CHAPTER 5

Finance

OUTLINE

Even though you're in college now, at some time, probably not too far in the future, you will be thinking of buying a house. And, unless you've won the lottery, you will need to borrow a substantial part of the purchase price in the form of a mortgage. You will quickly discover that mortgages are very complicated and come in many shapes and sizes. One choice you will need to make is whether "paying points" for a mortgage makes sense. The Chapter Project at the end of this chapter takes you through some of the possibilities that can occur when "points" are involved.

A LOOK BACK. A LOOK FORWARD

In earlier mathematics courses, you studied percents. In later courses, you studied exponential expressions and logarithmic expressions, which are reviewed in Appendix A, Section A.4. In this chapter we apply these topics to the field of finance, discussing simple interest, compound interest, simple annuities, sinking funds, and amortization.

5.1 Interest

OBJECTIVES
1. Solve problems involving percents
2. Solve problems involving simple interest
3. Solve problems involving discounted loans

Interest is money paid for the use of money. The total amount of money borrowed, whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account, is called the **principal**.

The **rate of interest** is the amount charged for the use of the principal for a given length of time, usually on a yearly, or *per annum*, basis. Rates of interest are usually expressed as percents: 10% per annum, 14% per annum, $7\frac{1}{2}\%$ per annum, and so on.

Solve problems involving percents 1

The word **percent** means "per hundred." The familiar symbol for percent means to divide by one hundred. For example,

$$1\% = \frac{1}{100} = 0.01 \quad 12\% = \frac{12}{100} = 0.12 \quad 0.3\% = \frac{0.3}{100} = \frac{3}{1000} = 0.003$$

By reversing these ideas, we can write decimals as percents.

$$0.35 = \frac{35}{100} = 35\% \quad 1.25 = \frac{125}{100} = 125\% \quad 0.005 = \frac{5}{1000} = \frac{0.5}{100} = 0.5\% = \frac{1}{2}\%$$

NOW WORK PROBLEMS 1 AND 9.

EXAMPLE 1 Working with Percents

(a) Find 12% of 80.
(b) What percent of 40 is 18?
(c) 8 is 15% of what number?

SOLUTION

(a) The English word "of" translates to "multiply" when percents are involved. For example,

$$12\% \text{ of } 80 = 12\% \text{ times } 80 = (0.12)\cdot(80) = 9.6$$

(b) Let x represent the unknown percent. Then

$$x\% \text{ of } 40 = 18 \qquad \text{What \% of 40 is 18}$$

$$\frac{x}{100}\cdot 40 = 18 \qquad x\% = \frac{x}{100}$$

$$40x = 1800 \qquad \text{Multiply both sides by 100}$$

$$x = \frac{1800}{40} \qquad \text{Divide both sides by 40}$$

$$x = 45$$

So, 45% of 40 is 18.

(c) Let x represent the number. Then

$$8 = 15\% \text{ of } x$$
$$8 = 0.15x \qquad 15\% = 0.15$$
$$x = \frac{8}{0.15} \qquad \text{Divide both sides by } 0.15$$
$$x = 53.33 \qquad \text{Use a calculator}$$

So, 8 is 15% of 53.33.

NOW WORK PROBLEMS 17, 23, AND 27.

EXAMPLE 2 Computing a State Income Tax

A resident of Illinois has base income, after adjustment for deductions, of $18,000. The state income tax on this base income is 3%. What tax is due?

SOLUTION We must find 3% of $18,000. We convert 3% to its decimal equivalent and then multiply by $18,000.

$$3\% \text{ of } \$18{,}000 = (0.03)(\$18{,}000) = \$540$$

The state income tax due is $540.00.

Simple Interest

Solve problems involving simple interest 2

The easiest type of interest to deal with is called *simple interest*.

Simple Interest

Simple interest is interest computed on the principal for the entire period it is borrowed.

Simple Interest Formula

If a principal P is borrowed at a simple interest rate of $r\%$ per annum (where $r\%$ is expressed as a decimal) for a period of t years, the interest charge I is

$$\boxed{I = Prt} \qquad (1)$$

The **amount** A owed at the end of t years is the sum of the principal P borrowed and the interest I charged: That is,

$$\boxed{A = P + I = P + Prt = P(1 + rt)} \qquad (2)$$

EXAMPLE 3 Computing Interest and the Amount Due

A loan of \$250 is made for 9 months at a simple interest rate of 10% per annum. What is the interest charge? What amount is due after 9 months?

SOLUTION The actual period the money is borrowed for is 9 months, which is $\frac{9}{12} = \frac{3}{4}$ of a year. The interest charge is the product of the amount borrowed, \$250; the annual rate of interest, 10%, expressed as a decimal, 0.10; and the length of time in years, $\frac{3}{4}$. We use Formula (1) to find the interest charge I.

$$I = Prt$$

$$\text{Interest charge} = (\$250)(0.10)\left(\frac{3}{4}\right) = \$18.75$$

Using Formula (2), the amount A due after 9 months is

$$A = P + I = 250 + 18.75 = \$268.75$$

NOW WORK PROBLEM 31.

EXAMPLE 4 Computing the Rate of Interest

A person borrows \$1000 for a period of 6 months. What simple interest rate is being charged if the amount A that must be repaid after 6 months is \$1045?

SOLUTION The principal P is \$1000, the period is $\frac{1}{2}$ year (6 months), and the amount A owed after 6 months is \$1045. We substitute the known values of A, P, and t in Formula (2), and then solve for r, the interest rate.

$$A = P + Prt \qquad \text{Formula (2)}$$

$$1045 = 1000 + 1000r\left(\frac{1}{2}\right) \qquad A = 1045;\ P = 1000;\ t = \frac{1}{2}$$

$$45 = 500r \qquad \text{Subtract 1000 from each side and simplify.}$$

$$r = \frac{45}{500} = 0.09 \qquad \text{Solve for } r.$$

The per annum rate of interest is 9%.

NOW WORK PROBLEM 37.

EXAMPLE 5 **Computing the Amount Due**

A company borrows \$1,000,000 for 1 month at a simple interest rate of 9% per annum. How much must the company pay back at the end of 1 month?

SOLUTION The principal P is \$1,000,000, the period t is $\frac{1}{12}$ year, and the interest rate r is 0.09. The amount A that must be paid back is

$$L = P(1 + rt) \qquad \text{Formula (2)}$$

$$\begin{aligned} A &= 1{,}000{,}000\left[1 + 0.09\left(\frac{1}{12}\right)\right] \\ &= 1{,}000{,}000(1.0075) \\ &= \$1{,}007{,}500 \end{aligned}$$

At the end of 1 month, the company must pay back \$1,007,500.

Discounted Loans

Solve problems involving discounted loans 3

If a lender deducts the interest from the amount of the loan at the time the loan is made, the loan is said to be **discounted**. The interest deducted from the amount of the loan is the **discount**. The amount the borrower receives is called the **proceeds**.

Discounted Loans

Let r be the per annum rate of interest, t the time in years, and L the amount of the loan. Then the proceeds R is given by

$$\boldsymbol{R = L - Lrt = L(1 - rt)} \qquad (3)$$

where Art is the discount, the interest deducted from the amount of the loan.

EXAMPLE 6 **Computing the Proceeds of a Discounted Loan**

A borrower signs a note for a discounted loan and agrees to pay the lender \$1000 in 9 months at a 10% rate of interest. How much does this borrower receive?

SOLUTION The amount of the loan is $L = 1000$. The rate of interest is $r = 10\% = 0.10$. The time is $t = 9$ months $= \frac{9}{12}$ year. The discount is

$$Lrt = \$1000(0.10)\left(\frac{9}{12}\right) = \$75$$

The discount is deducted from the loan amount of \$1000, so that the proceeds, the amount the borrower receives, is

$$R = L - Lrt = 1000 - 75 = \$925$$

NOW WORK PROBLEM 43.

EXAMPLE 7 Computing the Simple Interest on a Discounted Loan

What simple rate of interest is the borrower in Example 6 paying on the \$925 that was borrowed for 9 months and paid back in the amount of \$1000?

SOLUTION The principal P is \$925, t is $\frac{9}{12} = \frac{3}{4}$ of a year, and the amount A is \$1000. If r is the simple rate of interest, then, from Formula (2),

$$\begin{aligned} A &= P + Prt && \text{Formula (2)} \\ 1000 &= 925 + 925r\left(\frac{3}{4}\right) && A = 1000;\ P = 925;\ t = \frac{3}{4} \\ 75 &= 693.75r && \text{Simplify.} \\ r &= \frac{75}{693.75} = 0.108108 && \text{Solve for } r. \end{aligned}$$

The equivalent simple rate of interest for this loan is 10.81%.

EXAMPLE 8 Finding the Amount of a Discounted Loan

You wish to borrow \$10,000 for 3 months. If the person you are borrowing from offers a discounted loan at 8%, how much must you repay at the end of 3 months?

SOLUTION The amount you borrow, the proceeds is $R = \$10{,}000$; the interest rate is $r = 0.08$; and the time t is $\frac{3}{12} = \frac{1}{4}$ year. From Formula (3), the amount L you repay obeys the equation

$$\begin{aligned} R &= L(1 - rt) && \text{Formula (3)} \\ 10{,}000 &= L\left[1 - 0.08\left(\frac{1}{4}\right)\right] && R = 10{,}000;\ r = 0.08;\ t = \tfrac{1}{4} \\ 10{,}000 &= 0.98L && \text{Simplify.} \\ L &= \frac{10{,}000}{0.98} = \$10{,}204.08 && \text{Solve for } L. \end{aligned}$$

You will repay \$10,204.08 for this loan.

NOW WORK PROBLEM 47.

Treasury Bills

Treasury bills (T-bills) are short-term securities issued by the Federal Reserve. The bills do not specify a rate of interest. They are sold at public auction with financial institutions making competitive bids. For example, a financial institution may bid \$982,400 for a 3-month \$1 million treasury bill. At the end of 3 months the financial institution receives \$1 million, which includes the interest earned and the cost of the T-bill. This bidding process is an example of a discounted loan.

EXAMPLE 9 Bidding on Treasury Bills

How much should a bank bid if it wants to earn 1.2% simple interest on a 6-month \$1 million treasury bill?*

**On February 6, 2003, 6-month treasury bills were yielding 1.16%.*

SOLUTION The maturity value, the amount to be repaid to the bank by the government, is $L = \$1{,}000{,}000$. The rate of interest is $r = 1.2\% = 0.012$. The time is $t = 6$ months $= \frac{1}{2}$ year. The proceeds R to the government are

$$R = L(1 - rt) = \$1{,}000{,}000\left[1 - 0.012\left(\frac{1}{2}\right)\right]$$
$$= 1{,}000{,}000(0.994)$$
$$= 994{,}000$$

The bank should bid \$994,000 if it wants to earn 1.2%.

NOW WORK PROBLEM 61.

EXERCISE 5.1 Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–8, write each decimal as a percent.

1. 0.60 **2.** 0.40 **3.** 1.1 **4.** 1.2 **5.** 0.06 **6.** 0.07 **7.** 0.0025 **8.** 0.0015

In Problems 9–16, write each percent as a decimal.

9. 25% **10.** 15% **11.** 100% **12.** 300% **13.** 6.5% **14.** 4.3% **15.** 73.4% **16.** 92%

In Problems 17–30, calculate the indicated quantity.

17. 15% of 1000 **18.** 20% of 500 **19.** 18% of 100

20. 10% of 50 **21.** 210% of 50 **22.** 135% of 1000

23. What percent of 80 is 4? **24.** What percent of 60 is 5? **25.** What percent of 5 is 8?

26. What percent of 25 is 45? **27.** 20 is 8% of what number? **28.** 25 is 12% of what number?

29. 50 is 15% of what number? **30.** 40 is 18% of what number?

In Problems 31–36, find the interest due on each loan.

31. \$1000 is borrowed for 3 months at 4% simple interest.

32. \$100 is borrowed for 6 months at 8% simple interest.

33. \$500 is borrowed for 9 months at 12% simple interest.

