Functions and graphs

2.1 Kick off with CAS
2.2 Polynomial functions
2.3 Other algebraic functions
2.4 Combinations of functions
2.5 Non-algebraic functions
2.6 Modelling and applications
2.7 Review **eBook plus**
2.1 Kick off with CAS

Graph sketching

1 a Using the graph application on CAS, sketch
   \[ y = x^4 - x^3 - 23x^2 + 3x + 60, \quad x \in [-5, 5]. \]
   b Determine the \(x\)- and \(y\)-intercepts.
   c Find the coordinates of the turning points.
   d Find the coordinates of the end points.
   e Determine the solution(s) to the equation
      \(-40 = x^4 - x^3 - 23x^2 + 3x + 60, \quad x \in [-5, 5].\)
   f Hence, solve \(-40 < x^4 - x^3 - 23x^2 + 3x + 60, \quad x \in [-5, 5].\)
   g For what values of \(k\) does the equation
      \(k = x^4 - x^3 - 23x^2 + 3x + 60, \quad x \in [-5, 5]\) have:
      i 1 solution
      ii 2 solutions
      iii 3 solutions
      iv 4 solutions?

2 a Using CAS, sketch
   \[ f(x) = \begin{cases} 
   x^2 - 2, & x \leq 1 \\
   2, & 2 < x < 3 \\
   -2x + 8, & x \geq 3
   \end{cases} \]
   b Determine \(f(-1), f(1), f(2)\) and \(f(5)\).
   c Solve \(f(x) = 1\).
2.2 Polynomial functions

Familiar graphs include those of the straight line and the parabola. This section reviews how the key aspects of these graphs and the graphs of other polynomial functions can be deduced from their equations.

Functions

A function is a set of ordered pairs in which each x-value is paired to a unique y-value. A vertical line will intersect the graph of a function at most once. This is known as the vertical line test for a function.

A horizontal line may intersect the graph of a function once, in which case the function has a one-to-one correspondence, or the horizontal line may intersect the graph more than once, in which case the function has a many-to-one correspondence.

The domain of a function is the set of x-values in the ordered pairs, and the range is the set of the y-values of the ordered pairs.

As a mapping, a function is written \( f : D \rightarrow R \), \( f(x) = \ldots \), where the ordered pairs of the function \( f \) are formed using each of the x-values in the domain \( D \) and pairing them with a unique y-value drawn from the co-domain set \( R \) according to the function rule \( f(x) = \ldots \). Not all of the available y-values may be required for a particular mapping; this is dependent on the function rule.

For any polynomial function, the implied or maximal domain is \( R \). For example, the mapping or function notation for the straight line \( y = 2x \) is \( f : R \rightarrow R \), \( f(x) = 2x \).

Under this mapping, the image of 3, or the value of \( f \) at 3, is \( f(3) = 2 \times 3 = 6 \), and the ordered pair \((3, 6)\) lies on the line of the function.

If only that part of the line \( y = 2x \) where the x-values are positive was required, then this straight line function would be defined on a restricted domain, a subset of the maximal domain, and this would be written as \( g : R^+ \rightarrow R \), \( g(x) = 2x \).

WORKED EXAMPLE 1

Part of the graph of the parabola \( y = x^2 \) is shown in the diagram.

a Explain why the graph is a function and state the type of correspondence.

b State the domain and range.

c Express the given parabola using function notation.

d Calculate the value of \( y \) when \( x = -\sqrt{2} \).

THINK

a 1 Use the vertical line test to explain why the graph is of a function.

2 State the type of correspondence.

WRITE

a This is a function, because any vertical line that intersects the graph does so in exactly one place.

A horizontal line could cut the graph in up to two places. The correspondence is many-to-one.
**The linear polynomial function**

Two points are needed in order to determine the equation of a line. When sketching an oblique line by hand, usually the two points used are the \( x \)- and \( y \)-intercepts. If the line passes through the origin, then one other point needs to be determined from its equation.

**Gradient**

The gradient, or slope, of a line may be calculated from

\[
m = \frac{y_2 - y_1}{x_2 - x_1}.\]

This remains constant between any pair of points \((x_1, y_1)\) and \((x_2, y_2)\) on the line. The linear function either increases or decreases steadily.

Parallel lines have the same gradient, and the product of the gradients of perpendicular lines is equal to \(-1\). That is,

\[
m_1 = m_2 \text{ for parallel lines}
\]

\[
m_1m_2 = -1 \text{ for perpendicular lines.}
\]

The angle of inclination of an oblique line with the positive direction of the \( x \)-axis can be calculated from the gradient by the relationship \( m = \tan(\theta) \). The angle \( \theta \) is acute if the gradient is positive and obtuse if the gradient is negative.

**Equation of a line**

The equation of a straight line can be expressed in the form \( y = mx + c \), where \( m \) is the gradient of the line and \( c \) is the \( y \)-value of the intercept the line makes with the \( y \)-axis.

If a point \((x_1, y_1)\) and the gradient \( m \) are known, the equation of a line can be calculated from the point–gradient form \( y - y_1 = m(x - x_1) \).

Oblique lines are one-to-one functions.

Horizontal lines run parallel to the \( x \)-axis and have the equation \( y = c \). These are many-to-one functions.

Vertical lines rise parallel to the \( y \)-axis and have the equation \( x = k \). These lines are not functions.

The perpendicular bisector of a line segment is the line that passes through the midpoint of the line segment at right angles to the line segment. The midpoint of the line segment with end points \((x_1, y_1)\) and \((x_2, y_2)\) has coordinates \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).
Consider the line \( L \) where \( L = \{(x, y) : 2x + 3y = 12\} \).

**a** Sketch the line.

**b** Calculate the gradient of the line.

**c** Determine the coordinates of the point on the line that is closest to the origin.

**THINK**

**a** 1 Calculate the \( x \)- and \( y \)-intercepts.

**WRITE/DRAW**

**a** \( 2x + 3y = 12 \)

\( y \)-intercept: Let \( x = 0 \).
\( 3y = 12 \)
\( y = 4 \)
The \( y \)-intercept is (0, 4).

\( x \)-intercept: Let \( y = 0 \).
\( 2x = 12 \)
\( x = 6 \)
The \( x \)-intercept is (6, 0).

2 Sketch the graph.

**b** Rearrange the equation in the form \( y = mx + c \) and state the gradient.

*Note:* The gradient could also be calculated using \( m = \frac{\text{rise}}{\text{run}} \) from the diagram.

**c** 1 Draw a diagram describing the position of the required point.

**c** Let \( P \) be the point on the line which is closest to the origin. This means that \( OP \) is perpendicular to the line \( L \).
The quadratic polynomial function

The function \( f : R \rightarrow R, f(x) = ax^2 + bx + c \), where \( a, b, c \in R \) and \( a \neq 0 \), is the quadratic polynomial function. If \( a > 0 \), its graph is a concave-up parabola with a minimum turning point; if \( a < 0 \), its graph is a concave-down parabola with a maximum turning point.

General form \( y = ax^2 + bx + c \)

As the \( x \)-intercepts of the graph of \( y = ax^2 + bx + c \) are the roots of the quadratic equation \( ax^2 + bx + c = 0 \), there may be zero, one or two \( x \)-intercepts as determined by the discriminant \( \Delta = b^2 - 4ac \).
If $\Delta < 0$, there are no $x$-intercepts; the quadratic function is either positive or negative, depending whether $a > 0$ or $a < 0$ respectively.
If $\Delta = 0$, there is one $x$-intercept, a turning point where the graph touches the $x$-axis.
If $\Delta > 0$, there are two distinct $x$-intercepts and the graph crosses the $x$-axis at these places.

As the roots of the quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the axis of symmetry of the parabola has the equation $x = -\frac{b}{2a}$. This is also the $x$-coordinate of the turning point, so by substituting this value into the parabola’s equation, the $y$-coordinate of the turning point can be calculated.

**Turning point form,** $y = a(x - h)^2 + k$

The simplest parabola has the equation $y = x^2$. Its turning point is the origin, $(0, 0)$, which is unaltered by a dilation from the $x$-axis in the $y$-direction. However, if the graph of this parabola undergoes a horizontal translation of $h$ units and a vertical translation of $k$ units, the turning point moves to the point $(h, k)$.

Thus, $y = a(x - h)^2 + k$ is the equation of a parabola with turning point $(h, k)$ and axis of symmetry $x = h$.

If $y = a(x - h)^2 + k$ is expanded, then the general form $y = ax^2 + bx + c$ is obtained. Conversely, when the technique of completing the square is applied to the equation $y = ax^2 + bx + c$, the turning point form is obtained.
**x-intercept form, \( y = a(x - x_1)(x - x_2) \)**

When the equation of a quadratic function is expressed as the product of its two linear factors, the \( x \)-intercepts at \( x = x_1 \) and \( x = x_2 \) can be obtained by inspection. The axis of symmetry lies midway between the intercepts, so the equation for this axis must be

\[
x = \frac{x_1 + x_2}{2},
\]

and this gives the \( x \)-coordinate of the turning point. The \( y \)-coordinate of the turning point can be calculated from the equation once the \( x \)-coordinate is known. Expanding the equation \( y = a(x - x_1)(x - x_2) \) will return it to general form, and factorising the general equation \( y = ax^2 + bx + c \) will convert it to \( x \)-intercept form.

**Key features of the graph of a quadratic function**

When sketching the graph of a parabola by hand, identify:
- the \( y \)-intercept
- any \( x \)-intercepts
- the turning point
- the axis of symmetry, if it is helpful to the sketch
- any end point coordinates if the function is given on a restricted domain.

