed Fruit Juice costs $0.65 per 4-oz serving; each serving contains 60 calories, 15 g of carbohydrates, and 100% of the recommended daily allowance of vitamin C. Determine how many servings of each of the above foods would be needed to provide a child with at least 130 calories, 30 g of carbohydrates, and 60% of the recommended daily allowance of vitamin C at minimum cost. Fractions of servings are permitted.

**Source**: Gerber website and Safeway Stores, Inc.

38. **Maximizing Profit** A company makes two explosives: type I and type II. Due to storage problems, a maximum of 100 pounds of type I and 150 pounds of type II can be mixed and packaged each week. One pound of type I takes 60 hours to mix and 70 hours to package; 1 pound of type II takes 40 hours to mix and 40 hours to package. The mixing department has at most 7200 work-hours available each week, and packaging has at most 7800 work-hours available. If the profit for 1 pound of type I is $60 and for 1 pound of type II is $40, what is the maximum profit possible each week?

39. A pharmaceutical company makes two types of vitamins at its New Jersey plant—a high-potency, antioxidant vitamin and a vitamin enriched with added calcium. Each high-potency vitamin contains, among other things, 500 mg of vitamin C, 40 mg of calcium, and 100 mg of magnesium and generates a profit of $0.10 per tablet. A calcium-enriched vitamin tablet contains 100 mg of vitamin C, 400 mg of calcium, and 40 mg of magnesium and generates a profit of $0.05 per tablet. Each day the company has available 300 kg of vitamin C, 122 kg of calcium, and 65 kg of magnesium for use in the production of the vitamins. How many of each type of vitamin should be manufactured to maximize profit?

**Source**: Centrum Vitamin Supplements.

40. **Mixture** A company makes two kinds of animal food, A and B, which contain two food supplements. It takes 2 pounds of the first supplement and one pound of the second to make a dozen cans of food A, and 4 pounds of the first supplement and 5 pounds of the second to make a dozen cans of food B. On a certain day 80 pounds of the first supplement and 70 pounds of the second are available. How many cans of food A and food B should be made to maximize company profits, if the profit on a dozen cans of food A is $3.00 and the profit on a dozen cans of food B is $10.00.
Chapter 3 Project

BALANCING NUTRIENTS

In preparing a recipe you must decide what ingredients and how much of each ingredient you will use. In these health-conscious days, you may also want to consider the amount of certain nutrients in your recipe. You may even be interested in minimizing some quantities (like calories or fat) or maximizing others (like carbohydrates or protein). Linear programming techniques can help to do this.

For example, consider making a very simple trail mix from dry-roasted, unsalted peanuts and seedless raisins. Table 1 lists the amounts of various dietary quantities for these ingredients. The amounts are given per serving of the ingredient.

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Peanuts Serving Size = 1 Cup</th>
<th>Raisins Serving Size = 1 Cup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories (kcal)</td>
<td>850</td>
<td>440</td>
</tr>
<tr>
<td>Protein (g)</td>
<td>34.57</td>
<td>4.67</td>
</tr>
<tr>
<td>Fat (g)</td>
<td>72.50</td>
<td>.67</td>
</tr>
<tr>
<td>Carbohydrates (g)</td>
<td>31.40</td>
<td>114.74</td>
</tr>
</tbody>
</table>


Suppose that you want to make at most 6 cups of trail mix for a day hike. You don’t want either ingredient to dominate the mixture, so you want the amount of raisins to be at least $\frac{1}{2}$ of the amount of peanuts and the amount of peanuts to be at least $\frac{1}{2}$ of the amount of raisins. You want the entire amount of trail mix you make to have fewer than 4000 calories, and you want to maximize the amount of carbohydrates in the mix.

1. Let $x$ be the number of cups of peanuts you will use, let $y$ be the number of cups of raisins you will use, and let $c$ be the amount of carbohydrates in the mix. Find the objective function.

2. What constraints must be placed on the objective function?

3. Graph the set of feasible points for this problem.

4. Find the number of cups of peanuts and raisins that maximize the amount of carbohydrates in the mix.

5. How many grams of carbohydrates are in a cup of the final mix? How many calories?

6. Under all of the constraints given above, what recipe for trail mix will maximize the amount of protein in the mix? How many grams of protein are in a cup of this mix? How many calories?

7. Suppose you decide to eat at least 3 cups of the trail mix. Keeping the constraints given above, what recipe for trail mix will minimize the amount of fat in the mix?