34. \$800 is borrowed for 8 months at 5% simple interest.

35. \$1000 is borrowed for 18 months at 10% simple interest.

36. \$100 is borrowed for 24 months at 12% simple interest.

In Problems 37–42, find the simple interest rate for each loan.

37. \$1000 is borrowed; the amount owed after 6 months is \$1050.

38. \$500 is borrowed; the amount owed after 8 months is \$600.

39. \$300 is borrowed; the amount owed after 12 months is \$400.

40. \$600 is borrowed; the amount owed after 9 months is \$660.

41. \$900 is borrowed; the amount owed after 10 months is \$1000.

42. \$800 is borrowed; the amount owed after 3 months is \$900.

In Problems 43–46, find the proceeds for each discounted loan.

43. \$1200 repaid in 6 months at 10%.

44. \$500 repaid in 8 months at 9%.

45. \$2000 repaid in 24 months at 8%.

46. \$1500 repaid in 18 months at 10%.

In Problems 47–50, find the amount you must repay for each discounted loan. What is the equivalent simple interest for this loan?

47. The proceeds is \$1200 for 6 months at 10%.

48. The proceeds is \$500 for 8 months at 9%.

49. The proceeds is \$2000 for 24 months at 8%.

50. The proceeds is \$1500 for 18 months at 10%.

51. **Buying a Stereo** Madalyn wants to buy a \$500 stereo set in 9 months. How much should she invest at 3% simple interest to have the money then?

52. **Interest on a Loan** Mike borrows \$10,000 for a period of 3 years at a simple interest rate of 10%. Determine the interest due on the loan.

53. **Term of a Loan** Tami borrowed \$600 at 8% simple interest. The amount of interest paid was \$156. What was the length of the loan?

54. **Equipment Loan** The owner of a restaurant would like to borrow \$12,000 from a bank to buy some equipment. The bank will give the owner a discounted loan at an 11% rate of interest for 9 months. What maturity value should be used so that the owner will receive \$12,000?

55. **Comparing Loans** You need to borrow \$1000 right now, but can repay the loan in 6 months. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 9% per annum or a simple interest loan at 10% per annum?

56. **Comparing Loans** You need to borrow \$5000 right now, but can repay the loan in 9 months. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 8% per annum or a simple interest loan at 8.5% per annum?

57. **Comparing Loans** You need to borrow \$4000 right now, but can repay the loan in 1 year. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 6% per annum or a simple interest loan at 6.3% per annum?

58. **Comparing Loans** You need to borrow \$5000 right now, but can repay the loan in 18 months. Since you want to pay as little interest as possible, which type of loan should you take: a discounted loan at 5.3% per annum or a simple interest loan at 5.6% per annum?

59. **Comparing Loans** Ruth would like to borrow \$2000 for one year from a bank. She is given a choice of a simple interest loan at 12.3% or a discounted loan at 12.1%. What should she do?

60. Refer to Problem 59. If Ruth only needs to borrow the \$2000 for 3 months, what should she do?

61. **Bidding on Treasury Bills** A bank wants to earn 2% simple interest on a 3-month \$1 million treasury bill. How much should they bid?

62. **Bidding on Treasury Bills** How much should a bank bid on a 6-month \$3 million treasury bill to earn 3% simple interest?

63. **T-Bill Auctions** As the result of a treasury bill auction, 6-month T-bills were issued on January 16, 2003, at a price of \$993.78 per \$1000 face value. How much interest would be received on the maturity date, July 17, 2003, for an investment of \$10,000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?

Source: United States Treasury.

64. T-Bill Auctions As the result of a treasury bill auction, 3-month T-bills were issued on January 30, 2003, at a price of $997.12 per $1000 face value. How much interest would be received on the maturity date, May 1, 2003, for an investment of $100,000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?

Source: United States Treasury.

65. T-Bill Auctions As the result of a treasury bill auction, 1-month T-bills were issued on January 23, 2003, at a price of $999.12 per $1000 face value. How much interest would be received on the maturity date, February 20, 2003, for an investment of $5000? What is the per annum simple interest rate for this investment, rounded to the nearest thousandth of a percent?

Source: United States Treasury.

66. Price of Dell Computer Corporation Stock Year-end stock prices for one share of Dell stock are as follows:

1997:	$84.00
1998:	$73.19
1999:	$51.00
2000:	$17.44
2001:	$27.18
2002:	$26.74

Taking into account that there was a two-for-one stock split of Dell shares in March 1998, again in September 1998, and again in March 1999, for each year after 1997, determine the annual percent increase or decrease in the value of Dell stock beginning with one share of stock at the end of 1997. In which year was the percent increase the greatest? In which year was the percent decrease the greatest?

Source: Yahoo!; Finance; Historical Prices.

5.2 Compound Interest

PREPARING FOR THIS SECTION *Before getting started, review the following:*

> Exponents and Logarithms (Appendix A. Section A. 3 pp. xx–xx)

OBJECTIVES
1. Solve problems involving compound interest
2. Find the effective rate of interest
3. Solve problems involving present value

Compound Interest Formula

In working with problems involving interest, we use the term **payment period** as follows:

Annually	Once per year
Semiannually	Twice per year
Quarterly	4 times per year
Monthly	12 times per year
Daily	365 times per year*

If the interest due at the end of each payment period is added to the principal, so that the interest computed for the next payment period is based on this new amount of the old principal plus interest, then the interest is said to have been **compounded.** That is, **compound interest** is interest paid on the initial principal and previously earned interest.

**Some banks use 360 times per year.*

EXAMPLE 1 Computing Compound Interest

A bank pays interest of 4% per annum compounded quarterly. If \$200 is placed in a savings account and the quarterly interest is left in the account, how much money is in the account after 1 year?

SOLUTION At the end of the first quarter (3 months) the interest earned is

$$I = Prt = (\$200)(0.04)\left(\frac{1}{4}\right) = \$2.00$$

The new principal is $P + I = \$200 + \$2 = \$202$. The interest on this principal at the end of the second quarter is

$$I = (\$202)(0.04)\left(\frac{1}{4}\right) = \$2.02$$

The interest at the end of the third quarter on the principal of $\$202 + \$2.02 = \$204.02$ is

$$I = (\$204.02)(0.04)\left(\frac{1}{4}\right) = \$2.04$$

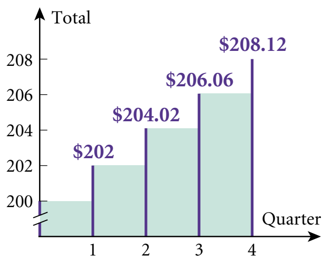
The interest at the end of the fourth quarter on the principal of $\$204.02 + \$2.04 = \$206.06$ is

$$I = (\$206.06)(0.04)\left(\frac{1}{4}\right) = \$2.06$$

After 1 year, the total in the savings account is $\$206.06 + \$2.06 = \$208.12$.

These results are shown in Figure 1.

FIGURE 1



Let's develop a formula for computing the amount when interest is compounded. Suppose r is the per annum rate of interest compounded each payment period. Then the rate of interest per payment period is

$$i = \frac{\text{Per annum rate of interest}}{\text{Number of payment periods}}$$

For example, if the annual rate of interest is 10% and the compounding is monthly, then there are 12 payment periods per year and

$$i = \frac{0.10}{12} = 0.00833$$

If 18% is the annual rate compounded daily (365 payment periods), then

$$i = \frac{0.18}{365} = 0.000493$$

If P is the principal and i is the interest rate per payment period, then the amount A_1 at the end of the first payment period is

$$A_1 = P + P_i = P(1 + i)$$

At the end of the second payment period, and subsequent ones, the amounts are

$$A_2 = A_1 + A_1 i = A_1(1 + i) = P(1 + i)(1 + i) = P(1 + i)^2$$
$$A_3 = A_2 + A_2 i = A_2(1 + i) = P(1 + i)^2(1 + i) = P(1 + i)^3$$
$$\vdots$$
$$A_n = A_{n-1} + A_{n-1} i = A_{n-1}(1 + i) = P(1 + i)^{n-1}(1 + i) = P(1 + i)^n$$

Compound Interest Formula

The amount A_n accrued on a principal P after n payment periods at i interest per payment period is

$$\boxed{A_n = P(1 + i)^n} \qquad (1)$$

In working with the Compound Interest Formula (1), we use a calculator with a $\boxed{y^x}$ key.* To use this key, enter the value of y, press $\boxed{y^x}$, enter the value of x, and press $\boxed{=}$ or $\boxed{\text{ENTER}}$.

1 **EXAMPLE 2 Working with the Compound Interest Formula**

If \$1000 is invested at an annual rate of interest of 10%, what is the amount after 5 years if the compounding takes place

(a) Annually? **(b)** Monthly? **(c)** Daily?

How much interest is earned in each case?

SOLUTION We use the Compound Interest Formula (1). The principal is $P = \$1000$.

(a) For annual compounding, $i = 0.10$ and $n = 5$. The amount A is

$$A = P(1 + i)^n = (\$1000)(1 + 0.10)^5 = (\$1000)(1.61051) = \$1610.51$$

The interest earned is

$$A - P = \$1610.51 - \$1000.00 = \$610.51$$

(b) For monthly compounding, there are $5 \cdot 12 = 60$ payment periods over 5 years. The interest rate per payment period is $i = \frac{0.10}{12}$. The amount A is

$$A = P(1 + i)^n = (\$1000)\left(1 + \frac{0.10}{12}\right)^{60} = (\$1000)(1.64531) = \$1645.31$$

The interest earned is

$$A - P = \$1645.31 - \$1000.00 = \$645.31$$

*On a graphing calculator, the key used is $\boxed{\wedge}$.

(c) For daily compounding, there are $5 \cdot 365 = 1825$ payment periods over 5 years. The interest rate per payment period is $i = \frac{0.10}{365}$. The amount A is

$$A = P(1 + i)^n = (\$1000)\left(1 + \frac{0.10}{365}\right)^{1825} = (\$1000)(1.64861) = \$1648.61$$

The interest earned is

$$A - P = \$1648.61 - \$1000.00 = \$648.61$$

The results of Example 2 are summarized in Table 1.

TABLE 1

Per Annum Rate	Compounding Method	Interest Rate per Payment Period	Initial Principal	Amount after 5 Years	Interest Earned
10%	Annual	0.10	$1000	$1610.51	$610.51
10%	Monthly	0.00833	$1000	$1645.31	$645.31
10%	Daily	0.000274	$1000	$1648.61	$648.61

NOW WORK PROBLEM 1.

EXAMPLE 3 Comparing Compound Rates of Interest with Simple Interest

(a) If $100 is invested at an annual rate of interest of 10% compounded monthly, what is the interest earned after 1 year?

(b) What simple rate of interest is required to obtain this amount of interest?

SOLUTION **(a)** We begin with a principal of $100 and proceed to compute the amount after 1 year at 10% compounded monthly. For monthly compounding at 10%, there are 12 payment periods, and the interest rate per period is $i = \frac{0.10}{12}$. The amount A is

$$A = P(1 + i)^n = \$100\left(1 + \frac{0.10}{12}\right)^{12} = \$110.47$$

The interest earned is $110.47 − $100 = $10.47.

(b) The simple interest rate r required to earn interest of $10.47 on a principal of $100 after one year obeys the equation

$$\begin{aligned} I &= Prt && \text{Simple Interest Formula} \\ 10.47 &= (100)r(1) && I = 10.47;\ P = 100;\ t = 1 \\ r &= .1047 && \text{Solve for } r \end{aligned}$$

So, a simple interest rate of 10.47% is required to obtain interest equal to that obtained using 10% compounded monthly.

Let's look at the effect of various compounding periods on a principal of $100 after 1 year using a rate of interest of 8% per annum.