The methods used to identify these features will depend on the form in which the equation of the graph is expressed.

Similarly, when determining the equation of a parabola given a key feature, you should select the form of the equation that emphasises that key feature.

- If the turning point is given, use the \( y = a(x - h)^2 + k \) form.
- If the \( x \)-intercepts are given, use the \( y = a(x - x_1)(x - x_2) \) form.
- Otherwise, use the \( y = ax^2 + bx + c \) form.

Three pieces of information are always required to determine the equation, as each form involves 3 constants or parameters.

---

**WORKED EXAMPLE 3**

**a** Sketch the graph of \( y = 9 - (2x + 1)^2 \) and state its domain and range.

**b** Determine the equation of the given graph and hence obtain the coordinates of the turning point.

**THINK**

- **a 1** Rewrite the equation so it is in a standard form.

**WRITE/DRAW**

- **a**
  - \( y = 9 - (2x + 1)^2 \)
  - \( y = -(2x + 1)^2 + 9 \)
  - Or
  - \( y = -\left(2\left(x + \frac{1}{2}\right)\right)^2 + 9 \)
  - \( y = -4\left(x + \frac{1}{2}\right)^2 + 9 \)
ubic functions

The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$, $a \neq 0$ is the cubic polynomial function. Although the shape of its graph may take several forms, for its maximal domain the function has a range of $\mathbb{R}$. Its long-term behaviour is dependent on the sign of the coefficient of the $x^3$ term.

If $a > 0$, then as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.

If $a < 0$, then as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

This behaviour is illustrated in the graph of $y = x^3$, the simplest cubic function, and that of $y = -x^3$.

### Practice Questions

2 State the coordinates and type of turning point. The graph has a maximum turning point at $\left(-\frac{1}{2}, 9\right)$.

3 Calculate the $y$-intercept. $y$-intercept: Let $x = 0$.

$y = 9 - (1)^2$

$y = 8$

The $y$-intercept is $(0, 8)$.

4 Calculate any $x$-intercepts. As the graph has a maximum turning point with a positive $y$-value, there will be $x$-intercepts.

Let $y = 0$.

$9 - (2x + 1)^2 = 0$

$(2x + 1)^2 = 9$

$2x + 1 = \pm 3$

$x = -2$ or $1$

The $x$-intercepts are $(-2, 0)$ and $(1, 0)$.

5 Sketch the graph.

6 State the domain and range. The domain is $\mathbb{R}$ and the range is $(-\infty, 9]$.

b 1 Select a form of the equation. As the two $x$-intercepts are known, the $x$-intercept form of the equation will be used.

2 Use the key features to partially determine the equation.

There is an $x$-intercept at $x = -5$. $\Rightarrow (x + 5)$ is a factor.

There is an $x$-intercept at $x = 8$. $\Rightarrow (x - 8)$ is a factor.

The equation is $y = a(x + 5)(x - 8)$. 
3 Use the third piece of information to fully determine the equation.

The point \((0, -4)\) lies on the graph. Substitute this point in \(y = a(x + 5)(x - 8)\).

\[-4 = a(5)(-8)\]
\[-4 = -40a\]
\[a = \frac{1}{10}\]

The equation is \(y = \frac{1}{10}(x + 5)(x - 8)\).

4 Determine the equation of the axis of symmetry.

The axis of symmetry lies midway between the \(x\)-intercepts. 
\[\therefore x = \frac{-5 + 8}{2}\]
\[= \frac{3}{2}\]

5 Calculate the coordinates of the turning point.

The turning point has \(x = \frac{3}{2}\).

Substitute \(x = \frac{3}{2}\) in the equation of the graph.

\[y = \frac{1}{10}\left(\frac{3}{2} + 5\right)\left(\frac{3}{2} - 8\right)\]
\[y = \frac{1}{10} \times \frac{13}{2} \times -\frac{13}{2}\]
\[y = -\frac{169}{40}\]

The turning point is \(\left(\frac{3}{2}, -\frac{169}{40}\right)\).

Cubic functions

The function \(f : R \to R, f(x) = ax^3 + bx^2 + cx + d, a, b, c, d \in R, a \neq 0\) is the cubic polynomial function. Although the shape of its graph may take several forms, for its maximal domain the function has a range of \(R\). Its long-term behaviour is dependent on the sign of the coefficient of the \(x^3\) term.

If \(a > 0\), then as \(x \to \infty\), \(y \to \infty\) and as \(x \to -\infty\), \(y \to -\infty\).

If \(a < 0\), then as \(x \to \infty\), \(y \to -\infty\) and as \(x \to -\infty\), \(y \to \infty\).

This behaviour is illustrated in the graph of \(y = x^3\), the simplest cubic function, and that of \(y = -x^3\).
Cubic functions of the form \( y = a(x - h)^3 + k \)

A significant feature of both of the graphs of \( y = x^3 \) and \( y = -x^3 \) is the stationary point of inflection at the origin. This point is constant under a dilation but becomes the point \((h, k)\) following a horizontal and vertical translation of \( h \) and \( k \) units respectively.

Cubic functions with equations of the form \( y = a(x - h)^3 + k \) have:
- a stationary point of inflection at \((h, k)\)
- one \( x \)-intercept
- long-term behaviour dependent on the sign of \( a \).

The coordinates of the stationary point of inflection are read from the equation in exactly the same way the turning points of a parabola are read from its equation in turning point form.

Cubic functions expressed in factorised form

A cubic function may have one, two or three \( x \)-intercepts, and hence its equation may have up to three linear factors. Where the equation can be expressed as the product of linear factors, we can readily deduce the behaviour of the function and sketch its graph without finding the positions of any turning points. Unlike the quadratic function, the turning points are not symmetrically placed between pairs of \( x \)-intercepts.

- If there are three linear factors, that is \( y = (x - m)(x - n)(x - p) \), the graph cuts the \( x \)-axis at \( x = m \), \( x = n \) and \( x = p \).
- If there is one factor of multiplicity 2 and one other linear factor, that is \( y = (x - m)^2(x - n) \), the graph touches the \( x \)-axis at a turning point at \( x = m \) and cuts the \( x \)-axis at \( x = n \).

If the equation of the cubic function has one linear factor and one irreducible quadratic factor, it is difficult to deduce its behaviour without either technology or calculus. For example, the diagram shows the graphs of \( y = (x + 3)(x^2 + 1) \) and \( y = (x^2 + 3)(x - 1) \).
a Sketch the graph of \( y = 2(x - 1)^3 + 8 \), labelling the intercepts with the coordinate axes with their exact coordinates.

b Determine the function \( f \) whose graph is shown in the diagram, expressing its rule as the product of linear factors with integer coefficients.

**THINK**

a 1 State the key feature that can be deduced from the equation.

**WRITE/DRAW**

a \( y = 2(x - 1)^3 + 8 \)

This equation shows there is a stationary point of inflection at \((1, 8)\).
2 Calculate the y-intercept.

\[ y = 2(-1)^3 + 8 \]
\[ y = 6 \]

The y-intercept is (0, 6).

3 Calculate the x-intercept in exact form.

\[ x = 0 \]
\[ 2(x - 1)^3 + 8 = 0 \]
\[ (x - 1)^3 = -4 \]
\[ x - 1 = \sqrt[3]{-4} \]
\[ x = 1 + \sqrt[3]{-4} \]
\[ x = 1 - \sqrt[3]{4} \]

The x-intercept is \((1 - \sqrt[3]{4}, 0)\).

4 Sketch the graph and label the intercepts with the coordinate axes.

5 State the required function.

b 1 Obtain a linear factor of the equation of the graph that has integer coefficients.

b The graph has an x-intercept at \(x = -0.75\).

In fraction form, this is \(x = -\frac{3}{4}\).

\[ (4x + 3) \] is a factor.

The graph has a turning point on the x-axis at \(x = 2\).

This means \((x - 2)^2\) is a factor.

The equation is of the form \(y = a(4x + 3)(x - 2)^2\).

The point \((0, -0.5)\) or \((0, -\frac{1}{2})\) lies on the graph.

Substitute this point into \(y = a(4x + 3)(x - 2)^2\).

\[ \frac{1}{2} = a(3)(-2)^2 \]
\[ \frac{1}{2} = 12a \]
\[ a = -\frac{1}{24} \]

The graph has the equation \(y = -\frac{1}{24}(4x + 3)(x - 2)^2\).

The domain of the graph is \(R\). Hence, the function \(f\) is given by \(f: R \rightarrow R, f(x) = -\frac{1}{24}(4x + 3)(x - 2)^2\).
Quartic and higher degree polynomial functions

The function \( f : \mathbb{R} \to \mathbb{R}, f(x) = ax^4 + bx^3 + cx^2 + dx + e \) where \( a, b, c, d, e \in \mathbb{R}, a \neq 0 \) is the general form of a quartic polynomial function. Its graph can take various shapes, but all of them exhibit the same long-term behaviour. If the \( x^4 \) term has a positive coefficient, \( y \to -\infty \) as \( x \to -\infty \); if the \( x^4 \) term has a negative coefficient, \( y \to +\infty \) as \( x \to -\infty \). Particular forms of the quartic equation enable some shapes of the graphs to be predicted.

Quartic functions of the form \( y = a(x - h)^4 + k \)

The simplest quartic function is \( y = x^4 \). It has a graph that has much the same shape as \( y = x^2 \), as shown in the diagram.

This leads to the conclusion that the graph of \( y = a(x - h)^4 + k \) will be much the same shape as that of \( y = a(x - h)^2 + k \) and will have the following characteristics.