8. How many grams of carbohydrates are in this mix?

9. How many grams of protein are in this mix?

10. Which of the three trail mixes would you use? Why?
MATHEMATICAL QUESTIONS FROM PROFESSIONAL EXAMS

Use the following information to do Problems 1–3:
The Random Company manufactures two products, Zeta and Beta. Each product must pass through two processing operations. All materials are introduced at the start of process 1. There are no work-in-process inventories. Random may produce either one product exclusively or various combinations of both products subject to the following constraints:

<table>
<thead>
<tr>
<th>Process</th>
<th>Process</th>
<th>Contribution Margin per Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>No. 2</td>
<td></td>
</tr>
<tr>
<td>Zeta</td>
<td>1 hour</td>
<td>1 hour</td>
</tr>
<tr>
<td>Beta</td>
<td>2 hours</td>
<td>3 hours</td>
</tr>
<tr>
<td>Total capacity in hours per day</td>
<td>1000 hours</td>
<td>1275 hours</td>
</tr>
</tbody>
</table>

A shortage of technical labor has limited Beta production to 400 units per day. There are no constraints on the production of Zeta other than the hour constraints in the above schedule. Assume that all relationships between capacity and production are linear, and that all of the above data and relationships are deterministic rather than probabilistic.

1. CPA Exam Given the objective to maximize total contribution margin, what is the production constraint for process 1?
   (a) Zeta + Beta ≤ 1000
   (b) Zeta + 2 Beta ≤ 1000
   (c) Zeta + Beta ≥ 1000
   (d) Zeta + 2 Beta ≥ 1000

2. CPA Exam Given the objective to maximize total contribution margin, what is the labor constraint for production of Beta?
   (a) Beta ≤ 400
   (b) Beta ≥ 400
   (c) Beta ≤ 425
   (d) Beta ≥ 425

3. CPA Exam What is the objective function of the data presented?
   (a) Zeta + 2 Beta = $9.25
   (b) $(4.00)Zeta + 3($5.25)Beta = Total contribution margin
   (c) $(4.00)Zeta + ($5.25)Beta = Total contribution margin
   (d) 2($4.00)Zeta + 3($5.25)Beta = Total contribution margin

4. CPA Exam Williamson Manufacturing intends to produce two products, X and Y. Product X requires 6 hours of time on machine 1 and 12 hours of time on machine 2. Product Y requires 4 hours of time on machine 1 and no time on machine 2. Both machines are available for 24 hours. Assuming that the objective function of the total contribution margin is $2X + $1Y, what product mix will produce the maximum profit?
   (a) No units of product X and 6 units of product Y.
   (b) 1 unit of product X and 4 units of product Y.
   (c) 2 units of product X and 3 units of product Y.
   (d) 4 units of product X and no units of product Y.

5. CPA Exam Quepea Company manufactures two products, Q and P, in a small building with limited capacity. The selling price, cost data, and production time are given below:

<table>
<thead>
<tr>
<th>Product Q</th>
<th>Product P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling price per unit</td>
<td>$20</td>
</tr>
<tr>
<td>Variable costs of producing and selling a unit</td>
<td>$12</td>
</tr>
<tr>
<td>Hours to produce a unit</td>
<td>3</td>
</tr>
</tbody>
</table>
Based on this information, the profit maximization objective function for a linear programming solution may be stated as

(a) Maximize $20Q + 17P$
(b) Maximize $12Q + 13P$
(c) Maximize $3Q + 1P$
(d) Maximize $8Q + 4P$

6. CPA Exam Patsy, Inc., manufactures two products, X and Y. Each product must be processed in each of three departments: machining, assembling, and finishing. The hours needed to produce one unit of product per department and the maximum possible hours per department follow:

<table>
<thead>
<tr>
<th>Department</th>
<th>X</th>
<th>Y</th>
<th>Maximum Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machining</td>
<td>2</td>
<td>1</td>
<td>420</td>
</tr>
<tr>
<td>Assembling</td>
<td>2</td>
<td>2</td>
<td>500</td>
</tr>
<tr>
<td>Finishing</td>
<td>2</td>
<td>3</td>
<td>600</td>
</tr>
</tbody>
</table>

Other restrictions follow:

\[ X \geq 50 \quad Y \geq 50 \]

The objective function is to maximize profits where profit = $4X + 2Y$. Given the objective and constraints, what is the most profitable number of units of X and Y, respectively, to manufacture?

(a) 150 and 100  (b) 165 and 90  (c) 170 and 80  (d) 200 and 50

7. CPA Exam Milford Company manufactures two models, medium and large. The contribution margin expected is $12 for the medium model and $20 for the large model. The medium model is processed 2 hours in the machining department and 4 hours in the polishing department. The large model is processed 3 hours in the machining department and 6 hours in the polishing department. How would the formula for determining the maximization of total contribution margin be expressed?