Annual compounding:	$A = \$100(1 + 0.08)^1$	$= \$108.00$
Semiannual compounding:	$A = \$100\left(1 + \frac{0.08}{2}\right)^2$	$= \$108.16$
Quarterly compounding(s)	$A = \$100\left(1 + \frac{0.08}{4}\right)^4$	$= \$108.24$
Monthly compounding:	$A = \$100\left(1 + \frac{0.08}{12}\right)^{12}$	$= \$108.30$
Daily compounding:	$A = \$100\left(1 + \frac{0.08}{365}\right)^{365}$	$= \$108.33$

With semiannual compounding, the amount \$108.16 could have been obtained with a simple interest of 8.16%. We describe this by saying that the *effective rate of interest* of 8% compounded semiannually is 8.16%.

Find the effective rate of interest **2**

In general, the **effective rate of interest** is the equivalent annual simple rate of interest that yields the same amount as compounding does after one year.

Table 2 summarizes the calculations given above for 8%.

TABLE 2

Rate/Compounding	Period Effective Rate of Interest
8% Compounded Semiannually	8.16%
8% Compounded Quarterly	8.24%
8% Compounded Monthly	8.3%
8% Compounded Daily	8.33%

NOW WORK PROBLEM 17.

EXAMPLE 4 Comparing Certificates of Deposit

Three local banks offer the following 1-year certificates of deposit (CDs):

(a) Simple interest of 5.2% per annum
(b) 5% per annum compounded monthly
(c) $4\frac{3}{4}$% per annum compounded daily

Which CD results in the most interest?

SOLUTION To compare the three CDs, we compute the amount \$1000 (any other amount could also be used) will grow to in each case.

(a) At simple interest of 5.2%, \$1000 will grow to

$$A = P + Prt = \$1000 + \$1000(0.052)(1) = \$1052.00$$

(b) There are 12 payment periods and the rate of interest per payment period is $i = \frac{0.05}{12}$. The amount A that \$1000 will grow to is

$$A = P(1 + i)^n = \$1000\left(1 + \frac{0.05}{12}\right)^{12} = \$1051.16$$

(c) There are 365 payment periods and the rate of interest per payment period is $i = \frac{0.0475}{365}$. The amount A that \$1000 will grow to is

$$A = P(1 + i)^n = \$1000\left(1 + \frac{0.0475}{365}\right)^{365} = \$1048.64$$

The CD offering 5.2% simple interest results in the most interest earned.

NOW WORK PROBLEM 31.

Present Value

Solve problems involving present value **3**

The Compound Interest Formula states that a principal P earning an interest rate per payment period i will, after n payment periods, be worth the amount A, where

$$A = P(1 + i)^n$$

If we solve for P, we obtain

$$P = \frac{A}{(1 + i)^n} = A(1 + i)^{-n} \quad (2)$$

In this formula P is called the **present value** of the amount A due at the end of n interest periods at i interest per payment period. In other words, P is the amount that must be invested for n interest periods at i interest per payment period in order to accumulate the amount A.

The Compound Interest Formula and the Present Value Formula can be used to solve many different kinds of problems. The examples below illustrate some of these applications.

EXAMPLE 5 Computing the Present Value of $10,000

How much money should be invested now at 8% per annum so that after 2 years the amount will be $10,000 when the interest is compounded

(a) Annually? **(b)** Monthly? **(c)** Daily?

SOLUTION In this problem we want to find the principal P needed now to get the amount $A =$ $10,000 after 2 years. That is, we want to find the present value of $10,000.

(a) Since compounding is once per year for 2 years, $n = 2$ and $i = 0.08$. The present value P of $10,000 is

$$P = A(1 + i)^{-n} = 10{,}000(1 + 0.08)^{-2} = 10{,}000(0.8573388) = \$8573.39$$

(b) Since compounding is 12 times per year for 2 years, $n = 24$ and $i = \frac{0.08}{12}$. The present value P of $10,000 is

$$P = A(1 + i)^{-n} = 10{,}000\left(1 + \frac{0.08}{12}\right)^{-24} = 10{,}000(0.852596) = \$8525.96$$

(c) Since compounding is 365 times per year for 2 years, $n = 730$ and $i = \frac{0.08}{365}$. The present value P of $10,000 is

$$P = A(1 + i)^{-n} = 10{,}000\left(1 + \frac{0.08}{365}\right)^{-730} = 10{,}000(0.8521587) = \$8521.59$$

NOW WORK PROBLEM 7.

EXAMPLE 6 Finding the Rate of Interest to Double an Investment

What annual rate of interest compounded annually should you seek if you want to double your investment in 5 years?

SOLUTION If P is the principal and we want P to double, the amount A will be $2P$. We use the Compound Interest Formula (1) with $n = 5$ to find i:

$$A = P(1+i)^n \quad \text{Formula (1)}$$
$$2P = P(1+i)^5 \quad A = 2P,\ n = 5$$
$$2 = (1+i)^5 \quad \text{Cancel the } Ps.$$
$$\sqrt[5]{2} = 1 + i \quad \text{Take the 5th root of each side.}$$
$$i = \sqrt[5]{2} - 1 = 1.148698 - 1 = 0.148698 \quad \text{Solve for } i.$$

$\sqrt[5]{2} = 2^{1/5} = 2^{0.2}$
Use the $\boxed{y^x}$ key on your calculator.

The annual rate of interest needed to double the principal in 5 years is 14.87%.

NOW WORK PROBLEM 19.

EXAMPLE 7 Finding the Time Required to Double/Triple an Investment*

(a) How long will it take for an investment to double in value if it earns 5% compounded monthly?
(b) How long will it take to triple at this rate?

SOLUTION **(a)** If P is the initial investment and we want P to double, the amount A will be $2P$. We use the Compound Interest Formula (1) with $i = \frac{0.05}{12}$. Then

$$A = P(1+i)^n \quad \text{Formula (1)}$$
$$2P = P\left(1 + \frac{0.05}{12}\right)^n \quad A = 2P,\ i = \frac{0.05}{12}$$
$$2 = (1.0041667)^n \quad \text{Simplify.}$$
$$n = \log_{1.0041667} 2 = \frac{\log_{10} 2}{\log_{10} 1.0041667} = 166.7 \text{ months}$$

Apply the definition of a logarithm. — Change-of-base formula — Payment period measured in months

It will take about 13 years 11 months to double the investment.

(b) To triple the investment, the amount A is $3P$. Thus,

$$A = P(1+i)^n \quad \text{Formula (1)}$$
$$3P = P\left(1 + \frac{0.05}{12}\right)^n \quad A = 3P,\ i = \frac{0.05}{12}$$
$$3 = (1.0041667)^n \quad \text{Simplify.}$$
$$n = \log_{1.0041667} 3 = \frac{\log_{10} 3}{\log_{10} 1.0041667} = 264.2 \text{ months}$$

Apply the definition of a logarithm

It will take about 22 years to triple the investment.

NOW WORK PROBLEM 21.

Requires a knowledge of logarithms, especially the Change-of-Base Formula. See Appendix A, Section A.3, for a review.

EXERCISE 5.2 Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–6, find the amount if:

1. \$1000 is invested at 4% compounded monthly for 36 months.
2. \$100 is invested at 6% compounded monthly for 20 months.
3. \$500 is invested at 5% compounded annually for 3 years.
4. \$200 is invested at 10% compounded annually for 10 years.
5. \$800 is invested at 6% compounded daily for 200 days.
6. \$400 is invested at 7% compounded daily for 180 days.

In Problems 7–10, find the principal needed now to get each amount.

7. To get \$100 in 6 months at 4% compounded monthly
8. To get \$500 in 1 year at 6% compounded annually
9. To get \$500 in 1 year at 7% compounded daily
10. To get \$800 in 2 years at 5% compounded monthly.

---

11. If \$1000 is invested at 4% compounded

    (a) Annually (b) Semiannually
    (c) Quarterly (d) Monthly

    what is the amount after 3 years? How much interest is earned?

12. If \$2000 is invested at 5% compounded

    (a) Annually (b) Semiannually
    (c) Quarterly (d) Monthly

    what is the amount after 5 years? How much interest is earned?

13. If \$1000 is invested at 6% compounded quarterly, what is the amount after

    (a) 2 years? (b) 3 years? (c) 4 years?

14. If \$2000 is invested at 4% compounded quarterly, what is the amount after

    (a) 2 years? (b) 3 years? (c) 4 years?

15. If a bank pays 3% compounded semiannually, how much should be deposited now to have \$5000

    (a) 4 years later? (b) 8 years later?

16. If a bank pays 2% compounded quarterly, how much should be deposited now to have \$10,000

    (a) 5 years later? (b) 10 years later?

17. Find the effective rate of interest for

    (a) 8% compounded semiannually
    (b) 4% compounded monthly

18. Find the effective rate of interest for

    (a) 6% compounded monthly
    (b) 14% compounded semiannually

19. What annual rate of interest compounded annually is required to double an investment in 3 years?
20. What annual rate of interest compounded annually is required to double an investment in 10 years?
21. Approximately how long will it take to triple an investment at 10% compounded annually?
22. Approximately how long will it take to triple an investment at 9% compounded annually?
23. Mr. Nielsen wants to borrow \$1000 for 2 years. He is given the choice of (a) a simple interest loan of 12% or (b) a loan at 10% compounded monthly. Which loan results in less interest due?
24. Rework Problem 23 if the simple interest loan is 15% and the other loan is at 14% compounded daily.
25. What principal is needed now to get \$1000 in 1 year at 9% compounded annually? How much should be invested to get \$1000 in 2 years?
26. Repeat Problem 25 using 9% compounded daily.
27. Find the effective rate of interest for $5\frac{1}{4}$% compounded quarterly.
28. Repeat Problem 27 using 6% compounded quarterly.
29. What interest rate compounded quarterly will give an effective interest rate of 7%?
30. Repeat Problem 29 using 10%.

In Problems 31–34, which of the two rates would yield the larger amount in 1 year? Hint: Start with a principal of $10,000 in each instance.

31. 6% compounded quarterly or $6\frac{1}{4}$% compounded annually

32. 9% compounded quarterly or $9\frac{1}{4}$% compounded annually

33. 9% compounded monthly or 8.8% compounded daily

34. 8% compounded semiannually or 7.9% compounded daily

35. Future Price of a Home If the price of homes rises an average of 5% per year for the next 4 years, what will be the selling price of a home that is selling for $90,000 today 4 years from today? Express your answer rounded to the nearest hundred dollars.

36. Amount Due on a Charge Card A department store charges 1.25% per month on the unpaid balance for customers with charge accounts (interest is compounded monthly). A customer charges $200 and does not pay her bill for 6 months. What is the bill at that time?

37. Amount Due on a Charge Card A major credit card company has a finance charge of 1.5% per month on the outstanding indebtedness. Caryl charged $600 and did not pay her bill for 6 months. What is the bill at that time?

38. Buying a Car Laura wishes to have $8000 available to buy a car in 3 years. How much should she invest in a savings account now so that she will have enough if the bank pays 8% interest compounded quarterly?

39. Cost of College—Public The average annual undergraduate college costs (tuition, room and board) for 2000–2001 for public 4-year institutions were $8655 (in-state tuition: $3506; room and board: $5149). The Bureau of Labor Statistics indicated that the inflation rate for tuition in 2001 was 5.09%, while the general inflation rate for room and board (Consumer Price Index) was 2.85%. Assuming that these inflation rates remain constant for the next 5 years, determine the projected annual undergraduate college costs for public 4-year institutions for 2005–2006.

Source: United States Department of Education, National Center for Education Statistics, Integrated Postsecondary Education Data System (IPEDS), "Fall Enrollment" and "Institutional Characteristics" survey (August 2001); FinAid.org, "The Smart Student Guide to Financial Aid"; and U. S. Department of Labor, Bureau of Labor Statistics.

40. Cost of College—Public The average annual undergraduate college costs (tuition, room and board) for 2000–2001 for private 4-year institutions were $21,907 (tuition: $15,531; room and board: $6376). The Bureau of Labor Statistics indicated that the inflation rate for tuition in 2001 was 5.09%, while the general inflation rate for room and board (Consumer Price Index) was 2.85%. Assuming that these inflation rates are constant for the next 5 years, determine the projected annual undergraduate college costs for private 4-year institutions for 2005–2006.