For \( y = a(x - h)^4 + k \):

- If \( a > 0 \), the graph will be concave up with a minimum turning point \( (h, k) \).
- If \( a < 0 \), the graph will be concave down with a maximum turning point \( (h, k) \).
- The axis of symmetry has the equation \( x = h \).
- There may be zero, one or two \( x \) intercepts.

Quartic functions with linear factors.

Not all quartic functions can be factorised. However, if it is possible to express the equation as the product of linear factors, then the multiplicity of each factor will determine the behaviour of its graph.

A quartic polynomial may have up to 4 linear factors as it is of fourth degree. The possible combinations of these linear factors are:

- four distinct linear factors: \( y = (x - a)(x - b)(x - c)(x - d) \)
- one repeated linear factor: \( y = (x - a)^2(x - b)(x - c) \), where the graph has a turning point that touches the \( x \)-axis at \( x = a \)
- two repeated linear factors: \( y = (x - a)^2(x - b)^2 \), where the graph has turning points that touch the \( x \)-axis at \( x = a \) and \( x = b \).
- one factor of multiplicity three: \( y = (x - a)^3(x - b) \), where the graph has a stationary point of inflection that cuts the \( x \)-axis at \( x = a \).

The factorised forms may be derived from the general equation using standard algebraic techniques. Technology or calculus is required to accurately identify the position of turning points that do not lie on the \( x \)-axis.
**a** Sketch the graph of \( y = -x^4 + 8x^2 - 7 \) and hence determine graphically the number of solutions to the equation \( x^4 - 8x^2 + 3 = 0 \).

**b** A quartic function has the equation \( y = a(x + b)^4 + c \). The points \((0, 5)\), \((-2, 9)\) and \((4, 9)\) lie on the graph of the function. Calculate the values of \( a \), \( b \) and \( c \) and state the coordinates of the turning point.

**THINK**

<table>
<thead>
<tr>
<th>WRITE/DRAW</th>
</tr>
</thead>
</table>
| **a**  
1. Express the equation in factorised form.  
   \( y = -x^4 + 8x^2 - 7 \)  
   This is a quadratic in \( x^2 \).  
   \( y = -(x^4 - 8x^2 + 7) \)  
   Let \( a = x^2 \).  
   \( y = -(a^2 - 8a + 7) \)  
   \( = -(a - 7)(a - 1) \)  
   Substitute back for \( a \):  
   \( y = -(x^2 - 7)(x^2 - 1) \)  
   \( = -(x + \sqrt{7})(x - \sqrt{7})(x + 1)(x - 1) \) |
| **b**  
2. State the \( x \)- and \( y \)-values of the intercepts with the axes.  
   \(-x^4 + 8x^2 - 7 \)  
   \( x \)-intercepts: Let \( y = 0 \).  
   \(-\sqrt{7}, 0 \)  
   \((-1, 0) \)  
   \((0, -7) \)  
   \((\sqrt{7}, 0) \)  
   \( y \)-intercept:  
   \( y = 0 \).  
   \( \therefore y = -x^4 + 8x^2 - 7 \)  
   Let \( x = 0 \).  
   \( \therefore y = -7 \). |
| **c**  
3. What will be the long-term behaviour?  
   As the coefficient of \( x^4 \) is negative, \( y \to -\infty \) as \( x \to \pm \infty \). |
| **d**  
4. Sketch the graph. |
| **e**  
5. Rearrange the given equation so that the graph’s equation appears on one of its sides.  
   The given equation is \( x^4 - 8x^2 + 3 = 0 \).  
   This rearranges to  
   \( 3 = -x^4 + 8x^2 \)  
   \( 3 - 7 = -x^4 + 8x^2 - 7 \)  
   \( -x^4 + 8x^2 - 7 = -4 \) |
| **f**  
6. Explain how the number of solutions to the equation could be solved graphically.  
   The number of intersections of the graph of \( y = -x^4 + 8x^2 - 7 \) with the horizontal line \( y = -4 \) will determine the number of solutions to the equation \( x^4 - 8x^2 + 3 = 0 \). |
7 Specify the number of solutions. The line \( y = -4 \) lies parallel to the \( x \)-axis between the origin and the \( y \)-intercept of the graph \( y = -x^4 + 8x^2 - 7 \).

\[
\begin{align*}
\text{There are four points of intersection, so there are four solutions to the equation } & x^4 - 8x^2 + 3 = 0. \\
\end{align*}
\]

\[y = a(x + b)^4 + c\]

As the points \((-2, 9)\) and \((4, 9)\) have the same \(y\)-value, the axis of symmetry must pass midway between them.

The axis of symmetry is the line \( x = \frac{-2 + 4}{2} \)

\[x = 1\]

\[\therefore b = -1\]

2 Use the given points given to form a pair of simultaneous equations.

The equation is \( y = a(x - 1)^4 + c \).

Substitute the point \((4, 9)\):

\[a(3)^4 + c = 9\]

\[81a + c = 9 \quad [1]\]

Substitute the point \((0, 5)\):

\[a(-1)^4 + c = 5\]

\[a + c = 5 \quad [2]\]

3 Solve the equations.

Subtract equation [2] from equation [1]:

\[80a = 4\]

\[a = \frac{1}{20}\]

\[\therefore c = 5 - \frac{1}{20}\]

\[c = \frac{99}{20}\]

4 State the values required.

\[a = \frac{1}{20}, \ b = -1 \text{ and } c = \frac{99}{20}\]

5 Give the coordinates of the turning point.

The equation is \( y = \frac{1}{20}(x - 1)^4 + \frac{99}{20} \).

The minimum turning point is \( \left(1, \frac{99}{20}\right) \).
The family of polynomial functions \( y = x^n \) where \( n \in \mathbb{N} \)

One classification of the polynomial functions is to group them according to whether their degree is even or odd.

The graph of \( y = x^n \) where \( n \) is an even positive integer

The similarities shown between the graphs of \( y = x^2 \) and \( y = x^4 \) continue to hold for all polynomial functions of even degree. A comparison of the graphs of \( y = x^2 \), \( y = x^4 \) and \( y = x^6 \) is shown in the diagram.

The graphs each have a minimum turning point at \((0, 0)\) and each contains the points \((–1, 1)\) and \((1, 1)\). They exhibit the same long-term behaviour that as \( x \to \pm \infty \), \( y \to \infty \).

The graph of the function with the highest degree, \( y = x^6 \), rises more steeply than the other two graphs for \( x < -1 \) and \( x > 1 \). However, for \(-1 < x < 0\) and \(0 < x < 1\), the function with the highest degree lies below the other graphs.

For \( y = a(x - h)^n + k \), where \( n \) is an even positive integer:

- If \( a > 0 \), the graph will be concave up with a minimum turning point \((h, k)\).
- If \( a < 0 \), the graph will be concave down with a maximum turning point \((h, k)\).
- The axis of symmetry has the equation \( x = h \).
- There may be zero, one or two \( x \)-intercepts.
- The shape of the graph will be similar to that of \( y = a(x - h)^2 + k \).
- If \( a > 0 \), the range is \([k, \infty)\).
- If \( a < 0 \), the range is \((-\infty, k)\).

The graph of \( y = x^n \) where \( n \) is an odd positive integer, \( n > 1 \)

Polynomials of odd degree also share similarities, as the graphs of \( y = x^3 \) and \( y = x^5 \) illustrate.

Both \( y = x^3 \) and \( y = x^5 \) have a stationary point of inflection at \((0, 0)\), and both pass through the points \((–1, –1)\) and \((1, 1)\), as does the linear function \( y = x \). The three graphs display the same long-term behaviour that as \( x \to \pm \infty \), \( y \to \pm \infty \).

As observed for even degree polynomials, the graph of the function with the highest degree, \( y = x^5 \), rises more steeply than the other two graphs for \( x < -1 \) and \( x > 1 \). However, for \(-1 < x < 0\) and \(0 < x < 1\), the function with the highest degree lies below the other graphs.
The graphs of \( y = a(x - h)^n + k \) where \( n \) is an odd positive integer, \( n \neq 1 \), have the following characteristics:

For \( y = a(x - h)^n + k \), where \( n \) is an odd positive integer and \( n \neq 1 \):
- There is a stationary point of inflection at \((h, k)\).
- If \( a > 0 \), the long-term behaviour is as \( x \to \pm \infty, y \to \pm \infty \).
- If \( a < 0 \), the long-term behaviour is as \( x \to \pm \infty, y \to \mp \infty \).
- There will be one \( x \)-intercept.
- The shape of the graph is similar to that of the cubic function \( y = a(x - h)^3 + k \).

Polynomial functions that can be expressed as the product of linear factors

A degree \( n \) polynomial function may have up to \( n \) linear factors and therefore up to \( n \) intercepts with the \( x \)-axis. Where the polynomial can be specified completely as the product of linear factors, its graph can be drawn by interpreting the multiplicity of each linear factor together with the long-term behaviour determined by the sign of the coefficient of \( x^n \).

For example, consider \( y = (x + 2)^3(1 - x)(x - 3)^2 \). The equation indicates there are \( x \)-intercepts at \(-2\), \(1\) and \(3\). The \( x \)-intercept \((-2, 0)\) has a multiplicity of 3, meaning that there is a stationary point of inflection at this point. The \( x \)-intercept \((3, 0)\) has a multiplicity of 2, so this point is a turning point. The point \((1, 0)\) is a standard \( x \)-intercept. The polynomial is of degree 6 and the coefficient of \( x^6 \) is negative; therefore, as \( x \to \pm \infty \), \( y \to -\infty \).
**WORKED EXAMPLE 6**

Sketch the graph of \( y = (x - 1)^5 - 32 \).

