NETWORK TOPOLOGY

A SUPPLEMENT TO ACCOMPANY
THE 3RD EDITION OF

THE ANALYSIS AND DESIGN OF LINEAR CIRCUITS

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1. Basic Concepts

This supplement introduces some basic concepts of network topology as they apply to circuits consisting of resistors and independent sources. There is a voltage and a current associated with each of these two-terminal circuit elements. A circuit containing a total of \( E \) such elements involves a total of \( 2E \) circuit variables. To solve for any or all of these variables we need \( 2E \) equations. At this point we assume that a unique solution of these equations exists. On paper it is easy to devise pathological circuits that do not have unique solutions. However, these pathologies do not often arise in physical situations and are remedied by using better device models. In sum, we lose nothing essential to our study with this assumption.

Collectively the element constraints (\( i-v \) characteristics of resistors and sources) provide a total of \( E \) independent equations, one for each element in the circuit. Since we assume the circuit has a unique solution, it must be possible to form exactly \( E \) independent connection equations using KCL and KVL. Two immediate questions are:

(1) How can we be sure that the connection equations we write are independent?

(2) How many independent equations can be generated by KCL and how many by KVL?

Network topology allows us to address these questions by concentrating our attention on the connection properties of circuits.

Network topology deals with a circuit model called a graph, which is a collection of line segments called branches and points called nodes. To construct the graph of a given circuit, we retain its nodes and replace each two-terminal element by a line segment called a branch. Figures 1(a) and 1(b) illustrate the relationship between a circuit diagram and its topological graph. Each branch is connected to two nodes indicated by black dots.

Figure 1. a. A circuit diagram, b. Its topological graph, c. Its oriented graph
We use \( N \) to denote the number of nodes and \( E \) to denote the number of branches \((N = 4 \text{ and } E = 6 \text{ in Figure 1})\). There is a current \( i_k \) and a voltage \( v_k \) associated with each branch, where \( k \) is the branch number \((k = 1, 2, \ldots, E)\). These variables are not explicitly shown on a topological graph. The graph in Figure 1(c) is called an oriented graph because each branch has an arrow head indicating the reference direction of its branch current. The reference marks for the branch voltages are not shown on an oriented graph but are implied by the passive sign convention.

The properties of a topological graph depend only on the branch-node connections. The two graphs in Figure 2 may seem different, but they are topologically equivalent to each other and to the graph in Figure 1(b) because the branch-node connections are the same in all three graphs. The intersection of two branches at the middle of Figure 2(a) does not imply a connection. Branches are only connected at the black dots indicating the nodes.

Exercise 1

If possible, assign branch numbers to the graph in Figure E-1 to make it equivalent to the graph in Figure 2(a).

![Figure E-1](image)

Answer:

Impossible. Four branches are connected to node B and only two are connected to node C.

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The following definitions are used to describe topological properties. A graph is said to be connected if there is at least one path, composed of branches in the graph, between any two
nodes. A **loop** is a connected subgraph that forms a closed path in which each branch is traversed only once and each node connects exactly two included branches. A **tree** is a connected subgraph containing all \( N \) nodes but no loops. A graph may have many different trees unless it happens to be a tree to begin with. The graph in Figure 1(b) has 30 different trees, three of which are shown in Figure 3. The solid lines in the figure indicate branches included in the tree and the dashed lines indicate those that are not included. Not surprisingly, the branches in a specified tree are called **tree branches** while those not included are called **links**.  

![Three trees](image)

**Figure 3. Three of the trees in the graph in Figure 1(b).**

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**Exercise 2**

(a) Identify all of the trees in Figure E-2 that include branches 1, 3, and 6.

(b) Identify all of the trees in Figure E-2 that exclude branches 1, 3, and 6.

Answers:

(a) There are three such trees whose branches are \((1, 3, 4, 6), (1, 3, 5, 6), \& (1, 3, 6, 7)\).

(b) There is only one such tree whose branches are \((2, 4, 5, 7)\).