Source: United States Department of Education, National Center for Education Statistics, Integrated Postsecondary Education Data System (IPEDS), "Fall Enrollment" and "Institutional Characteristics" surveys (August, 2001); FinAid.org. "The Smart Student Guide to Financial Aid"; and U.S. Department of Labor, Bureau of Labor Statistics.

41. Down Payment on a House Tami and Todd will need $40,000 for a down payment on a house in 4 years. How much should they invest in a savings account now so that they will be able to do this? The bank pays 8% compounded quarterly.

42. Saving for College A newborn child receives a $3000 gift toward a college education. How much will the $3000 be worth in 17 years if it is invested at 10% compounded quarterly?

43. Gifting A child's grandparents have opened a $6000 savings account for the child on the day of her birth. The account pays 8% compounded semiannually. The child will be allowed to withdraw the money when she reaches the age of 25. What will the account be worth at that time?

44. Future Price of a House What will a $90,000 house cost 5 years from now if the inflation rate over that period averages 5% compounded annually? Express your answer rounded to the nearest hundred dollars.

45. Deciding on a Stock Purchase Jack is considering buying 1000 shares of a stock that sells at $15 per share. The stock pays no dividends. From the history of the stock, Jack is certain that he will be able to sell it 4 years from now at $20 per share. Jack's goal is not to make any investment unless it returns at least 7% compounded quarterly. Should Jack buy the stock?

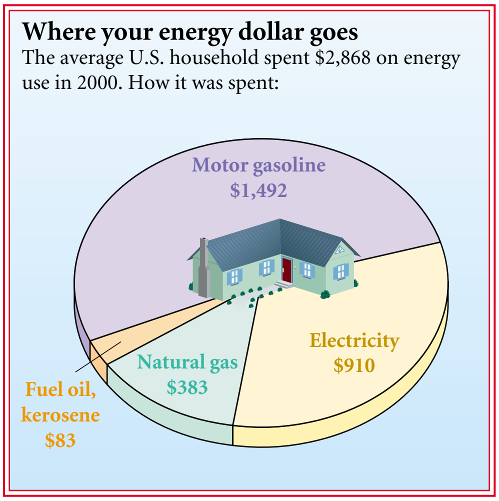
46. Repeat Problem 47 if Jack requires a return of at least 14% compounded quarterly.

47. Value of an IRA An Individual Retirement Account (IRA) has $2000 in it, and the owner decides not to add any more money to the account other than the interest earned at 9% compounded quarterly. How much will be in the account 25 years from the day the account was opened?

48. Return on Investment If Jack sold a stock for $35,281.50 (net) that cost him $22,485.75 three years ago, what annual compound rate of return did Jack make on his investment?

49. National Debt The national debt was approximately $4.8 trillion on January 1, 1995, and approximately $6.406 trillion on January 1, 2003. Find the annual rate of growth of the national debt in this 8-year period, assuming a constant annual rate of growth. If this rate of growth continues, what would be the projected national debt on January 1, 2010?

Source: United States Treasury Department.



Source: Energy Information Administration

50. Cost of Energy The average U.S. household spent $2868 on energy use in the year 2000. This included $1492 for gasoline, $910 for electricity, $383 for natural gas, and $83 for fuel oil and kerosene. Assuming a constant annual inflation rate of 2.85%, which was the general inflation rate in 2001, determine the projected amounts that the average U. S. household will spend on each of the forms of energy listed above in the year 2005.

Source: *USA Today*, 12/1/02, based on Energy Information Administration data; and the U.S. Department of Labor, Bureau of Labor Statistics.

*For Problems 51–53, zero coupon bonds are used. A **zero coupon bond** is a bond that is sold now at a discount and will pay its face value at some time in the future when it matures; no interest payments are made.*

51. Saving for College Tami's grandparents are considering buying a $40,000 face value zero coupon bond at birth so that she will have enough money for her college education 17 years later. If money is worth 8% compounded annually, what should they pay for the bond?

52. Price of a Bond How much should a $10,000 face value zero coupon bond, maturing in 10 years, be sold for now if its rate of return is to be 8% compounded annually?

53. Rate of Return of a Bond If you pay $12,485.52 for a $25,000 face value zero coupon bond that matures in 8 years, what is your annual compound rate of return?

54. Effective Rates of Interest A bank advertises that it pays interest on saving accounts at the rate of 6.25% compounded daily.

(a) Find the effective rate if the bank uses 360 days in determining the daily rate.
(b) What if 365 days is used?

Problems 55 and 56 require logarithms.

55. Length of Investment How many years will it take for an initial investment of $10,000 to grow to $25,000? Assume a rate of interest of 6% compounded daily.

56. Length of Investment How many years will it take for an initial investment of $25,000 to grow to $80,000? Assume a rate of interest of 7% compounded daily.

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*Use the following discussion for Problems 57–64. **Inflation** erodes the purchasing power of money. For example, suppose there is an annual rate of inflation of 3%. Then $1000 worth of purchasing power now will be worth only $970 in one year. In general, for an annual rate of inflation of r%, the amount A that $P will purchase after n years is*

$$A = P(1 - r)^n$$

where r is a decimal.

57. Suppose the inflation rate is 3%. After 2 years, how much will $1000 purchase?

58. Suppose the inflation rate is 4%. After 2 years, how much will $1000 purchase?

59. Suppose the inflation rate is 3%. After 5 years, how much will $1000 purchase?

60. Suppose the inflation rate is 4%. After 5 years, how much will $1000 purchase?

Problems 61–64 require logarithms.

61. Suppose the inflation rate is 3%. How long is it until purchasing power is halved?

62. Suppose the inflation rate is 4%. How long is it until purchasing power is halved?

63. Suppose the inflation rate is 6%. How long is it until purchasing power is halved?

64. Suppose the inflation rate is 9%. How long is it until purchasing power is halved?

5.3 Annuities; Sinking Funds

PREPARING FOR THIS SECTION *Before getting started, review the following:*

> Exponents and Logarithms (Appendix A, Section A.3, pp. xx–xx)

> Geometric Sequences (Appendix A, Section A.2, pp. xx–xx)

OBJECTIVES
1. Solve problems involving annuities
2. Solve problems involving sinking funds

Annuities

Solve problems involving annuities 1

In the previous section we saw how to compute the future value of an investment when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, however, people and financial institutions do not deposit money and then sit back and watch it grow. Rather, money is invested in small amounts at periodic intervals. Examples of such investments are annual life insurance premiums, monthly deposits in a bank, installment loan payments, and dollar averaging in the stock market with 401(k) or 403(b) accounts.

An **annuity** is a sequence of equal periodic deposits. The periodic deposits can be annual, semiannual, quarterly, monthly, or any other fixed length of time. When the deposits are made at the same time the interest is credited, the annuity is termed **ordinary.** We shall concern ourselves only with ordinary annuities in this book.

The **amount of an annuity** is the sum of all deposits made plus all interest accumulated.

EXAMPLE 1 Finding the Amount of an Annuity

Find the amount of an annuity after 5 deposits if each deposit is equal to \$100 and is made on an annual basis at an interest rate of 10% per annum compounded annually.

SOLUTION After 5 deposits the first \$100 deposit will have accumulated interest compounded annually at 10% for 4 years. Using the Compound Interest Formula, the value A_1 of the first deposit of \$100 after 4 years is

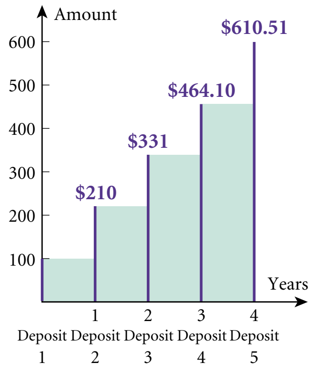
$$A_1 = \$100(1 + 0.10)^4 = \$100(1.4641) = \$146.41$$

The second deposit of \$100, made 1 year after the first deposit, will accumulate interest compounded at 10% for 3 years. Its value A_2 after 3 years is

$$A_2 = \$100(1 + 0.10)^3 = \$100(1.331) = \$133.10$$

Similarly, the third, fourth, and fifth deposits will have the values

FIGURE 2



$$A_3 = \$100(1 + 0.10)^2 = \$100(1.21) = \$121.00$$
$$A_4 = \$100(1 + 0.10)^1 = \$100(1.10) = \$110.00$$
$$A_5 = \$100$$

The amount of the annuity after 5 deposits is

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 + A_5 &= \$146.41 + \$133.10 + \$121.00 \\ &\quad + \$110.00 + \$100.00 \\ &= \$610.51 \end{aligned}$$

Figure 2 illustrates the growth of the annuity described above.

We seek a formula for finding the amount of an annuity. Suppose that the interest an annuity earns is i percent per payment period (expressed as a decimal). For example, if an account pays 12% compounded monthly (12 times a year) then $i = \frac{0.12}{12} = 0.01$. If an account pays 8% compounded quarterly (4 times a year) then $i = \frac{0.08}{4} = 0.02$.

Now, suppose that n deposits of \$$P$ each are made at the beginning of each payment period. When the last deposit is made, the first deposit of \$$P$ has earned interest compounded for $n - 1$ payment periods, the second deposit of \$$P$ has earned interest compounded for $n - 2$ payment periods, and so on. Table 3 shows the value of each deposit after n deposits have been made.

TABLE 3

Deposit	1	2	3	. . .	$n - 1$	n
Amount	$P(1 + i)^{n-1}$	$P(1 + i)^{n-2}$	$P(1 + i)^{n-3}$	. . .	$P(1 + i)$	P

The amount A of the annuity is the sum of the amounts shown in Table 3; that is,

$$\begin{aligned} A &= P(1 + i)^{n-1} + P(1 + i)^{n-2} + \cdots + P(1 + i) + P \\ &= P[1 + (1 + i) + \cdots + (1 + i)^{n-1}] \end{aligned}$$

The expression in brackets is the sum of a geometric sequence* with n terms and common ratio $1 + i$. As a result,

$$1 + (1 + i) + \cdots + (1 + i)^{n-1} = \frac{1 - (1 + i)^n}{1 - (1 + i)} = \frac{1 - (1 + i)^n}{-i} = \frac{(1 + i)^n - 1}{i}$$

We have established the following result.

Amount of an Annuity

Suppose P is the deposit in dollars made at each payment period for an annuity paying i percent interest per payment period. The amount A of the annuity after n deposits is

$$\mathbf{A = P\frac{(1 + i)^n - 1}{i}} \qquad (1)$$

**The sum of the first n terms of the geometric sequence with common ratio r is*

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

See Appendix A, Section A.2, for a more detailed discussion.

Be careful in using Formula (1). Formula (1) gives the amount A of an annuity after n deposits. The nth deposit is made after $n - 1$ compounding periods. Refer to Table 3.

Let's do some examples.

EXAMPLE 2 Finding the Amount of an Annuity

Find the amount of an annuity after 5 years if a deposit of \$100 is made each year, at 10% compounded annually. How much interest is earned?

SOLUTION The deposit is $P = \$100$. The number of deposits is $n = 5$ and the interest per payment period is $i = 0.10$. Using Formula (1), the amount A after 5 years is

$$A = P\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{(1+0.10)^5 - 1}{0.10}\right] = \$100(6.1051) = \$610.51$$

The interest earned is the amount after 5 years less the 5 annual payments of \$100 each:

$$\text{Interest earned} = A - 500 = 610.51 - 500 = \$110.51$$

NOW WORK PROBLEM 1.

EXAMPLE 3 Finding the Amount of an Annuity

Mary decides to put aside \$100 every month in an insurance fund that pays 8% compounded monthly. After making 8 deposits, how much money does Mary have?

SOLUTION This is an annuity with $P = \$100$, $n = 8$ deposits, and interest $i = \frac{0.08}{12}$ per payment period. Using Formula (1), the amount A after 8 deposits is

$$A = P\left[\frac{(1+i)^n - 1}{i}\right] = 100\left[\frac{\left(1 + \frac{0.08}{12}\right)^8 - 1}{\frac{0.08}{12}}\right] = \$100(8.1892) = \$818.92$$

Mary has \$818.92 after making 8 deposits.