**THINK**

1. State whether the graph has a turning point or a point of inflection, and give the coordinates of the key point.

2. Calculate the intercepts with the coordinate axes.

3. Sketch the graph.

**WRITE/DRAW**

\[ y = (x - 1)^5 - 32 \]

As the degree is odd, the graph will have a stationary point of inflection at \((1, -32)\).

**y-intercept:** Let \( x = 0 \).

\[ y = (-1)^5 - 32 = -33 \]

The \( y \)-intercept is \((0, -33)\).

**x-intercepts:** Let \( y = 0 \).

\[ 0 = (x - 1)^5 - 32 \]

\[ (x - 1)^5 = 32 \]

\[ x - 1 = 2 \]

\[ x = 3 \]

The \( x \)-intercept is \((3, 0)\).

**EXERCISE 2.2**

**Polynomial functions**

1. **WE1** Part of the graph of the parabola \( y = x^2 \) is shown in the diagram.
   
   a. Explain why the graph shows a function and state the type of correspondence.
   
   b. State the domain and range.
   
   c. Express the given parabola using function notation.
   
   d. Calculate the value of \( y \) when \( x = -2\sqrt{3} \).

2. Given \( f(x) = x^2 - 4 \):
   
   a. calculate the value of \( y \) when \( x = \frac{2}{3} \)
   
   b. calculate \( f(2a) \)
   
   c. state the implied domain of the function.
3. Consider the line $L$ where $L = \{(x, y) : 3x - 4y = 12\}$.
   a. Sketch the line.
   b. Calculate the gradient of the line.
   c. Determine the coordinates of the point on the line that is closest to the origin.

4. Consider the points A $(-1, -3)$ and B $(5, -7)$.
   a. Form the equation of the line that passes through points A and B.
   b. Calculate the equation of the perpendicular bisector of the line segment joining the points A and B.
   c. Calculate the angle at which the perpendicular bisector of AB cuts the x-axis.

5. a. Sketch the graph of $y = 2(3x - 2)^2 - 8$ and state its domain and range.
   b. Determine the equation of the given graph and hence obtain the coordinates of the turning point.

6. Consider the quadratic function $f: R^+ \cup \{0\} \rightarrow R$, $f(x) = 4x^2 - 8x + 7$.
   a. Determine the number of intercepts the graph of $y = f(x)$ makes with the x-axis.
   b. Express the equation of the function in the form $f(x) = a(x + b)^2 + c$.
   c. Sketch the graph of $y = f(x)$ and state its domain and range.

7. a. Sketch the graph of $y = -4(x + 2)^3 + 16$, labelling the intercepts with the coordinate axes with their exact coordinates.
   b. Determine the function $f$ whose graph is shown in the diagram, expressing its rule as the product of linear factors with integer coefficients.

8. Consider the function $f: [-2, 4] \rightarrow R$, $f(x) = 4x^3 - 8x^2 - 16x + 32$.
   a. Factorise $4x^3 - 8x^2 - 16x + 32$.
   b. Sketch the graph of $y = f(x)$.
   c. State the maximum and minimum values of the function $f$.

9. a. Sketch the graph of $y = x^2 - x^4$ and hence determine graphically the number of solutions to the equation $x^4 - x^2 + x - 2 = 0$.
   b. A quartic function has the equation $y = a(x + b)^4 + c$. The graph of the function cuts the x-axis at $x = -9$ and $x = -3$. The range of the graph is $(-\infty, 7]$. Calculate the values of $a$, $b$ and $c$ and state the coordinates of the turning point.

10. Sketch the graph of $y = x^4 - 6x^3$ and hence state the number of intersections the graph of $y = x^4 - 6x^3 - 1$ would make with the x-axis.

11. Sketch the graph of $y = (x + 1)^6 + 10$.

12. Sketch the graph of $y = (x + 4)(x + 2)^2(x - 2)^3(x - 5)$.
13 For each of the following, state:
   i the type of correspondence
   ii the domain and the range
   iii whether or not the relation is a function.

   a \[
   \begin{align*}
   (\frac{-3}{2}, 7) \\
   (-3, 5) \\
   (0, 4) \\
   (1, 0) \\
   (6, -6) \\
   (\frac{9}{2}, -9)
   \end{align*}
   \]

   b \[
   \begin{align*}
   (0, 0) \\
   (6, -6) \\
   (9, -9)
   \end{align*}
   \]

   c \[
   \begin{align*}
   (0, 2) \\
   (-2, 0) \\
   (0, -2) \\
   (2, 0)
   \end{align*}
   \]

   d \[
   \begin{align*}
   (0, 2) \\
   (3, 0)
   \end{align*}
   \]

   e \[
   \begin{align*}
   (0, 2)
   \end{align*}
   \]

   f \[
   \begin{align*}
   (2, 8) \\
   (-2, -8)
   \end{align*}
   \]

14 Sketch the following linear functions and state the range of each.
   a \( f: R \to R, \ f(x) = 9 - 4x \)
   b \( g: (-3, 5] \to R, \ g(x) = \frac{3x}{5} \)

15 Consider the three points A \((5, -3)\), B \((7, 8)\) and C \((-2, p)\). The line through
   A and C is parallel to \(9x + 7y = 24\).
   a Calculate the value of \(p\).
   b Determine the equation of the line through B that is perpendicular to AC.
   c Calculate the shortest distance from B to AC, expressing the value to
      1 decimal place.
16 a Express \(-x^2 + 2x - 5\) in the form \(a(x + b)^2 + c\).

b Hence, state the coordinates of the turning point of the graph of \(y = -x^2 + 2x - 5\).

c Sketch the graph of \(y = -x^2 + 2x - 5\) and state its range.

d Use a graphical method to show that the graphs of \(y = x + 3\) and \(y = -x^2 + 2x - 5\) never intersect.

e Determine the value of \(k\) so that the graphs of \(y = x + k\) and \(y = -x^2 + 2x - 5\) will intersect exactly once.

17 Determine the equations of the following quadratic functions.

a The turning point has coordinates \((-6, 12)\) and the graph of the function passes through the point \((4, -3)\).

b The points \((-7, 0)\), \((0, -20)\) and \((-2 \frac{1}{2}, 0)\) lie on the graph.

c The minimum value of the function is \(-5\) and it contains the points \((-8, 11)\) and \((8, 11)\).

18 Sketch the graphs of the following cubic functions without attempting to locate any turning points that do not lie on the coordinate axes.

a \(y = x^3 - x^2 - 6x\)

b \(y = 1 - \frac{1}{8}(x + 1)^3, x \in [-3, 2)\)

c \(y = 12(x + 1)^2 - 3(x + 1)^3\)

19 Form a possible equation for the cubic graph shown.

20 a Show that the graph of \(y = f(x)\) where \(f(x) = -2x^3 + 9x^2 - 24x + 17\) has exactly one \(x\)-intercept.

b Show that there is no stationary point of inflection on the graph.

c State the long-term behaviour of the function.

d Given the function has a one-to-one correspondence, draw a sketch of the graph.

21 a A quartic function has exactly one turning point at \((-5, 12)\) and also contains the point \((-3, -36)\). Form its equation.

b Sketch the graph of \(y = (2 + x)(1 - x)^3\).

c i Factorise \(-x^4 + x^3 + 10x^2 - 4x - 24\).

ii Hence sketch \(y = -x^4 + x^3 + 10x^2 - 4x - 24\).
22 a  i Sketch the graphs of \( y = x^6 \) and \( y = x^7 \) on the same set of axes, labelling any points of intersection with their coordinates.

ii Hence state the solutions to \( \{ x : x^6 - x^7 \geq 0 \} \).

b Sketch the graphs of \( y = 16 - (x + 2)^4 \) and \( y = 16 - (x + 2)^5 \) on the same set of axes, identifying the key features of each graph and any points of intersection.

c Consider the graph of the polynomial function shown.

i Assuming the graph is a monic polynomial that maintains the long-term behaviour suggested in the diagram, give a possible equation for the graph and state its degree.

ii In fact the graph cuts straight through the \( x \)-axis once more at \( x = 10 \). This is not shown on the diagram. Given this additional information, state the degree and a possible equation for the function.

23 Use CAS technology to sketch the graphs of \( y = x^4 - 2 \) and \( y = 2 - x^3 \), and hence state to 2 decimal places the values of the roots of the equation \( x^4 + x^3 - 4 = 0 \).

24 Use CAS technology to obtain the coordinates of any turning points or stationary points of inflection on the graphs of:

a \( y = (x^2 + x + 1)(x^2 - 4) \)

b \( y = 1 - 4x - x^2 - x^3 \)

c \( y = \frac{1}{4}(x - 2)^5(x + 3) + 80 \).

Express answers to 3 decimal places, where appropriate.

2.3 Other algebraic functions

The powers of the variable in a polynomial function must be natural numbers. In this section we consider functions where the power of the variable may be rational.

Maximal domain

The maximal domain of any function must exclude:

* any value of \( x \) for which the denominator would become zero
* any value of \( x \) which would create a negative term under a square root sign.

The maximal or implied domain of rational functions of the form \( y = \frac{g(x)}{f(x)} \), where both \( f(x) \) and \( g(x) \) are polynomials, must exclude any values of \( x \) for which \( f(x) = 0 \). The domain would be \( \mathbb{R} \setminus \{ x : f(x) = 0 \} \).

Likewise, the maximal domain of square root functions of the form \( y = \sqrt{f(x)} \) would be \( \{ x : f(x) \geq 0 \} \).