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2. **Basic Topological Properties**

A tree partitions the branches into two disjoint sets, one containing the tree branches and the other the links. While a graph may have several trees they all include the same number of branches. The first property defines the number of tree branches and links in a connected graph.

**Property (1): In a connected graph with \( E \) branches and \( N \) nodes,**

**every tree includes \( N - 1 \) tree branches and excludes \( E - N + 1 \) links.**

To construct a tree we start with one branch and then add branches without forming any
loops. Each new branch is added by connecting one end to a node in the partial tree already formed. To avoid creating a loop, the other end is connected to a node not previously included. This process produces no loops and continues until every node is connected, thereby creating a tree. A different tree could be created by choosing a different starting branch and adding the other branches in a different sequence. But regardless of the starting place and sequence, the initial branch connects two nodes and each subsequent addition connects only one node. Thus, the number of branches needed to construct any tree is \( N - 1 \). Since the links are the branches not included in the tree, the number of links is the total number of branches \( E \) minus the number of tree branches \( (N - 1) \), or \( E - (N - 1) = E - N + 1 \)

Exercise 3

(a) How many links are there in a circuit made up of 22 resistors connected in parallel?
(b) Repeat (a) for 22 resistors connected in series?
Answers: (a) 21. (b) None. The circuit is a tree to begin with.

The following property of a tree is crucial to finding unique loops in a circuit.

**Property (2): There is only one tree branch path between any two nodes in a tree.**

By definition, a connected graph has at least one path along its branches between any two nodes. Since a tree is a connected subgraph there is at least one path along the tree branches between any two nodes. Can there be more than one? We use contradiction to prove that there is only one such path. *Assume* there are two tree branch paths between two nodes. If this assumption were true, then traveling from one node to the other node by one of the paths and returning using the other would form a closed loop, which is not possible in a tree. Thus, we have a contradiction, the assumption is incorrect, and property (2) is true.

Applying KCL is straightforward because it is easy to identify the nodes at which currents are to be summed. On the other hand, KVL involves voltage sums around loops and there may
be many loops in a circuit. The following property helps us to identify unique loops.

**Property (3): There is a unique loop associated with every link in a tree.**

To prove this property we select a tree and disconnect all of its links. Reconnecting one of the links forms a loop consisting of the link, together with the unique tree branch path between its two nodes. Removing this link and connecting a different link forms another loop. Proceeding in this way until all links have been reconnected once, we generate one loop for each link, or a total of $E - N + 1$ loops. Each loop thus formed is unique because it includes at least one branch (the link) that is not included in any of the other loops. Each such loop consists of tree branches and a single "missing link," hence the name given to branches not included in a tree.

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**Exercise 4**

Identify the unique tree branch path associated with each link in Figure E-4.

**Answers:** The tree branch paths are branches:

- (1, 4, 7) for link branch 2,
- (1, 3, 4, 7, 8) for link branch 5, and
- (1, 3, 7, 8) for link branch 6.

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### 3. Independent Connection Constraints

Let $N_i$ represent the number of independent KCL equations and $N_v$ the number of independent KVL equations in a connected graph. Our assumption that the circuit has a unique solution means that connection constraints provide a total of $E$ independent equations. That is

$$N_i + N_v = E$$ \hspace{1cm} (1)

The question addressed here is: what can network topology tell us about $N_i$ and $N_v$?

KVL involves the algebraic sum of voltages around loops in a graph. Since a graph may have many loops we need to know how to write independent KVL equations. An algorithm that always produces independent equations starts by selecting a tree thereby identifying $E - N + 1$
links. We then invoke property (3) and write one KVL equation around each of the \( E - N + 1 \) unique loops associated with the links. Each of these loops consists of the link together with the unique tree branch path between its nodes. This algorithm automatically generates independent KVL equations because each equation includes at least one branch voltage (the link voltage) not included in the others. This does not prove that the process yields the maximum number of independent KVL equations that are available. But, we do know that there are at least \( E - N + 1 \) because the algorithm shows us how to generate that many. In sum, a lower bound on \( N_v \) is

\[
N_v \geq E - N + 1
\]  

(2)

Exercise 4

Write KVL equations around the unique loop associated with the links in the oriented graph shown in Figure E-5.