EXAMPLE 4 Saving for College

To save for his son's college education, Mr. Graff decides to put \$50 aside every month in a credit union account paying 10% interest compounded monthly. If he begins this savings program when his son is 3 years old, how much will he have saved by the time his son is 18 years old?

SOLUTION When his son is 18 years old, Mr. Graff will have made his 180th deposit (15 years × 12 deposits per year). This is an annuity with $P = \$50$, $n = 180$ deposits, and interest $i = \frac{0.10}{12}$ per payment period. The amount A saved is

$$A = 50\left[\frac{\left(1 + \frac{0.10}{12}\right)^{180} - 1}{\frac{0.10}{12}}\right] = \$50(414.4703) = \$20{,}723.52$$

When his son is 18 years old, Mr. Graff will have saved \$20,723.52.

NOW WORK PROBLEM 21.

EXAMPLE 5 Funding an IRA

Joe, at age 35, decides to invest in an IRA. He will put aside \$2000 per year for the next 30 years. How much will he have at age 65 if his rate of return is assumed to be 10% per annum?

SOLUTION This is an annuity with $P = \$2000$, $n = 30$ deposits, and interest $i = 0.10$ per payment period. The amount A in Joe's IRA after 30 years is

$$A = 2000\left[\frac{(1 + 0.10)^{30} - 1}{0.10}\right] = \$2000(164.49402) = \$328{,}988.05$$

Under this plan, Joe will have \$328,988.05 in his IRA when he is 65.

EXAMPLE 6 Funding an IRA

If, in Example 5, Joe had begun his IRA at age 25, instead of 35, what would his IRA be worth at age 65?

SOLUTION Now there are $n = 40$ deposits of \$2000 per year. The amount A of this annuity is

$$A = 2000\left[\frac{(1 + 0.10)^{40} - 1}{0.10}\right] = \$2000(442.59256) = \$885{,}185.11$$

Joe will have \$885,185.11 in his IRA when he is 65 if he begins the IRA at age 25.

NOW WORK PROBLEM 23.

EXAMPLE 7 Finding the Time it Takes to Reach a Certain Savings Goal*

How long does it take to save \$500,000 if you place \$500 per month in an account paying 6% compounded monthly?

SOLUTION This is an annuity in which the amount A is \$500,000, each deposit is $P = \$500$, and the interest is $i = \frac{0.06}{12}$ per payment period. We seek the number n of deposits needed to reach \$500,000.

**This example requires a knowledge of logarithms, especially the Change-of-Base Formula. A review of logarithms appears in Appendix A, Section A.3.*

$$A = P\frac{(1+i)^n - 1}{i}$$ Formula (1)

$$500{,}000 = 500\,\frac{\left(1+\frac{0.06}{12}\right)^n - 1}{\frac{0.06}{12}}$$ $A = 500{,}000;\ P = 500;\ i = \frac{0.06}{12}$

$$5 = \left(1+\frac{0.06}{12}\right)^n - 1$$ Simplify

$$6 = (1.005)^n$$ Simplify

$$n = \log_{1.005} 6 = \frac{\log_{10} 6}{\log_{10} 1.005} = 359.25 \text{ months}$$

↑ Apply the definition of a logarithm. ↑ Change-of-base formula ↑ Payment period in months

It takes $359\frac{1}{4}$ months (almost 30 years) to save the $500,000.

NOW WORK PROBLEM 35.

Sinking Funds

Solve problems involving sinking funds 2

A person with a debt may decide to accumulate sufficient funds to pay off the debt by agreeing to set aside enough money each month (or quarter, or year) so that when the debt becomes payable, the money set aside each month plus the interest earned will equal the debt. The fund created by such a plan is called a **sinking fund**.

In working with sinking funds, we generally seek the deposit or payment required to reach a certain goal. In other words, we seek the payment P required to obtain the amount A in Formula (1).

Let's look at an example.

EXAMPLE 8 Funding a Bond Obligation

The Board of Education received permission to issue $4,000,000 in bonds to build a new high school. The board is required to make payments every 6 months into a sinking fund paying 8% compounded semiannually. At the end of 12 years the bond obligation will be retired. What should each payment be?

SOLUTION This is an example of a sinking fund. The payment P required twice a year to accumulate $4,000,000 in 12 years (24 payments at a rate of interest of $i = \frac{0.08}{2}$ per payment period) obeys Formula (1).

$$A = P\frac{(1+i)^n - 1}{i}$$ Formula (1)

$$4{,}000{,}000 = P\,\frac{\left(1+\frac{0.08}{2}\right)^{24} - 1}{\frac{0.08}{2}}$$ $A = 4{,}000{,}000;\ n = 24;\ i = \frac{0.08}{2}$

$$4{,}000{,}000 = P(39.082604)$$ Simplify

$$P = \$102{,}347.33$$ Solve for P.

The board will need to make a payment of \$102,347.33 every 6 months to redeem the bonds in 12 years.

NOW WORK PROBLEM 11.

EXAMPLE 9 Funding a Loan

A woman borrows \$3000, which will be paid back to the lender in one payment at the end of 5 years. She agrees to pay interest monthly at an annual rate of 12%. At the same time she sets up a sinking fund in order to repay the loan at the end of 5 years. She decides to make monthly payments into her sinking fund, which earns 8% interest compounded monthly.

(a) What is the monthly sinking fund payment?
(b) Construct a table that shows how the sinking fund grows over time.
(c) How much does she need each month to be able to pay the interest on the loan and make the sinking fund payment?

SOLUTION **(a)** The sinking fund payment is the value of P in Formula (1):

$$A = P\frac{(1+i)^n - 1}{i} \qquad \text{Formula (1)}$$

Here A equals the amount to be accumulated, namely, $A = \$3000$, $n = 60$ (5 years of monthly payments), and $i = \frac{0.08}{12}$. The sinking fund payment P obeys

$$3000 = P\left[\frac{\left(1 + \frac{0.08}{12}\right)^{60} - 1}{\frac{0.08}{12}}\right]$$

$$300 = P(73.476856)$$

$$P = \$40.83$$

The monthly sinking fund payment is \$40.83

(b) Table 4 shows the growth of the sinking fund over time. For example, the total in the account after payment number 12 is obtained by using Formula (1). Since the monthly payment of \$40.83 has been made for 12 months at 8% compounded monthly, the total amount in the account at this point in time is

TABLE 4

Payment Number	Sinking Fund Deposit, \$	Cumulative Deposits	Accumulated Interest, \$	Total, \$
1	40.83	40.83	0	40.83
12	40.83	489.96	18.37	508.83
24	40.83	979.92	78.93	1058.85
36	40.83	1469.88	185.19	1655.07
48	40.83	1959.84	340.93	2300.77
60	40.77	2449.74	550.26	3000.00

$$\text{Total} = \$40.83\left[\frac{\left(1+\frac{0.08}{12}\right)^{12}-1}{\frac{0.08}{12}}\right] = \$40.83(12.449926) = \$508.33$$

The deposit for payment number 60, the final payment, is only \$40.77 because a deposit of \$40.83 results in a total of \$3000.06.

(c) The monthly interest payment due on the loan of \$3000 at 12% interest is found using the simple interest formula.

$$I = 3000(0.12)\left(\frac{1}{12}\right) = \$30$$

The woman needs to be able to pay \$40.83 + \$30 = \$70.83 each month.

NOW WORK PROBLEM 27.

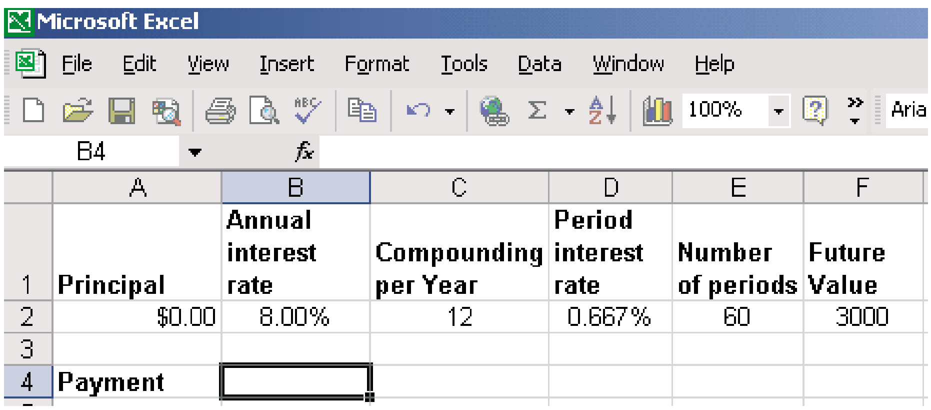
Use Excel to solve Example 9, parts (a) and (b).

SOLUTION **(a)** Use the Payment Value function in Excel.
The syntax of the Payment Value function is:

PMT(rate, nper, pv, fv, type)

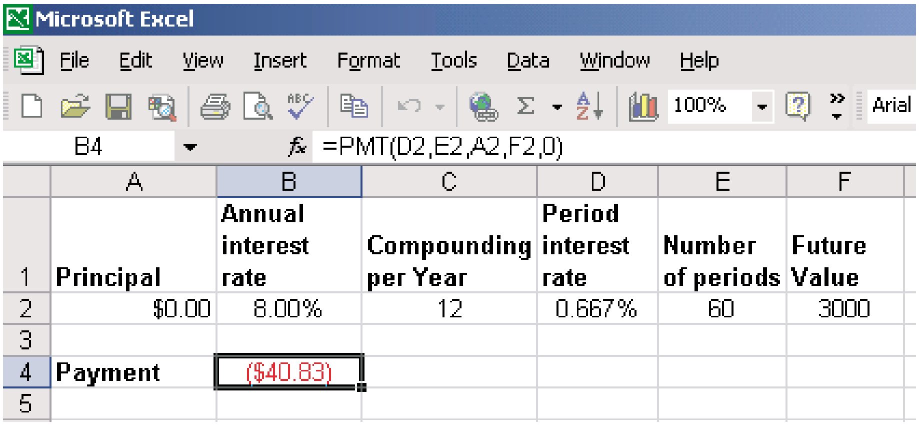
rate is the interest rate per period.
nper is the total number of payments.
pv is the principal value.
fv is the balance after the last payment is made.
type is the number 0 or 1 and indicates when payments are due. Use 0 for an ordinary annuity.

STEP 1 Set up the Excel spreadsheet as shown below:



	A	B	C	D	E	F
1	Principal	Annual interest rate	Compounding per Year	Period interest rate	Number of periods	Future Value
2	$0.00	8.00%	12	0.667%	60	3000
3						
4	Payment					

STEP 2 Insert the Payment function in B4 as shown below in the *fx* bar.

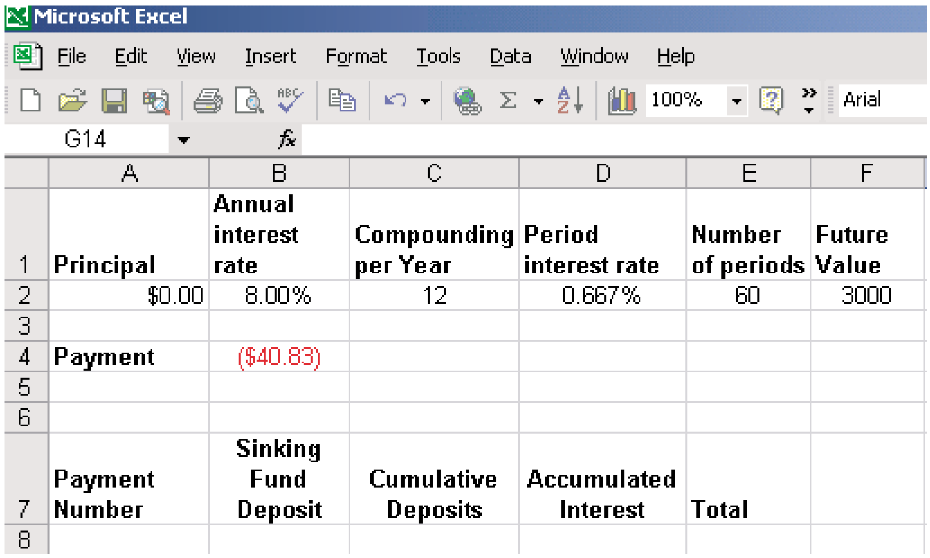


	A	B	C	D	E	F
1	Principal	Annual interest rate	Compounding per Year	Period interest rate	Number of periods	Future Value
2	$0.00	8.00%	12	0.667%	60	3000
3						
4	Payment	($40.83)				
5						

She has to deposit $40.83 each month into her sinking fund.