For a function of the form \( y = \frac{g(x)}{\sqrt{f(x)}} \), the maximal domain would be \( \{ x : f(x) > 0 \} \).
The rectangular hyperbola

The equation of the simplest hyperbola is \( y = \frac{1}{x} \). In power form this is written as \( y = x^{-1} \). Its maximal domain is \( \mathbb{R}\{0\} \), as the function is undefined if \( x = 0 \).

The graph of this function has the following characteristics.

- There is a vertical asymptote with equation \( x = 0 \).
- There is a horizontal asymptote with equation \( y = 0 \).
- As \( x \to \infty \), \( y \to 0 \) from above the horizontal asymptote, and as \( x \to -\infty \), \( y \to 0 \) from below the horizontal asymptote.
- As \( x \to 0^+ \), \( y \to \infty \), and as \( x \to 0^- \), \( y \to -\infty \).
- The function has one-to-one correspondence.
- The domain is \( \mathbb{R}\{0\} \) and the range is \( \mathbb{R}\{0\} \).

As the asymptotes are perpendicular to each other, the graph is called a rectangular hyperbola. The graph lies in the first and third quadrants formed by its asymptotes. The graph of \( y = \frac{1}{x} \) would lie in the second and fourth quadrants.

Hyperbolas of the form \( y = \frac{a}{x-h} + k \)

The asymptotes are the key feature of the graph of a hyperbola. Their positions are unaffected by a dilation, but if the graph of \( y = \frac{1}{x} \) is horizontally or vertically translated, then the vertical and horizontal asymptotes are moved accordingly.

The graph of \( y = \frac{a}{x-h} + k \) has:

- a vertical asymptote \( x = h \)
- a horizontal asymptote \( y = k \)
- a domain of \( \mathbb{R}\{h\} \)
- a range of \( \mathbb{R}\{k\} \).

If \( a > 0 \), the graph lies in quadrants 1 and 3 as formed by its asymptotes.

If \( a < 0 \), then the graph lies in quadrants 2 and 4 as formed by its asymptotes.

Identifying the asymptotes

The presence of a vertical asymptote at \( x = h \) on the graph of \( y = \frac{a}{x-h} + k \) could also be recognised by solving \( x - h = 0 \). The hyperbola \( y = \frac{a}{bx + c} \) has a vertical asymptote when \( bx + c = 0 \), and its maximal domain is \( \mathbb{R}\left\{-\frac{c}{b}\right\} \).
The horizontal asymptote is identified from the equation of a hyperbola expressed in proper rational form, that is, when the numerator is of lower degree than the denominator. The equation \( y = \frac{1 + 2x}{x} \) should be rewritten as \( y = \frac{1}{x} + 2 \) in order to identify the horizontal asymptote \( y = 2 \).

**WORKED EXAMPLE 7**

a Determine an appropriate equation for the hyperbola shown.

![Graph of a hyperbola](image)

b i Obtain the maximal domain of \( y = \frac{2x + 5}{x + 1} \).

ii Sketch the graph of \( y = \frac{2x + 5}{x + 1} \) and state its range.

**THINK**

1. Write the general equation of a hyperbola.

2. Identify the asymptotes and enter them into the equation.

3. Identify the known point through which the graph passes and use this to fully determine the equation.

**WRITE/DRAW**

a Let the equation be \( y = \frac{a}{x - h} + k \).

The graph shows there is a vertical asymptote at \( x = 2 \).

\[ \therefore y = \frac{a}{x - 2} + k \]

There is a horizontal asymptote at \( y = -4 \).

\[ \therefore y = \frac{a}{x - 2} - 4 \]

The graph passes through the origin.

Substitute \((0, 0)\):

\[ 0 = \frac{a}{-2} - 4 \]

\[ 4 = \frac{a}{2} \]

\[ a = -8 \]

The equation is \( y = \frac{-8}{x - 2} - 4 \).

b i Identify what must be excluded from the domain.

The function is undefined if its denominator is zero.

When \( x + 1 = 0, x = -1 \). This value must be excluded from the domain.

2 State the maximal domain.

The maximal domain is \( \mathbb{R}\{\{-1\} \).
Express the equation in proper rational form.

\[
\frac{2x + 5}{x + 1} = \frac{2(x + 1) + 3}{x + 1}
\]

\[
= \frac{2(x + 1)}{x + 1} + \frac{3}{x + 1}
\]

\[
= 2 + \frac{3}{x + 1}
\]

The equation is \( y = \frac{3}{x + 1} + 2 \).

State the equations of the asymptotes.

The graph has a vertical asymptote \( x = -1 \) and a horizontal asymptote \( y = 2 \).

Calculate any intercepts with the coordinate axes.

\( x \)-intercept: Let \( y = 0 \) in \( y = \frac{2x + 5}{x + 1} \).

\[
0 = \frac{2x + 5}{x + 1}
\]

\[
0 = 2x + 5
\]

\[
x = -\frac{5}{2}
\]

The \( x \)-intercept is \( \left( -\frac{5}{2}, 0 \right) \).

\( y \)-intercept: Let \( x = 0 \).

\[
y = \frac{5}{1}
\]

\[
y = 5
\]

The \( y \)-intercept is \( (0, 5) \).

Sketch the graph.

The range is \( R\{2\} \).
The truncus

The graph of the function \( y = \frac{1}{x^2} \) is called a truncus. Its rule can be written as a power function, \( y = x^{-2} \).

The graph of this function has the following characteristics.

- There is a vertical asymptote with equation \( x = 0 \).
- There is a horizontal asymptote with equation \( y = 0 \).
- The domain is \( \mathbb{R}\{0\} \).
- The range is \( \mathbb{R}^+ \).
- The function has many-to-one correspondence.
- The graph is symmetric about its vertical asymptote.

The graph of \( y = \frac{1}{x^2} \) lies in the first and second quadrants that are created by its asymptotes. The graph of \( y = -\frac{1}{x^2} \) would lie in the third and fourth quadrants.

The truncus is steeper than the hyperbola for \( x / \in (-1, 0) \) and \( x / \in (0, 1) \). However, a similar approach is taken to sketching both functions.

The general form of the truncus \( y = \frac{a}{(x - h)^2} + k \)

The graph of the truncus with the equation \( y = \frac{a}{(x - h)^2} + k \) has the following characteristics.

- There is a vertical asymptote at \( x = h \).
- There is a horizontal asymptote at \( y = k \).
- The domain is \( \mathbb{R}\{h\} \).
- If \( a > 0 \), then the range is \( (k, \infty) \).
- If \( a < 0 \), then the range is \( (-\infty, k) \).

WORKED EXAMPLE 8 Sketch the graph of \( y = 8 - \frac{2}{(x - 3)^2} \) and state its domain and range.

THINK

1 State the equations of the asymptotes.

\[ y = 8 - \frac{2}{(x - 3)^2} \]

The vertical asymptote is \( x = 3 \).

The horizontal asymptote is \( y = 8 \).

2 Calculate the y-intercept.

y-intercept: Let \( x = 0 \).

\[ y = 8 - \frac{2}{(-3)^2} \]

\[ y = \frac{7}{9} \]

The y-intercept is \( (0, \frac{7}{9}) \).
3 Calculate any $x$-intercepts.

$x$-intercepts: Let $y = 0$.

\[ 0 = 8 - \frac{2}{(x - 3)^2} \]
\[ \frac{2}{(x - 3)^2} = 8 \]
\[ 2 = 8(x - 3)^2 \]
\[ (x - 3)^2 = \frac{1}{4} \]
\[ x - 3 = \pm \frac{1}{2} \]
\[ x = \frac{5}{2} \text{ or } x = \frac{7}{2} \]

The $x$-intercepts are $\left(\frac{5}{2}, 0\right)$, $\left(\frac{7}{2}, 0\right)$.

4 Sketch the graph.

5 State the domain and range.

The domain is $\mathbb{R}\setminus\{3\}$ and the range is $(-\infty, 8)$.

The square root and cube root functions

The square root function has the rule $y = \sqrt{x}$, and the rule for the cube root function is $y = \sqrt[3]{x}$. As power functions these rules can be expressed as $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ respectively.

The maximal domain of $y = \sqrt{x}$ is $[0, \infty)$, because negative values under a square root must be excluded. However, cube roots of negative numbers are real, so the maximal domain of the cube root function $y = \sqrt[3]{x}$ is $\mathbb{R}$.

The graph of the square root function

The function $y = \sqrt{x}$ is the top half of the ‘sideways’ parabola $y^2 = x$. The bottom half of this parabola is the function $y = -\sqrt{x}$. 
The parabola \( y^2 = x \) is not a function, but its two halves are. The equation \( y^2 = x \) could also be written as \( y = \pm \sqrt{x} \). The turning point or vertex of the parabola is the end point for the square root functions \( y = \sqrt{x} \) and \( y = -\sqrt{x} \). These functions both have domain \([0, \infty)\), but their ranges are \([0, \infty)\) and \((-\infty, 0]\) respectively.

The parabola \( y^2 = -x \) would open to the left of its vertex. Its two branches would be the square root functions \( y = \sqrt{-x} \) and \( y = -\sqrt{-x} \), with domain \((-\infty, 0]\) and ranges \([0, \infty)\) and \((-\infty, 0]\) respectively.

The four square root functions show the different orientations that can be taken. Calculation of the maximal domain and the range will identify which form a particular function takes.

Square root functions of the form \( y = a\sqrt{x - h} + k \) have the following characteristics.
- The end point is \((h, k)\).
- The domain is \([h, \infty)\).
- If \(a > 0\), the range is \([k, \infty)\); if \(a < 0\), the range is \((-\infty, k]\).