Answers: The KVL equations are:

\[ v_2 + v_7 - v_1 + v_4 = 0 \] for link branch 2, \[ v_3 + v_5 + v_8 + v_7 - v_1 + v_4 = 0 \] for link branch 5, and \[ v_5 + v_3 + v_8 + v_7 - v_1 = 0 \] for link branch 6.

KCL is easy to apply. We simply form the algebraic sum of the branch currents at each of the \( N \) nodes and equate them all to zero. While it is easy to generate a complete set of KCL equations, we need to know how many are independent. Consider the graph in Figure 4 in which one node (node A) has been pulled out for special consideration leaving a connected subgraph with \( N - 1 \) nodes. There are three branches (1, 2, & 3) connecting node A with the rest of the graph. We now show that the KCL equation at node A is superfluous (dependent) because its satisfaction is guaranteed by the KCL equations at the other \( N - 1 \) nodes.
Suppose we use branch currents to write KCL equations at the \( N \) \(-1\) nodes in the connected subgraph. If we sum all of these KCL equations over all of these \( N \) \(-1\) nodes the currents for branches entirely inside the subgraph cancel out. Each such branch is connected to two nodes and the reference direction is directed in at one and directed out at the other. Therefore, each interior branch current appears twice in the KCL sum, once with a positive sign and once with a negative sign.

However, the branch currents for branches 1, 2, and 3 in Figure 4 do not cancel out. The reason is that each of these branches is connected to only one of the subgraph nodes. Hence, each only appears once in the subgraph KCL equations. The reference directions in Figure 4 tell us that branches 1 and 3 are directed away (-i) from their subgraph nodes and branch 2 is directed toward (+i) its subgraph node. As a result, the sum of the KCL equations over the \( N \) \(-1\) subgraph nodes reduces to

\[-i_1 + i_2 - i_3 = 0\]

which is the negative of the KCL equation at node A in Figure 4.

Since node A could be any node, there is at least one dependent equation among the KCL equations written at all \( N \) nodes. This does not prove that there are \( N \) \(-1\) independent KCL equations. We can only say that there are at most \( N \) \(-1\) independent KCL equations. Hence, an upper bound on \( N_1 \) is

\[N_1 \leq N - 1\]  \hspace{1cm} (3)

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**Exercise 6**

Show that the KCL equation at node A Figure E-6 is the negative sum of the KCL equations at the other nodes

Answer:

\[-(i_1 + i_3 - i_5) \cdot (-i_2 + i_4 + i_3) \cdot (-i_3 - i_2) = i_1 + i_2\]  \hspace{1cm} \text{Figure E-6}
Figure 5 shows the graphical representation of Eqs. (1), (2), and (3). The unique solution requirement means that \( N_i \) and \( N_v \) must lie along the diagonal line defined by Eq. (1). The KVL requirement in Eq. (2) means that \( N_v \) must lie in the closed region above the line \( N_v = E - N + 1 \). The KCL requirement in Eq. (3) means that \( N_i \) must lie in the closed region to the left of the line \( N_i = N - 1 \). The intersection of these lines is the only point at which all three requirements are met simultaneously. We conclude that

**In a connected graph with \( N \) nodes and \( E \) branches there are exactly**

\( E - N + 1 \) independent KVL constraints and \( N - 1 \) independent KCL constraints.

Advanced treatments of network topology prove this result without assuming the existence of a unique solution. Those proofs are a good deal more complicated that the method used here.