(b) Construct a table that shows how the sinking fund grows over time.

STEP 1 Set up the column headings in row 7.



	A	B	C	D	E	F
1	Principal	Annual interest rate	Compounding per Year	Period interest rate	Number of periods	Future Value
2	$0.00	8.00%	12	0.667%	60	3000
3						
4	Payment	($40.83)				
5						
6						
7	Payment Number	Sinking Fund Deposit	Cumulative Deposits	Accumulated Interest	Total	
8						

STEP 2 Set up row 8.

- Payment number will be 1.
- Sinking fund deposit is $40.83. This is stored in cell B4. $40.83 is the deposit for each month; it must be positive and not change when the rest of the table is completed. Enter the formula =−1*B4 in B8.
- Cumulative deposits is number of payments times $40.83. Enter the formula =A8*B8 in C8.
- The total is the future value of the sinking fund. The inputs are fixed except for the payment number, A8. Enter the formula =FV(D2,A8,B4,A2,0).
- Accumulated interest is found by subtracting the cumulative deposits from the total. Enter the formula =E8 − C8 into D8.

The formulas are given in the table below.

7	Payment Number	Sinking Fund Deposit	Cumulative Deposits	Accumulated Interest	Total
8	1	=-1*B4	=A8*B8	=E8-C8	=FV(D2,A8,B4,A2,0)
9					

STEP 3 To complete the table, capture row 8 and drag down two rows. Change the payment number in A9 to 6 and A10 to 12. Highlight rows 9 and 10 and click and drag through row 18.

The completed table is given below.

Microsoft Excel

File Edit View Insert Format Tools Data Window Help

Arial 10

J25 =

	A	B	C	D	E	F
1	Principal	Annual interest rate	Compounding per year	Period interest rate	Number of periods	Future value
2	$0.00	8.00%	12	0.667%	60	3000
3						
4	Payment	($40.83)				
5						
6						
7						
8						
9						
10	Payment Number	Sinking Fund Deposit	Cumulative Deposits	Accumulated Interest	Total	
11	1	$40.83	$40.83	($0.00)	$40.83	
12	6	$40.83	$244.97	$4.12	$249.09	
13	12	$40.83	$489.95	$18.37	$508.32	
14	18	$40.83	$734.92	$43.17	$778.08	
15	24	$40.83	$979.89	$78.93	$1,058.82	
16	30	$40.83	$1,224.86	$126.12	$1,350.98	
17	36	$40.83	$1,469.84	$185.19	$1,655.03	
18	42	$40.83	$1,714.81	$256.63	$1,971.44	
19	48	$40.83	$1,959.78	$340.94	$2,300.72	
20	54	$40.83	$2,204.75	$438.64	$2,643.39	
21	60	$40.83	$2,449.73	$550.27	$3,000.00	
22						

NOW WORK PROBLEM 27 USING EXCEL.

EXAMPLE 10 Depletion Investment

A gold mine is expected to yield an annual net income of \$200,000 for the next 10 years, after which it will be worthless. An investor wants an annual return on his investment of 18%. If he can establish a sinking fund earning 10% annually, how much should he be willing to pay for the mine?

SOLUTION Let x denote the price to be paid for the mine. Then $0.18x$ represents an 18% annual Return On Investment (ROI).

The annual Sinking Fund Contribution (SFC) needed to recover the purchase price x in 10 years obeys Formula (1), where $A = x$, $n = 10$, $i = 0.10$, and $P = \text{SFC}$. Then

$$A = P\frac{(1+i)^n - 1}{i} \qquad \text{Formula (1)}$$

$$x = \text{SFC}\frac{(1+0.10)^{10} - 1}{0.10} \qquad A = x,\ n = 10,\ i = 0.10,\ P = \text{SFC}$$

$$x = \text{SFC}\,(15.9374246)$$

$$\text{SFC} = 0.0627454\,x \qquad \text{Solve for SFC.}$$

The required annual Return On Investment (ROI) plus the annual Sinking Fund Contribution equals the annual net income of \$200,000.

$$\begin{pmatrix}\text{ROI}\\ \text{Annual return}\\ \text{on investment}\end{pmatrix} + \begin{pmatrix}\text{SFC}\\ \text{Annual sinking}\\ \text{fund contribution}\end{pmatrix} = \text{Annual income}$$

$$0.18x + (0.0627453)x = \$200{,}000$$

$$0.2427453x = \$200{,}000$$

$$x = \$823{,}909$$

A purchase price of \$823,909 will achieve the investor's goals.

NOW WORK PROBLEM 29.

EXERCISE 5.3 Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–10, find the amount of each annuity.

1. After ten annual deposits of \$100 at 10% compounded annually.
2. After twelve monthly deposits of \$200 at 5% compounded monthly.
3. After twelve monthly deposits of \$400 at 12% compounded monthly.
4. After five annual deposits of \$1000 at 10% compounded annually.
5. After thirty-six monthly deposits of \$200 at 6% compounded monthly.
6. After forty semiannual deposits of \$2000 at 5% compounded semiannually.
7. After sixty monthly deposits of \$100 at 6% compounded monthly.
8. After eight quarterly deposits of \$1000 at 4% compounded quarterly.
9. After ten annual deposits of \$9000 at 5% compounded annually.
10. After twenty annual deposits of \$5000 at 4% compounded annually.

In Problems 11–20, find the payment required for each sinking fund.

11. The amount required is \$10,000 after 5 years at 5% compounded monthly. What is the monthly payment?
12. The amount required is \$5000 after 180 days at 4% compounded daily. What is the daily payment?
13. The amount required is \$20,000 after $2\frac{1}{2}$ years at 6% compounded quarterly. What is the quarterly payment?

14. The amount required is \$50,000 after 10 years at 7% compounded semiannually. What is the semiannual payment?

15. The amount required is \$25,000 after 6 months at $5\frac{1}{2}$% compounded monthly. What is the monthly payment?

16. The amount required is \$100,000 after 25 years at $4\frac{1}{4}$% compounded annually. What is the annual payment?

17. The amount required is \$5000 after 2 years at 4% compounded monthly. What is the monthly payment?

18. The amount required is \$5000 after 2 years at 4% compounded semiannually. What is the semiannual payment?

19. The amount required is \$9000 after 4 years at 5% compounded annually. What is the annual payment?

20. The amount required is \$9000 after 2 years at 5% compounded quarterly. What is the quarterly payment?

21. **Market Value of a Mutual Fund** Al invests \$2500 a year in a mutual fund for 15 years. If the market value of the fund increases 7% per year, what will be the value of the fund after 15 deposits?

22. **Saving for a Car** Sheila wants to invest an amount every 3 months so that she will have \$12,000 in 3 years to buy a new car. The account pays 8% compounded quarterly. How much should she deposit each quarter to have \$12,000 after 12 deposits?

23. **Value of an Annuity** Todd and Tami pay \$300 every 3 months for 6 years into an ordinary annuity paying 8% compounded quarterly. What is the value of the annuity after the 24 deposits?

24. **Saving for a House** In 4 years Colleen and Bill would like to have \$30,000 for a down payment on a house. How much should they deposit each month into an account paying 9% compounded monthly to have \$30,000 after 48 deposits?

25. **Funding a Pension** Dan wishes to have \$350,000 in a pension fund 20 years from now. How much should he deposit each month into an account paying 9% compounded monthly to have \$350,000 after 240 deposits?

26. **Funding a Keogh Plan** Pat has a Keogh retirement plan (this type of plan is tax-deferred until money is withdrawn). If deposits of \$7500 are made each year into an account paying 8% compounded annually, how much will be in the account after 25 years?

27. **Sinking Fund Payment** A company establishes a sinking fund to provide for the payment of a \$100,000 debt maturing in 4 years. Contributions to the fund are to be made each year. Find the amount of each annual deposit if interest is 8% per annum. Prepare a table showing the annual growth of the sinking fund.

28. **Paying Off Bonds** A state has \$5,000,000 worth of bonds that are due in 20 years. A sinking fund is established to pay off the debt. If the state can earn 10% annually on its money, what is the annual sinking fund deposit needed? Prepare a table showing the growth of the sinking fund every 5 years.

29. **Depletion Investment** An investor wants to know the amount she should pay for an oil well expected to yield an annual return of \$30,000 for the next 30 years, after which the well will be dry. Find the amount she should pay to yield a 14% annual return if a sinking fund earns 10% annually.

30. **Time Needed for a Million Dollars** If you deposit \$10,000 every year into an account paying 8% compounded annually, how long will it take to accumulate \$1,000,000?

31. **Bond Payments** A city has issued bonds to finance a new library. The bonds have a total face value of \$1,000,000 and are payable in 10 years. A sinking fund has been opened to retire the bonds. If the interest rate on the fund is 8% compounded quarterly, what are the quarterly payments?

32. **Value of an IRA**

(a) Tanya invested \$2000 in an IRA each year for 10 years earning 8% compounded annually. At the end of 10 years she ceased the IRA payments, but continued to invest her accumulated amount at 8% compounded annually, for the next 30 years. What was the value of her IRA investment at the end of 10 years? What was the value of her investment at the end of the next 30 years?

(b) Carol started her IRA investment in the 11th year and invested \$2000 per year for the next 30 years at 8% compounded annually. What was the value of her investment at the end of 30 years?

(c) Who had more money at the end of the period?

33. **Managing a Condo** The Crown Colony Condo Association is required by law to set aside funds to replace its roof. The current cost to replace the roof is \$100,000 and it will need to be replaced in 20 years. The cost of a roof is expected to increase at the rate of 3% per year. The Condo can invest in Treasuries yielding 6% paid semiannually.

(a) What will the roof cost in 20 years?

(b) If the Condo invests in the Treasuries, what semiannual payment is required to have the funds to replace the roof in 20 years?

34. **Managing a Condo** The Crown Colony Condo Association is required by law to set aside funds to replace its common-area carpet. The current cost to replace the carpet is \$20,000 and it will need to be replaced in 6 years. The cost of carpet is expected to increase at the rate of 2% per year. The Condo can invest in Treasuries yielding 5% paid semiannually.

(a) What will the carpet cost in 6 years?

(b) If the Condo invests in the Treasuries, what semiannual payment is required to have the funds to replace the carpet in 6 years?

Problems 35 and 36 require logarithms.

35. How many years will it take to save $1,000,000 if you place $600 per month in an account that earns 7% compounded monthly?

36. How many years will it take to save $1,000,000 if you place $1000 per month in an account that earns 6% compounded monthly?

37. **Saving for a Car** In January 2003, a new Honda Accord EX with manual transmission was listed at $21,600. Assuming a constant annual inflation rate of 1.58%, which was the annual percent increase in the U. S. Consumer Price Index for 2002, calculate a projection of what a new Honda Accord EX with manual transmission would cost in January 2007. Assume that the sales tax rate where you plan to purchase this vehicle in January 2007 will be 9.25%. Beginning in January 2003, suppose that you want to make a monthly payment into a sinking fund that earns interest at a 2.75% annual rate, compounded monthly, in order to accumulate funds to purchase a new Honda in 4 years. What monthly payment will produce a balance in this sinking fund in January 2007, which would be equal to the projected cost of a 2007 Honda Accord EX as calculated above plus sales tax?

Source: American Honda Motor Co., Inc., and U. S. Department of Labor, Bureau of Labor Statistics.