Square root functions of the form \( y = a\sqrt{-x + h} + k \) have the following characteristics.
- The end point is \((h, k)\).
- The domain is \((-\infty, h]\).
- If \(a > 0\), the range is \([k, \infty)\); if \(a < 0\), the range is \((-\infty, k]\).

The graph of the cube root function

The graph of the cubic function \( y = x^3 \) has a stationary point of inflection at the origin. The graph of \( y^3 = x \) has a ‘sideways’ orientation but still has a point of inflection at the origin.

The rule \( y^3 = x \) can also be expressed as \( y = \sqrt[3]{x} \). The graph of \( y = \sqrt[3]{x} \) is shown in the diagram.
The graph \( y = \sqrt[3]{x} \) has the following characteristics.
- There is a point of inflexion at \((0, 0)\) where the tangent drawn to the curve would be vertical.
- The domain is \( R \) and the range is \( R \).
- The function has one-to-one correspondence.

The graph of \( y = -\sqrt[3]{x} \) would be the reflection of \( y = \sqrt[3]{x} \) in the \( x \) axis.

This would also be the graph of \( y = \sqrt[3]{-x} \), as \( \sqrt[3]{-x} = -\sqrt[3]{x} \).

The general equation \( y = a\sqrt[3]{x - h} + k \) shows the graph has the following characteristics.
- There is a point of inflection at \((h, k)\).
- The domain is \( R \) and the range is \( R \).
- One \( x \)-intercept can be located by solving \( a\sqrt[3]{x - h} + k = 0 \).
- If \( a > 0 \), the long-term behaviour is \( x \to \pm\infty, y \to \pm\infty \).
- If \( a < 0 \), the long-term behaviour is \( x \to \pm\infty, y \to \mp\infty \).

The long-term behaviour of the cube root function resembles that of the cubic function.

**WORKED EXAMPLE 9**

**a**  
**i** State the maximal domain of \( y = \sqrt{4 - x - 1} \).

**ii** Sketch the graph of \( y = \sqrt{4 - x - 1} \) and state its range.

**b** The graph of a cube root function has its point of inflection at \((1, 5)\) and the graph cuts the \( y \)-axis at \((0, 2)\). Determine the rule and sketch the graph.

**THINK**

**a**  
**i** Form the maximal domain.

**ii**  
1 State the coordinates of the end point.
2 Calculate the \( y \)-intercept, if there is one.

**WRITE/DRAW**

**a**  
**i** \( y = \sqrt{4 - x - 1} \)  
The term under the square root cannot be negative.  
\[ 4 - x \geq 0 \]
\[ x \leq 4 \]

The maximal domain is \((-\infty, 4]\).

**ii** The end point is \((4, -1)\).  
With the domain \((-\infty, 4]\), the graph opens to the left, so it will cut the \( y \)-axis.  
\( y \)-intercept: Let \( x = 0 \).  
\[ y = \sqrt{4} - 1 \]
\[ y = 1 \]

The \( y \)-intercept is \((0, 1)\).
3 Calculate the \( x \) intercept, if there is one.

The end point lies below the \( x \)-axis and the \( y \)-intercept lies above the \( x \)-axis. There will be an \( x \)-intercept.

\[ x \text{-intercept: Let } y = 0. \]
\[ 0 = \sqrt{4 - x} - 1 \]
\[ \sqrt{4 - x} = 1 \]
\[ 4 - x = 1 \]
\[ x = 3 \]

The \( x \)-intercept is \( (3, 0) \).

4 Sketch the graph.

5 State the range.

b 1 Write the general equation of a cube root function.

2 Insert the information about the point of inflection.

3 Fully determine the equation using the other piece of information given.

4 Calculate the \( x \)-intercept.

\[ x \text{-intercept: Let } y = 0. \]
\[ 0 = 3\sqrt[3]{x - 1} + 5 \]
\[ \sqrt[3]{x - 1} = \frac{-5}{3} \]
\[ x - 1 = \left( \frac{-5}{3} \right)^3 \]
\[ x = 1 - \frac{125}{27} \]
\[ x = \frac{98}{27} \]

The \( x \)-intercept is \( \left( \frac{98}{27}, 0 \right) \).

5 Sketch the graph.
Power functions of the form $y = x^{\frac{p}{q}}$, $p, q \in N$

The square root and cube root functions are examples of power functions of the form $y = x^{\frac{p}{q}}$, $p, q \in N$. For the square root function, $y = \sqrt{x} = x^{\frac{1}{2}}$ so $p = 1$ and $q = 2$; for the cube root function, $y = \sqrt[3]{x} = x^{\frac{1}{3}}$, so $p = 1$ and $q = 3$.

In this section we consider some other functions that have powers which are positive rational numbers and deduce the shape of their graphs through an analysis based on index laws.

Index laws enable $x^{\frac{p}{q}}$ to be expressed as $\sqrt[q]{x^p}$.

With $p, q \in N$, the function is formed as the $q$th root of the polynomial $x^p$. As polynomial shapes are known, this interpretation allows the shape of the graph of the function to be deduced. Whichever is the larger of $p$ and $q$ will determine whether the polynomial or the root shape will be the dominant function.

For the graph of $y = x^{\frac{p}{q}}$, $p, q \in N$:
- If $p > q$, the polynomial shape dominates, because the index $\frac{p}{q} > 1$.
- If $q > p$, the root shape dominates, because the index must be in the interval $0 < \frac{p}{q} < 1$.
- If $p = q$, the index is 1 and the graph is that of $y = x$.
- Even roots of the polynomial $x^p$ cannot be formed in any section where the polynomial graph is negative.
- The points $(0, 0)$ and $(1, 1)$ will always lie on the graph.

The basic polynomial or root shape for the first quadrant is illustrated for $p > q$ ⇒ index $> 1$, $p = q$ ⇒ index $= 1$ and $q > p$ ⇒ index $< 1$.

Note that the polynomial shape lies below $y = x$ for $0 < x < 1$ and above $y = x$ for $x > 1$, whereas the root shape lies above $y = x$ for $0 < x < 1$ and below $y = x$ for $x > 1$. It is always helpful to include the line $y = x$ when sketching a graph of the form $y = x^{\frac{p}{q}}$.

WORKED EXAMPLE 10

Give the domain and deduce the shape of the graph of:

a $y = x^{\frac{2}{3}}$

b $y = x^{\frac{3}{2}}$.

THINK

a 1 Express the function rule in surd form and deduce how the function can be formed.

WRITE/DRAW

a $y = x^{\frac{2}{3}} = \sqrt[3]{x^2}$

The function is formed as the cube root of the quadratic polynomial $y = x^2$. 
2 Use the nature of the operation forming the function to determine the domain of the function.

Cube roots of both positive and negative numbers can be calculated. However, the graph of \( y = x^2 \) lies in quadrants 1 and 2 and is never negative. Therefore, there will be two non-negative branches to the power function, giving it a domain of \( \mathbb{R} \).

As \( 3 > 2 \) (or as the index is less than 1), the root shape dominates the graph. This means the graph lies above \( y = x \) for \( 0 < x < 1 \) and below it for \( x > 1 \).

The points \((0, 0)\) and \((1, 1)\) lie on the graph, and by symmetry the graph will also pass through the point \((-1, 1)\).

4 Draw the required graph, showing its position relative to the line \( y = x \).

Note: There is a sharp point at the origin.

b 1 Express the function rule in surd form and deduce how the function can be formed.

The function is formed as the square root of the cubic polynomial \( y = x^3 \).

The graph of \( y = x^3 \) is positive in quadrant 1 and negative in quadrant 3, so the square root can only be taken of the section in quadrant 1. There will be one branch and its domain will be \( \mathbb{R}^+ \cup \{0\} \).

As \( 3 > 2 \) (or as the index is greater than 1), the polynomial shape dominates. The graph will lie below \( y = x \) for \( 0 < x < 1 \) and above it for \( x > 1 \).

The points \((0, 0)\) and \((1, 1)\) lie on the graph.
**Other algebraic functions**

1. a) **WE7** Determine an appropriate equation for the hyperbola shown.
   
   b) i) Find the maximal domain of \( y = \frac{5x - 2}{x - 1} \).
       
       ii) Sketch the graph of \( y = \frac{5x - 2}{x - 1} \) and state the range.

2. Sketch the graph of \( y = \frac{4}{1 - 2x} \), stating its domain and range.

3. **WE8** Sketch the graph of \( y = \frac{8}{(x + 2)^2} - 2 \) and state its domain and range.

4. Determine an appropriate equation for the truncus shown.

5. a) i) **WE9** State the maximal domain of \( y = -\sqrt{x + 9} + 2 \)
       
       ii) Sketch the graph of \( y = -\sqrt{x + 9} + 2 \) and state its range.
   
   b) The graph of a cube root function has its point of inflection at (1, 3) and the graph cuts the y-axis at (0, 1). Determine its rule and sketch its graph, locating its x-intercept.

6. a) Determine the maximal domain and the range of \( y = 3\sqrt{4x - 9} - 6 \), and sketch its graph.
   
   b) State the coordinates of the point of inflection of the graph of \( y = (10 - 3x)^{\frac{1}{3}} \) and sketch the graph.