![Graphical representation of the constraints on \( N_i \) and \( N_v \).](image)

4. **Selecting Solution Variables**

The breakdown of \( 2E \) equations required to completely describe a circuit is:

- Independent Connection Equations \( E \)
- Independent KVL Equations \( E - N + 1 \)
- Independent KCL Equations \( N - 1 \)

Although we must take all of these equations into account, we never deal with this many equations simultaneously. Instead, we use a set of voltages or currents called solution variables that greatly reduce the number of equations that must be solved simultaneously. These solution
variables must meet the following requirements.

1. The solution variables must be independent; that is, it must be impossible to relate these variables to each other using only Kirchhoff's laws.

2. The solution variables must be complete; that is, once we have solved for these variables it must be both possible and easy to find any other circuit variable.

3. The solution variables must be solvable: that is, it must be possible to write a solvable set of independent equations in the unknown solution variables.

We will discuss the first two requirements here. The text itself treats the third requirement in detail.

The flip side of the independent connection constraints is that the number of independent circuit variables is known. Specifically, since there are $E$ voltages in a circuit and $E - N + 1$ independent KVL equations, there must be $E - (E - N + 1) = N - 1$ independent voltages that are not relatable by KVL alone. By similar reasoning, there must be $E - (N - 1) = E - N + 1$ independent currents that are not relatable by KCL alone. Thus, a set of solution variables can be either $N - 1$ voltages or $E - N + 1$ currents, as long as they are independent.

The node-voltage method is a general method of circuit analysis using $N - 1$ voltages. Under this approach we select one node as a reference node and define the solution variables to be the voltages between the reference node and the remaining $N - 1$ nodes. Figure 6 shows how the node voltages are defined in a circuit with $N = 5$. The arrows defining node voltages fan out from the reference node and all terminate on a unique node. The arrows can not form any closed paths around which to write KVL equations involving only node voltages. Thus, it is impossible to relate the node voltages to each other using KVL. In other words,

The node voltages provide a set of $N - 1$ independent solution variables.

Given the node voltages, we can find the voltage between any two nodes by a simple KVL
equation involving no more than two node voltages. Since branches are connected between two nodes, we can easily solve for any branch voltage and use the element equation to find the corresponding branch current. In sum,

The N - 1 node voltages provide a complete set of solution variables.

A detailed explanation of the steps used to formulate a solvable set of independent equations in the unknown node voltages begins in Section 3-1 of Chapter 3.

The \textbf{loop-current} method is a general approach to circuit analysis using \( E - N + 1 \) currents as solution variables. Under this approach we select a tree and use the links to identify \( E - N + 1 \) distinct loops. We then assign a fictitious loop current that circulates around each of these loops. Figure 7(a) shows how the links define a set of loop currents. Since they are circulating currents, any loop current that enters a node via one branch must exit the node on another. At any node the loop currents cancel out and KCL equations written in terms of loop currents all lead to the conclusion that \( 0 = 0 \). Thus, it impossible to relate the loop currents to each other using KCL. In other words,

![Diagram of loop currents and mesh currents.](image)

\textit{The loop currents provide a set of} \( E - N + 1 \) \textit{independent solution variables.}

Since loop currents circulate through the branches, we can easily relate branch currents to one or more loop currents using KCL. Given a branch current, the element equation yields the corresponding branch voltage. Hence,

\textit{The} \( E - N + 1 \) \textit{loop currents provide a complete set of solution variables.}

The loop-current method is perfectly general. In the text, however, we use a related
approach called the mesh-current method that is restricted to **planar circuits**. A circuit is said to be planar if its graph can be drawn on a plane without any of its branches intersecting. Key words in this definition are "can be." The graph is Figure 2(a) has intersecting branches, but this does not make the circuit nonplanar because Figure 2(b) shows that the graph "can be" drawn on a plane without intersections.

Figure 7(b) shows how mesh currents are defined by simply inserting a circulating current in each of the enclosed open areas of a graph. Of course, this only works for planar circuits. The mesh-current and loop-current methods differ mainly in the manner in which the solution variables are defined. Essentially the same methods are used to formulate independent equations in either set of solution variables. The discussion of mesh-current equation formulation begins in Section 3-2 of Chapter 3.