(a) $5X + 10Y$  (b) $6X + 9Y$  (c) $12X + 20Y$  (d) $12X(2 + 4) + 20Y(3 + 6)$

8. CPA Exam Hale Company manufactures products A and B, each of which requires two processes, polishing and grinding. The contribution margin is $3 for product A and $4 for product B. The illustration shows the maximum number of units of each product that may be processed in the two departments.

Consider the constraints (restrictions) on processing, which combination of products A and B maximizes the total contribution margin?

(a) 0 units of A and 20 units of B.  (b) 20 units of A and 10 units of B.  (c) 30 units of A and 0 units of B.  (d) 40 units of A and 0 units of B.

9. CPA Exam Johnson, Inc., manufactures product X and product Y, which are processed as follows:

<table>
<thead>
<tr>
<th>Machine A</th>
<th>Machine B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product X</td>
<td>6 hours</td>
</tr>
<tr>
<td>Product Y</td>
<td>9 hours</td>
</tr>
</tbody>
</table>

The contribution margin is $12 for product X and $7 for product Y. The available time daily for processing the two products is 120 hours for machine A and 80 hours for machine B. How would the restriction (constraint) for machine B be expressed?

(a) $4X + 5Y$  (b) $4X + 5Y \leq 80$  (c) $6X + 9Y \leq 120$  (d) $12X + 7Y$

10. CPA Exam A small company makes only two products, with the following two production constraints representing two machines and their maximum availability:

\[ 2X + 3Y \leq 18 \]
\[ 2X + Y \leq 10 \]

where \( X = \) Units of the first product
\( Y = \) Units of the second product

If the profit equation is \( Z = 4X + 2Y \), the maximum possible profit is

(a) $20  (b) $21  (c) $18  (d) $24  (e) Some profit other than those given above
Questions 11–13 are based on the Jarten Company, which manufactures and sells two products. Demand for the two products has grown to such a level that Jarten can no longer meet the demand with its facilities. The company can work a total of 600,000 direct labor-hours annually using three shifts. A total of 200,000 hours of machine time is available annually. The company plans to use linear programming to determine a production schedule that will maximize its net return.

The company spends $2,000,000 in advertising and promotion and incurs $1,000,000 for general and administrative costs. The unit sale price for model A is $27.50; model B sells for $75.00 each. The unit manufacturing requirements and unit cost data are as shown below. Overhead is assigned on a machine-hour (MH) basis.

<table>
<thead>
<tr>
<th></th>
<th>Model A</th>
<th>Model B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw material</td>
<td>$3</td>
<td>$7</td>
</tr>
<tr>
<td>Direct labor</td>
<td>1 DLH @ $8</td>
<td>1.5 DLH @ $8</td>
</tr>
<tr>
<td>Variable overhead</td>
<td>0.5 MH @ $12</td>
<td>2.0 MH @ $12</td>
</tr>
<tr>
<td>Fixed overhead</td>
<td>0.5 MH @ $4</td>
<td>2.0 MH @ $4</td>
</tr>
<tr>
<td></td>
<td>$19</td>
<td>$51</td>
</tr>
</tbody>
</table>

11. CMA Exam  The objective function that would maximize Jarten’s net income is
- (a) $10.50A + 32.00B
- (b) $8.50A + 24.00B
- (c) $27.50A + 75.00B
- (d) $19.00A + 51.00B
- (e) $17.00A + 43.00B

12. CMA Exam  The constraint function for the direct labor is
- (a) $A + 1.5B \leq 200,000$
- (b) $8A + 12B \leq 600,000$
- (c) $8A + 12B \leq 200,000$
- (d) $1A + 1.5B \leq 4,800,000$
- (e) $1A + 1.5B \leq 600,000$

13. CMA Exam  The constraint function for the machine capacity is
- (a) $6A + 24B \leq 200,000$
- (b) $(1/0.5)A + (1.5/2.0)B \leq 800,000$
- (c) $0.5A + 2B \leq 200,000$
- (d) $(0.5 + 0.5)A + (2 + 2)B \leq 200,000$
- (e) $(0.5 \times 1) + (1.5 \times 2.0) \leq (200,000 \times 600,000)$

14. CPA Exam  Boaz Company manufactures two models, medium (X) and large (Y). The contribution margin expected is $24 for the medium model and $40 for the large model. The medium model is processed 2 hours in the machining department and 4 hours in the polishing department. The large model is processed 3 hours in the machining department and 6 hours in the polishing department. If total contribution margin is to be maximized, using linear programming, how would the objective function be expressed?
- (a) $24X(2 + 4) + 40Y(3 + 6)$
- (b) $24X + 40Y$
- (c) $6X + 9Y$
- (d) $5X + 10Y$