7. **WE10** Give the domain and deduce the shape of the graph of:
   
   a) \( y = x^{\frac{3}{4}} \)
   
   b) \( y = x^{\frac{4}{3}} \)

8. Give the domain and deduce the shape of the graph of:
   
   a) \( y = x^{\frac{1}{5}} \)
   
   b) \( y = x^{\frac{1}{8}} \)

9. Determine the maximal domains of each of the following functions.
   
   a) \( y = \frac{x - 6}{x + 9} \)
   
   b) \( y = \sqrt{1 - 2x} \)
   
   c) \( y = \frac{-2}{(x + 3)^2} \)
   
   d) \( y = \frac{1}{x^2 + 3} \)
10 Sketch the following hyperbolas and state the domain and range of each.

a \[ y = \frac{4}{x} + 5 \]

b \[ y = 2 - \frac{3}{x + 1} \]

c \[ y = \frac{4x + 3}{2x + 1} \]

d \[ xy + 2y + 5 = 0 \]

e \[ y = \frac{10}{5 - x} - 5 \]

11 a The graph of a hyperbola has a vertical asymptote at \( x = -3 \) and a horizontal asymptote at \( y = 6 \). The point \((-4, 8)\) lies on the graph. Form the equation of this graph.

b Form a possible equation for the graph shown.

12 Sketch each of the following and state the domain and range of each.

a \[ y = \frac{2}{(3 - x)^2} + 1 \]

b \[ y = \frac{-3}{4(x - 1)^2} - 2 \]

c \[ y = \frac{1}{(2x + 3)^2} - 1 \]

d \[ y = \frac{25x^2 - 1}{5x^2} \]

13 a The diagram shows the graph of a truncus. Form its equation.

b A function \( f \) defined on its maximal domain has a graph \( y = f(x) \) in the shape of a truncus with range \((-4, \infty)\). Given \( f(-1) = 8 \) and \( f(2) = 8 \), determine the equation of the graph and state the function \( f \) using function notation.

14 a Give the equations of the two square root functions that form the branches of each of the following ‘sideways’ parabolas, and state the domain and range of each function.

i \[ (y - 2)^2 = 4(x - 3) \]

ii \[ y^2 + 2y + 2x = 5 \]

b Sketch the following square root functions and state the domain and range of each.

i \[ y = 1 - \sqrt{3x} \]

ii \[ y = 2\sqrt{-x} + 4 \]

iii \[ y = 2\sqrt{4 + 2x + 3} \]

iv \[ y = -\sqrt{3} - \sqrt{12 - 3x} \]
15 a The graph of the function \( f: [5, \infty) \to R \), 
\( f(x) = a\sqrt{x + b} + c \) is shown in the diagram. 
Determine the values of \( a, b \) and \( c \).

b The graph of the function \( f: (-\infty, 2] \to R \), 
\( f(x) = \sqrt{ax + b} + c \) is shown in the diagram. 
i i Determine the values of \( a, b \) and \( c \). 
ii If the graph of \( y = f(x) \) is reflected in the \( x \)-axis, what would the equation of the reflection be?

16 a Sketch the graph of 
\( \{ (x, y) : y = \sqrt{x + 2} - 1 \} \), labelling the intercepts with the coordinate axes with their exact coordinates.

b Sketch the graph of \( y = f(x) \) where 
\( f(x) = \frac{1 - \sqrt[3]{x} + 8}{2} \), stating its implied domain and range.

c Sketch the graph of \( g: [-3, 6] \to R \), 
\( g(x) = \sqrt[3]{-x} + 5 \) and state its domain and range.

d Form a possible equation for the cube root function whose graph is shown.

e The graph of a cube root function passes through the points \((-9, 5)\) and \((-1, -2)\). 
At the point \((-1, -2)\), the tangent drawn to the curve is vertical. Determine the equation of the graph.

f Express \( y \) as the subject of the equation \((y + 2)^3 = 64x - 128\) and hence state the coordinates of the point of inflection of its graph.

17 a Explain how the graph of \( y = x^{\frac{1}{3}} \) could be drawn using the graph of \( y = x \).

b On the same set of axes, sketch the graphs of \( y = x \) and \( y = x^{\frac{1}{3}} \).

c Hence, obtain \( \{ x : x^{\frac{1}{3}} - x > 0 \} \).
For each of the following, identify the domain and the quadrants in which the graph lies, and sketch the graph, showing its position relative to the line $y = x$.

a) $y = x^3$

b) $y = x^2$

c) $y = x^3$

d) $y = x^{0.25}$

Use CAS technology to draw the graphs of $y = \frac{1}{x^2 - 4}$, $y = \frac{1}{x^2 + 4}$ and $y = \frac{1}{(x - 4)^2}$. Hence or otherwise, determine the domain and range of each graph, and the equations of the asymptotes. Which graph is a truncus?

What is the maximal domain of the function $y = \sqrt{(2 - x)(x + 3)}$? Use CAS technology to investigate the shape of the graph.

### Combinations of functions

By combining together pieces of different functions defined over restricted domains, a ‘piecewise’ function can be created. By combining together different functions using arithmetic operations, other functions can be created. In this section we consider some of these combinations.

#### Hybrid functions

A **hybrid function**, or piecewise function, is a function whose rule takes a different form over different subsets of its domain. An example of a hybrid function is the one defined by the rule

$$f(x) = \begin{cases} \sqrt{x}, & x \leq 0 \\ 2, & 0 < x < 2 \\ x, & x \geq 2 \end{cases}$$

To sketch its graph, the three functions that combine to form its branches, $y = \sqrt{x}$, $y = 2$ and $y = x$, are drawn on their respective restricted domains on the same set of axes. If the branches do not join, then it is important to indicate which end points are open and which are closed, as each of the $x$-values of any function must have a unique $y$-value. The graph of this hybrid function $y = f(x)$ is shown in the diagram.

The function is not continuous when $x = 0$ as the branches do not join for that value of $x$. It is said to be discontinuous at that point of its domain. As the rule shows, $x = 0$ lies in the domain of the cube root section, the point $(0, 0)$ is closed and the point $(0, 2)$ is open.

The function is continuous at $x = 2$ as there is no break or gap in the curve. There is no need for a closed point to be shown at $x = 2$, because its two neighbouring branches run ‘naturally’ into each other at this point.
To calculate the value of the function for a given value of \( x \), choose the function rule of that branch defined for the section of the domain to which the \( x \)-value belongs.

**WORKED EXAMPLE 11**

Consider the function for which \( f(x) = \begin{cases} \sqrt{-x}, & x \leq -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \geq 1 \end{cases} \)

**a** Evaluate \( f(-1) \), \( f(0) \) and \( f(4) \).

**b** Sketch the graph of \( y = f(x) \).

**c** State:

i. any value of \( x \) for which the function is not continuous

ii. the domain and range.

**THINK**

a For each \( x \)-value, decide which section of the domain it is in and calculate its image using the branch of the hybrid function’s rule applicable to that section of the domain.

b 1 Obtain the information needed to sketch each of the functions forming the branches of the hybrid function.

**WRITE/DRAW**

a \( f(x) = \begin{cases} \sqrt{-x}, & x \leq -1 \\ 2 - x^2, & -1 < x < 1 \\ \sqrt{x} + 1, & x \geq 1 \end{cases} \)

\( f(-1) \): Since \( x = -1 \) lies in the domain section \( x \leq -1 \), use the rule \( f(x) = \sqrt{-x} \).

\( f(-1) = \sqrt{-(-1)} = \sqrt{1} = 1 \)

\( f(0) \): Since \( x = 0 \) lies in the domain section \( -1 < x < 1 \), use the rule \( f(x) = 2 - x^2 \).

\( f(0) = 2 - 0^2 = 2 - 0 = 2 \)

\( f(4) \): Since \( x = 4 \) lies in the domain section \( x \geq 1 \), use the rule \( f(x) = \sqrt{x} + 1 \).

\( f(4) = \sqrt{4} + 1 = 2 + 1 = 3 \)

b \( y = \sqrt{-x}, x \leq -1 \) is a square root function. The points \((-1, 1)\) and \((-4, 2)\) lie on its graph.

\( y = 2 - x^2, -1 < x < 1 \) is a parabola with maximum turning point \((0, 2)\). At \( x = -1 \) or \( x = 1 \), \( y = 1 \). The points \((-1, 1)\) and \((1, 1)\) are open for the parabola.

\( y = \sqrt{x} + 1, x \geq 1 \) is a square root function. The points \((1, 2)\) and \((4, 3)\) lie on its graph.
2 Sketch each branch on the same set of axes to form the graph of the hybrid function.

\[
\begin{align*}
(0, 2) & \quad (4, 3) \\
(-1, 1) & \quad (1, 1) \\
(-4, 2) & \quad (1, 2)
\end{align*}
\]

\[y = f(x)\]

\[
\begin{align*}
\text{c i} & \quad \text{State any value of } x \text{ where the branches of the graph do not join} \\
\text{c ii} & \quad \text{State the domain and range.}
\end{align*}
\]

\[
\text{c The function is not continuous at } x = 1.
\]

The domain is \(R\).
The range is \([1, \infty)\).

Sums, differences and products of functions

New functions are formed when two given functions are combined together under the operations of addition, subtraction and multiplication. The given functions can only be combined where they both exist, so the domain of the new function formed must be the domain common to both the given functions. For functions \(f\) and \(g\) with domains \(d_f\) and \(d_g\) respectively, the common domain is \(d_f \cap d_g\).

- The sum and difference functions \(f \pm g\) are defined by \((f \pm g)(x) = f(x) \pm g(x)\) with domain \(d_f \cap d_g\).
- The product function \(fg\) is defined by \((fg)(x) = f(x)g(x)\) with domain \(d_f \cap d_g\).