38. **Saving for a Down Payment on a Home** The median price of an existing single-family detached home in California was $270,210 in November 2001 and increased to $328,310 in November 2002. If the median price of an existing single-family detached home in California continues to increase at the annual rate experienced between November 2001 and November 2002, determine the projected median price of an existing single-family detached home in California in November 2006. Suppose that you wish to make monthly payments into a sinking fund between November 2002 and November 2006 to generate a down payment of 10% of this projected median price. Find the monthly payment required for this sinking fund, assuming that this sinking fund earns 2.9%, compounded monthly.

Source: San Francisco *Business Times* based on data from the California Association of Realtors.

39. **Saving for College** The average annual undergraduate college in-state tuition nationally for 2000–2001 for public 4-year institutions was $3506. The Bureau of Labor Statistics indicated that the inflation rate for tuition in 2001 was 5.09%. Assuming that this inflation rate is constant for the next 16 years, determine the projected annual undergraduate college tuition for public 4-year institutions for 2012–2013, 2013–2014, 2014–2015, and 2015–2016. Next, determine a quarterly sinking fund payment, beginning with the fourth quarter of 2002, so that at the end of 2012 the sinking fund will have the amount needed to fund 4 years of tuition, as calculated above. Assume that this sinking fund earns interest at a 4.2% annual rate, compounded quarterly. Assume that no additional quarterly payments to this sinking fund will be made after the fourth quarter of 2012, that tuition is due to be paid at the end of the fourth quarter of 2012 for the year 2012–2013, at the end of the fourth quarter of 2013 for the year 2013–2014, and so on and that funds which remain in this sinking fund after 2012 will continue to earn interest at a 4.2% annual rate, compounded quarterly.

Source: United States Department of Education, National Center for Education Statistics, Integrated Postsecondary Education Data System (IPEDS), "Fall Enrollment" and "Institutional Characteristics" surveys (August, 2001), FinAid.org, "The Smart Student Guide to Financial Aid"; and U. S. Department of Labor, Bureau of Labor Statistics.

40. **Saving for College** The average annual undergraduate college tuition nationally for 2000–2001 for private 4-year institutions was $15,531. The Bureau of Labor Statistics indicated that the inflation rate for tuition in 2001 was 5.09%. Assuming that this inflation rate is constant for the next 16 years, determine the projected annual undergraduate college tuition for private 4-year institutions for 2012–2013, 2013–2014, 2014–2015, and 2015–2016. Next, determine a quarterly sinking fund payment, beginning with the fourth quarter of 2002, so that at the end of 2012 the sinking fund will have the amount needed to fund 4 years of tuition, as calculated above. Assume that this sinking fund earns interest at a 4.2% annual rate, compounded quarterly. Assume that no additional quarterly payments to this sinking fund will be made after the fourth quarter of 2012, that tuition is due to be paid at the end of the fourth quarter of 2012 for the year 2012–2013, at the end of the fourth quarter of 2013 for the year 2013–2014, and so on and that funds which remain in this sinking fund after 2012 will continue to earn interest at a 4.2% annual rate, compounded quarterly.

Source: United States Department of Education, National Center for Education Statistics, Integrated Postsecondary Education Data System (IPEDS), "Fall Enrollment" and "Institutional Characteristics" surveys (August, 2001), FinAid.org, "The Smart Student Guide to Financial Aid"; and U. S. Department of Labor, Bureau of Labor Statistics.

5.4 Present Value of an Annuity; Amortization

PREPARING FOR THIS SECTION *Before getting started, review the following:*

> Exponents and Logarithms (Appendix A, Section A.3, pp. xx–xx)

OBJECTIVES
1. Solve problems involving the present value of an annuity
2. Solve problems involving amortization

In Section 5.2 we defined present value (as it relates to the compound interest formula) as the amount of money needed now to obtain an amount A in the future. A similar idea is used for periodic withdrawals.

Suppose you want to withdraw \$10,000 per year each year for the next 5 years from a retirement account that earns 10% compounded annually. How much money is required initially in this account for this to happen? In fact, the amount is the *present value* of each of the \$10,000 withdrawals. This leads to the following definition.

The **present value** of an annuity is the sum of the present values of the withdrawals. In other words, the present value of an annuity is the amount of money needed now so that if it is invested at i percent, n equal dollar amounts can be withdrawn without any money left over.

EXAMPLE 1 Finding the Present Value of an Annuity

Compute the amount of money required to pay out \$10,000 per year for 5 years at 10% compounded annually.

SOLUTION For the first \$10,000 withdrawal, the present value V_1 (the dollars needed now to withdraw \$10,000 in one year) is

$$V_1 = \$10{,}000(1 + 0.10)^{-1} = \$10{,}000(0.9090909) = \$9090.91$$

For the second \$10,000 withdrawal, the present value V_2 (the dollars needed now to withdraw \$10,000 in two years) is

$$V_2 = \$10{,}000(1 + 0.10)^{-2} = \$10{,}000(0.826446) = \$8264.46$$

Similarly,

$$V_3 = \$10{,}000(1 + 0.10)^{-3} = \$10{,}000(0.7513148) = \$7513.15$$
$$V_4 = \$10{,}000(1 + 0.10)^{-4} = \$10{,}000(0.6830134) = \$6830.13$$
$$V_5 = \$10{,}000(1 + 0.10)^{-5} = \$10{,}000(0.6209213) = \$6209.21$$

The present value V for 5 withdrawals of \$10,000 each is

$$\begin{aligned} V &= V_1 + V_2 + V_3 + V_4 + V_5 \\ &= \$9090.91 + \$8264.46 + \$7513.15 + \$6830.13 + \$6209.21 \\ &= \$37{,}907.86 \end{aligned}$$

A person would need \$37,907.86 now, invested at 10% per annum in order to withdraw \$10,000 per year for the next 5 years.

Table 5 summarizes these results. Table 6 lists the amount at the beginning of each year.

TABLE 5

Withdrawal	Present Value
1st	$\$10,000(1.10)^{-1} = \$9,090.91$
2nd	$\$10,000(1.10)^{-2} = \$8,264.46$
3rd	$\$10,000(1.10)^{-3} = \$7,513.15$
4th	$\$10,000(1.10)^{-4} = \$6,830.13$
5th	$\$10,000(1.10)^{-5} = \$6,209.21$
Total	\$37,907.86

TABLE 6

Year	Amount at the Beginning of the Year	Add Interest	Subtract Withdrawal
1	37,907.86	3,790.79	10,000.00
2	31,698.65	3,169.87	10,000.00
3	24,868.52	2,486.85	10,000.00
4	17,355.37	1,735.54	10,000.00
5	9,090.91	909.09	10,000.00
6	0		

We seek a formula for the present value of an annuity. Suppose an annuity earns interest of i percent per payment period and suppose we wish to make n withdrawals of $\$P$ at each payment period. The amount V_1 required for the first withdrawal (the present value of $\$P$) is

$$V_1 = P(1 + i)^{-1}$$

The amount V_2 required for the second withdrawal is

$$V_2 = P(1 + i)^{-2}$$

The amount V_n required for the nth withdrawal is

$$V_n = P(1 + i)^{-n}$$

The present value V of the annuity is the sum of the amounts $V_1, V_2, \ldots, V_n$

$$\begin{aligned} V = V_1 + \cdots + V_n &= P(1 + i)^{-1} + \cdots + P(1 + i)^{-n} \\ &= P(1 + i)^{-n}[1 + (1 + i) + \cdots + (1 + i)^{n-1}] \end{aligned}$$

The expression in brackets is the sum of the first n terms of a geometric sequence, whose ratio is $1 + i$. As a result,

$$V = P(1 + i)^{-n}\frac{1 - (1 + i)^n}{1 - (1 + i)} = P\frac{(1 + i)^{-n} - 1}{-i} = P\frac{1 - (1 + i)^{-n}}{i}$$

Present Value of an Annuity

Suppose an annuity earns interest at the rate of i percent per payment period. If n withdrawals of $\$P$ are made at each payment period, the amount V required obeys

$$\boxed{V = P\frac{1 - (1 + i)^{-n}}{i}} \qquad (1)$$

Here V is called the present value of the annuity.

1 **EXAMPLE 2** **Getting By in College**

A student requires \$200 each month for the next 10 months to cover miscellaneous expenses at school. A money market fund will pay her interest of 2% per annum each month. How much money should she ask for from her parents so that she can withdraw \$200 each month for 10 months?

SOLUTION We seek the present value of an annuity. The monthly withdrawal is \$200, the interest rate is $i = \frac{0.02}{12}$ per month, and the number of withdrawals is $n = 10$. The money required for this is given by Formula (1):

$$V = \$200 \left[\frac{1 - \left(1 + \frac{0.02}{12}\right)^{-10}}{\frac{0.02}{12}} \right] = \$200\,(9.90894) = \$1981.79$$

She should ask her parents for \$1981.79

NOW WORK PROBLEM 1.

EXAMPLE 3 **Finding the Cost of a Car**

A man agrees to pay \$300 per month for 48 months to pay off a car loan. If interest of 12% per annum is charged monthly, how much did the car originally cost? How much interest was paid?

SOLUTION This is the same as asking for the present value V of an annuity of \$300 per month at 12% for 48 months. The original cost of the car is

$$V = 300 \left[\frac{1 - \left(1 + \frac{0.12}{12}\right)^{-48}}{\frac{0.12}{12}} \right] = \$300(37.9739595) = \$11,392.19$$

The total payment is (\$300)(48) = \$14,400. The interest paid is

$$\$14{,}400 - \$11{,}392.19 = \$3007.81$$

NOW WORK PROBLEM 15.

Amortization

We can look at Example 3 differently. What it also says is that if the man pays \$300 per month for 48 months with an interest of 12% compounded monthly, then the car is his. In other words, he *amortized* the debt in 48 equal monthly payments. (The Latin word *mort* means "death." Paying off a loan is regarded as "killing" it.) A loan with a fixed rate of interest is said to be **amortized** if both principal and interest are paid by a sequence of equal payments made over equal periods of time.

When a loan of V dollars is amortized at a rate of interest i per payment period over n payment periods, the customary question is, "What is the payment P?" In other words, in amortization problems, we want to find the amount of payment P that, after n payment periods at the rate of interest i per payment period, gives us a present value equal to the amount of the loan. We need to find P in the formula

$$V = P\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Solving for P, we find

$$P = V\left[\frac{1-(1+i)^{-n}}{i}\right]^{-1} = V\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Amortization

The payment P required to pay off a loan of V dollars borrowed for n payment periods at a rate of interest i per payment period is

$$\boldsymbol{P = V\left[\frac{i}{1-(1+i)^{-n}}\right]} \tag{2}$$

2 **EXAMPLE 4 Finding the Payment for an Amortized Loan**

What monthly payment is necessary to pay off a loan of \$800 at 10% per annum **(a)** In 2 years? **(b)** In 3 years? **(c)** What total amount is paid out for each loan?

SOLUTION

(a) For the 2-year loan $V = \$800$, $n = 24$, and $i = \frac{0.10}{12}$. The monthly payment P is

$$P = 800\left[\frac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-24}}\right] = \$800(0.04614493) = \$36.92$$

(b) For the 3-year loan $V = \$800$, $n = 36$, and $i = \frac{0.10}{12}$. The monthly payment P is

$$P = 800\left[\frac{\frac{0.10}{12}}{1-\left(1+\frac{0.10}{12}\right)^{-36}}\right] = \$800(0.03226719) = \$25.81$$

(c) For the 2-year loan, the total amount paid out is $(36.92)(24) = \$886.08$; for the 3-year loan, the total amount paid out is $(\$25.81)(36) = \$929.16$.

NOW WORK PROBLEM 9.

EXAMPLE 5 Mortgage Payments

Mr. and Mrs. Corey have just purchased a \$300,000 house and have made a down payment of \$60,000. They can amortize the balance ($\$300,000 - \$60,000 = \$240,000$) at 6% for 30 years.

(a) What are the monthly payments?
(b) What is their total interest payment?
(c) After 20 years, what equity do they have in their house (that is, what is the sum of the down payment and the amount paid on the loan)?

SOLUTION **(a)** The monthly payment P needed to pay off the loan of \$240,000 at 6% for 30 years (360 months) is

$$P = \$240{,}000\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-360}}\right] = \$240{,}000(0.0059955) = \$1438.92$$

(b) The total paid out for the loan is

$$(\$1438.92)(360) = \$518{,}011.65$$

The interest on this loan amounts to

$$\$518{,}011.65 - \$240{,}000 = \$278{,}011.65$$

(c) After 20 years (240 months) there remains 10 years (or 120 months) of payments. The present value of the loan is the present value of a monthly payment of \$1438.92 for 120 months at 6%, namely,

$$V = \$1438.92\left[\frac{1-\left(1+\frac{0.06}{12}\right)^{-120}}{\frac{0.06}{12}}\right]$$
$$= (\$1438.92)(90.073453) = \$129{,}608.49$$

The amount paid on the loan is

(Original loan amount) − (Present value) = \$240,000 − \$129,608.49 = \$110,391.51

The equity after 20 years is

(Down payment) + (Amount paid on loan) = \$60,000 + \$110,391.51 = \$170,391.51

This equity does not include any appreciation in the value of the house over the 20 year period.

Table 7 gives a partial schedule of payments for the loan in Example 5. It is interesting to observe how slowly the amount paid on the loan increases early in the payment schedule, with very little of the payment used to reduce principal, and how quickly the amount paid on the loan increases during the last 5 years.

TABLE 7

Payment Number	Monthly Payment	Principal	Interest	Amount Paid on Loan
1	\$1,438.92	\$ 238.92	\$1,200.00	\$ 238.92
60	\$1,438.92	\$ 320.67	\$1,118.26	\$ 16,669.54
120	\$1,438.92	\$ 432.53	\$1,006.39	\$ 39,154.26
180	\$1,438.92	\$ 538.42	\$ 855.50	\$ 69,482.77
240	\$1,438.92	\$ 786.94	\$ 651.98	\$110,391.39
300	\$1,438.92	\$1,061.47	\$ 377.45	\$165,570.99
360	\$1,438.92	\$1,431.76	\$ 7.16	\$240,000.00

NOW WORK PROBLEM 11.

Use EXCEL to solve Example 5, parts (a) and (c).

SOLUTION **(a)** Use the Excel function PMT to find the monthly payments. The function has the following syntax:

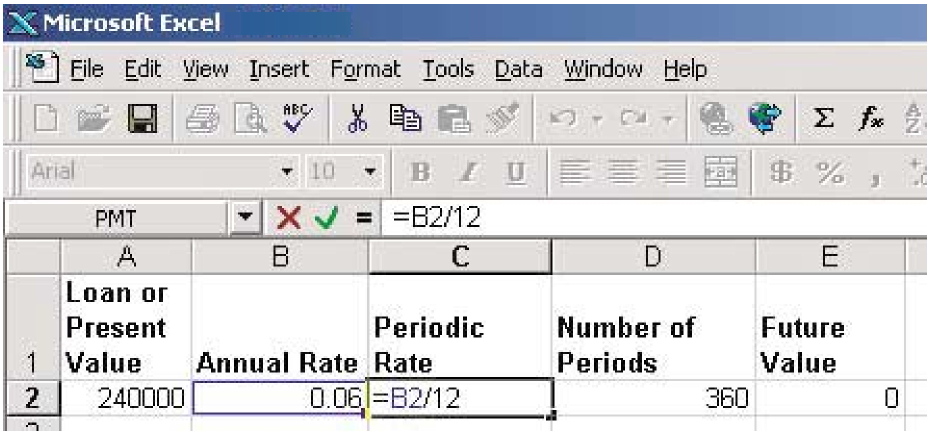
$$\text{PMT}(rate, nper, pv, fv, type)$$

rate is the interest rate per period.
nper is the total number of payment periods
pv is the initial deposit or present value
fv is the future value
type is the number 0 or 1: 0 for ordinary annuity, 1 for annuity due

The values of the parameters of the PMT function are:

rate is .09/12
nper is 25*12 = 300
pv is \$55,000
fv is 0
type is 0

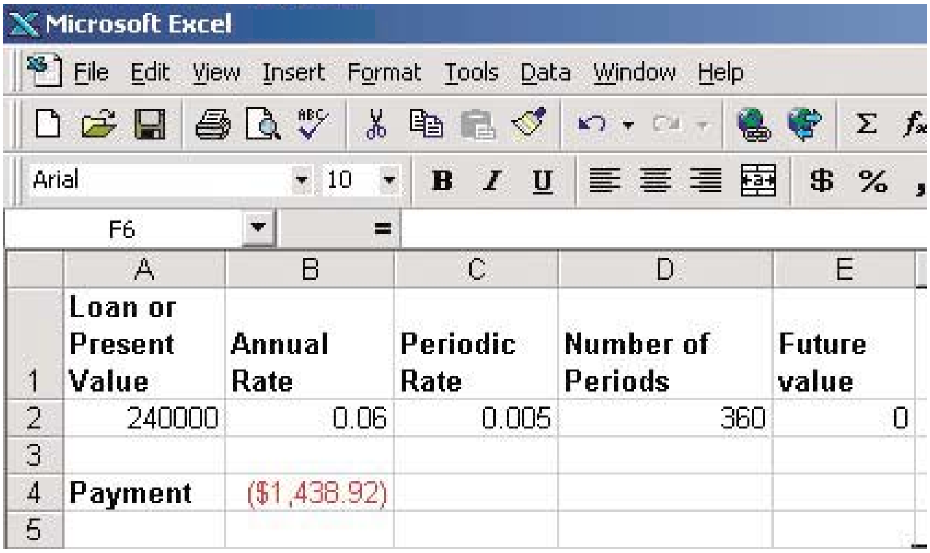
The Excel spreadsheet is given below.



	A	B	C	D	E
1	Loan or Present Value	Annual Rate	Periodic Rate	Number of Periods	Future Value
2	240000	0.06	=B2/12	360	0

In cell B4 type =PMT(*C2,D2,A2,E2,0*)

The final Excel spreadsheet for part (a) is given below.



	A	B	C	D	E
1	Loan or Present Value	Annual Rate	Periodic Rate	Number of Periods	Future value
2	240000	0.06	0.005	360	0
3					
4	Payment	($1,438.92)			
5					

(b) Mr. And Mrs. Corey's monthly payments are $461.56.

(c) To create an amortization table, use the Excel function PPMT to compute the monthly principal.

- Set up the table headings in row 7.
- In A8 enter 1 for payment number 1.
- In B8 enter the formula = −1 *B4. (Recall the $ keeps the cell constant.)
- In C8 enter the formula = −1 *PPMT(C2,A8,D2,A2,E2).
- In D8 enter the formula =B8 − C8.
- In E8 enter the formula = 240,000 + FV(C2, A8,B4,A2, 0).
- Highlight row 8, click and drag 2 rows (filling rows 9 and 10).
- Change A9 to 60 and A10 to 120.
- Highlight rows 9 and 10; click and drag the two rows to row 14 to get the rest of the table.

The completed table is given below.

Microsoft Excel

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	A	B	C	D	E	F
1	Principal	Annual interest rate	Compounding per Year	Period interest rate	Number of Periods	Future Value
2	240000	0.06	0.005	360	0	0
3						
4	Payment	($1,438.92)				
5						
6						
7	Payment Number	Monthly Payment	Principal	Interest	Amount Paid on Loan	
8	1	$1,438.92	$238.92	$1,200.00	$238.92	
9	60	$1,438.92	$320.67	$1,118.26	$16,669.54	
10	120	$1,438.92	$432.53	$1,006.39	$39,154.26	
11	180	$1,438.92	$583.42	$855.50	$69,482.77	
12	240	$1,438.92	$786.94	$651.98	$110,391.39	
13	300	$1,438.92	$1,061.47	$377.45	$165,570.99	
14	360	$1,438.92	$1,431.76	$7.16	$240,000.00	

The row corresponding to payment number 240(12*20) can be used to find out how much equity they have in the house after 20 years. They have paid $32,765.19 of the loan, so their equity is

$$\text{Down payment} + \text{Amount paid on loan} = \$15{,}000 + \$32{,}765.19 = \$47{,}765.19$$

This equity does not include any appreciation in the value of the house over the 20-year period.

NOW WORK PROBLEM 11 USING EXCEL.

EXAMPLE 6 Inheritance

When Mr. Nicholson died, he left an inheritance of $15,000 for his family to be paid to them over a 10-year period in equal amounts at the end of each year. If the $15,000 is invested at 10% per annum, what is the annual payout to the family?

SOLUTION This example asks what annual payment is needed at 10% for 10 years to disperse $15,000. That is, we can think of the $15,000 as a loan amortized at 10% for 10 years. The payment needed to pay off the loan is the yearly amount Mr. Nicholson's family will receive. The yearly payout P is

$$P = \$15{,}000\left[\frac{0.10}{1 - (1 + 0.10)^{-10}}\right]$$
$$= \$15{,}000(0.16274539) = \$2441.18$$

EXAMPLE 7 Determining Retirement Income

Joan is 20 years away from retiring and starts saving $100 a month in an account paying 6% compounded monthly. When she retires, she wishes to withdraw a fixed amount each month for 25 years. What will this fixed amount be?

SOLUTION After 20 years the amount A accumulated in her account is the amount of an annuity with a monthly payment of $100 and an interest rate of $i = \frac{0.06}{12}$.

$$A = 100\left[\frac{\left(1 + \frac{0.06}{12}\right)^{240} - 1}{\frac{0.06}{12}}\right]$$

$$= 100 \cdot (462.0408951) = \$46{,}204.09$$

The amount P she can withdraw for 300 payments (25 years) at 6% compounded monthly is

$$P = 46{,}204.09\left[\frac{\frac{0.06}{12}}{1 - \left(1 + \frac{0.06}{12}\right)^{-300}}\right] = \$297.69$$

NOW WORK PROBLEM 7.

EXERCISE 5.4 Answers to Odd-Numbered Problems Begin on Page AN-00.

In Problems 1–6, find the present value of each annuity.

1. The withdrawal is to be $500 per month for 36 months at 10% compounded monthly.
2. The withdrawal is to be $1000 per year for 3 years at 8% compounded annually.
3. The withdrawal is to be $100 per month for 9 months at 12% compounded monthly.
4. The withdrawal is to be $400 per month for 18 months at 5% compounded monthly.
5. The withdrawal is to be $10,000 per year for 20 years at 10% compounded annually.
6. The withdrawal is to be $2000 per month for 3 years at 4% compounded monthly.
7. **Value of a IRA** A husband and wife contribute $4000 per year to an IRA paying 10% compounded annually for 20 years. What is the value of their IRA? How much can they withdraw each year for 25 years at 10% compounded annually?
8. Rework Problem 7 if the interest rate is 8%.
9. **Loan Payments** What monthly payment is needed to pay off a loan of $10,000 amortized at 12% compounded monthly for 2 years?
10. **Loan Payments** What monthly payment is needed to pay off a loan of $500 amortized at 12% compounded monthly for 2 years?
11. In Example 5, if Mr. and Mrs. Corey amortize the $240,000 loan at 6% for 20 years, what is their monthly payment?
12. In Example 5, if Mr. and Mrs. Corey amortize their $240,000 loan at 5% for 15 years, what is their monthly payment?
13. In Example 6, if Mr. Nicholson left $15,000 to be paid over 20 years in equal yearly payments and if this amount were invested at 12%, what would the annual payout be?
14. Joan has a sum of $30,000 that she invests at 10% compounded monthly. What equal monthly payments can she receive over a 10-year period? Over a 20-year period?
15. **Planning Retirement** Mr. Doody, at age 65, can expect to live for 20 years. If he can invest at 10% per annum compounded monthly, how much does he need now to guarantee himself $250 every month for the next 20 years?
16. **Planning Retirement** Sharon, at age 65, can expect to live for 25 years. If she can invest at 10% per annum compounded monthly, how much does she need now to guarantee herself $300 every month for the next 25 years?