Graphs of the functions \(f \pm g\) and \(fg\) may be able to be recognised from their rules. If not, the graphs may be deduced by sketching the graphs of \(f\) and \(g\) and combining by addition, subtraction or multiplication, as appropriate, the values of \(f(x)\) and \(g(x)\) for selected \(x\)-values in their common domain. The difference function \(f - g\) can be considered to be the sum function \(f + (-g)\).

\[
\begin{align*}
\text{Consider the functions } f \text{ and } g \text{ defined by } f(x) = \sqrt{4 + x} \text{ and } g(x) = \sqrt{4 - x} \text{ respectively.} \\
\text{a Form the rule for the sum function } f + g, \text{ stating its domain, and sketch the graph of } y = (f + g)(x). \\
\text{b Form the rule for the product function } fg \text{ and state its domain and range.}
\end{align*}
\]
THINK

a 1 Write the domains of the functions $f$ and $g$.

2 State the common domain.

3 Form the sum function and state its domain.

4 Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same set of axes. Add the y-coordinates of key points together to form the graph of $y = (f + g)(x)$.

b 1 Form the product function and state its domain.

2 State the range of the function.

WRITE/DRAW

a $f(x) = \sqrt{4 + x}$
   Domain: $4 + x \geq 0$
   $x \geq -4$
   $d_f = [-4, \infty)$

$g(x) = \sqrt{4 - x}$
   Domain: $4 + x \geq 0$
   $x \leq 4$
   $d_g = (-\infty, 4]$  

$df \cap dg = [-4, 4]$  

$(f + g)(x) = f(x) + g(x)$
   $= \sqrt{4 + x} + \sqrt{4 - x}$
   $d_{f+g} = [-4, 4]$

b $(fg)(x) = f(x)g(x)$
   $= (\sqrt{4 + x}) \times (\sqrt{4 - x})$
   $= \sqrt{(4 + x)(4 - x)}$
   $= \sqrt{16 - x^2}$
   $d_{fg} = [-4, 4]$  

The rule $(fg)(x) = \sqrt{16 - x^2}$ is that of the top half of a semicircle with centre $(0, 0)$ and radius 4. Therefore, the range is $[0, 4]$.

Graphical techniques

Given the graphs of functions whose rules are not necessarily known, it may be possible to deduce the shape of the graph of the function that is the sum or other combination of the functions whose graphs are given.
Addition of ordinates

Given the graphs of $y_1 = f(x)$ and $y_2 = g(x)$, the graphing technique known as addition of ordinates adds together the $y$-values, or ordinates, of the two given graphs over the common domain to form the graph of the sum function $y = y_1 + y_2 = f(x) + g(x)$.

Note the following points when applying this technique over the common domain $d_f \cap d_g$:

- If the graphs of $f$ and $g$ intersect at $(a, b)$, then the point $(a, 2b)$ lies on the graph of $f + g$.
- Where $f(x) = -g(x)$, the graph of $f + g$ cuts the $x$-axis.
- If one of $f(x)$ or $g(x)$ is positive and the other is negative, the graph of $f + g$ lies between the graphs of $f$ and $g$.
- If one of $f(x)$ or $g(x)$ is zero, then the graph of $f + g$ cuts the other graph.
- If $f(x) \rightarrow 0^+$, then the graph of $f + g$ approaches the graph of $g$ from above.
- If $f(x) \rightarrow 0^-$, then the graph of $f + g$ approaches the graph of $g$ from below.
- Any vertical asymptote of $f$ or $g$ will be a vertical asymptote on the graph of $f + g$.

The subtraction of ordinates is usually simpler to achieve as the addition of the ordinates of $y_1 = f(x)$ and $y_2 = -g(x)$.

Squaring ordinates

Given the graph of $y = f(x)$, the graph of $y = (f(x))^2$ can be deduced by squaring the $y$-values, or ordinates, noting in particular that $0^2 = 0$, $1^2 = 1$ and $(-1)^2 = 1$.

- The graph of $f$ and its square will intersect at any point on $f$ where $y = 0$ or $y = 1$.
- If the point $(a, -1)$ is on the graph of $f$, then $(a, 1)$ lies on the graph of the squared function.
- The squared function’s graph can never lie below the $x$-axis.
- Where $0 < f(x) < 1$, $(f(x))^2 < f(x)$, and where $f(x) > 1$ or $f(x) < -1$, $(f(x))^2 > f(x)$.

Similar reasoning about the ordinates and their square roots and the domain will allow the graph of $y = \sqrt{f(x)}$ to be deduced.

These graphing techniques can be applied to combinations of known functions where the first step would be to draw their graphs.

WORKED EXAMPLE 13

The graphs of the functions $f$ and $g$ are shown.

Draw the graph of $y = (f + g)(x)$. 
Consider the function for which \( f(x) = \begin{cases} \sqrt[3]{x}, & x < -1 \\ x^3, & -1 \leq x \leq 1 \\ 2 - x, & x > 1 \end{cases} \)

a. Evaluate \( f(-8), f(-1) \) and \( f(2) \).

b. Sketch the graph of \( y = f(x) \).

c. State:
   i. any value of \( x \) for which the function is not continuous
   ii. the domain and range.

2 Form the rule for the hybrid function shown in the diagram.
3 WE12 Consider the functions \( f \) and \( g \) defined by \( f(x) = -\sqrt{1 + x} \) and \( g(x) = -\sqrt{1 - x} \) respectively.

a Form the rule for the sum function \( f + g \), stating its domain, and sketch the graph of \( y = (f + g)(x) \).

b Form the rule for the product function \( fg \), stating its domain and range.

4 Given \( f(x) = x^3 \) and \( g(x) = x^2 \), form the rule \( (f - g)(x) \) for the difference function and sketch the graphs of \( y = f(x) \), \( y = -g(x) \) and \( y = (f - g)(x) \) on the same set of axes. Comment on the relationship of the graphs at the places where \( y = (f - g)(x) \) cuts the axes.

5 WE13 The graphs of the functions \( f \) and \( g \) are shown. Draw the graph of \( y = (f + g)(x) \).

6 Sketch the graph of \( y = x^2 - 1 \) and hence draw the graph of \( y = (x^2 - 1)^2 \), stating the domain and range.

7 Sketch the graphs of each of the following hybrid functions and state their domains, ranges and any points of discontinuity.

a \( y = \begin{cases} -2x, & x \leq 0 \\ 4 - x^2, & x > 0 \end{cases} \)

b \( y = \begin{cases} \sqrt[3]{x}, & x < 1 \\ \frac{1}{x^2}, & x \geq 1 \end{cases} \)

8 A hybrid function is defined by

\[
 f(x) = \begin{cases} \frac{1}{(x + 1)^2}, & x < -1 \\ x^2 - x, & -1 \leq x \leq 2 \\ 8 - 2x, & x > 2 \end{cases}
\]

a Evaluate:

i \( f(-2) \) 

ii \( f(2) \).

b Sketch the graph of \( y = f(x) \).

c State the domain over which the hybrid function is continuous.

9 Consider the function

\[
 f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{9}x^3 + 5, & x < -3 \\ \sqrt{1 - x}, & -3 \leq x \leq 1 \\ x - 2, & x > 1 \end{cases}
\]

a Show the function is not continuous at \( x = 1 \).

b Sketch the graph of \( y = f(x) \) and state the type of correspondence it displays.

c Determine the value(s) of \( x \) for which \( f(x) = 4 \).
10 Form the rule for the function whose graph is shown in the diagram.

11 A hybrid function is defined by

\[ f(x) = \begin{cases} 
  x + a, & x \in (-\infty, -8] \\
  \sqrt[3]{x} + 2, & x \in (-8, 8] \\
  b, & x \in (8, \infty) 
\end{cases} \]

a Determine the values of \( a \) and \( b \) so that the function is continuous for \( x \in \mathbb{R} \), and for these values, sketch the graph of \( y = f(x) \).

b Determine the values of \( a \) and \( b \) from part a for the remainder of this question.

c Find \( \{ x : f(x) = 1 \} \).

12 Consider the functions \( f \) and \( g \) defined by \( f(x) = 5 - 2x \) and \( g(x) = 2x - 2 \) respectively. For each of the following, give the rule, state the domain and the range, and sketch the graph.

a \( y = (f + g)(x) \)

b \( y = (f - g)(x) \)

c \( y = (fg)(x) \)

13 Consider the functions \( f(x) = x^2 - 1 \) and \( g(x) = \sqrt{x + 1} \).

a Evaluate:

i \( (g - f)(3) \)

ii \( (gf)(8) \)

b State the domain of the function \( f + g \).

c Draw a possible graph for each of the following functions.

i \( f + g \)

ii \( g - f \)

iii \( fg \)

14 Use addition of ordinates to sketch \( y = x + \sqrt{-x} \).

15 The graphs of two functions \( y = f(x) \) and \( y = g(x) \) are drawn in the following diagrams. Use the addition of ordinates technique to sketch \( y = f(x) + g(x) \) for each diagram.

a

b
Consider the function defined by \( g(x) = (2x - 1)^3 \). Sketch the graph of \( y = g(x) \) and hence sketch \( y = (g(x))^2 \).

b Calculate the coordinates of the points of intersection of the graphs of \( y = f(x) \) and \( y = (f(x))^2 \) if \( f(x) = x^3 - 2x \).

Use CAS technology to draw on screen the hybrid function defined by the rule
\[
 f(x) = \begin{cases} 
 -x, & x < -1 \\
 1, & -1 \leq x \leq 1 \\
 (2x - 1)(x - 3), & x > 1 
\end{cases}
\]
State the range of the function.

Draw the graph of \( y = \sqrt[3]{f(x)} \) for the function \( f(x) = x^2(x + 4)(x - 2) \) using CAS technology and describe the shape of the graph. How many points of intersection are there for the two graphs \( y = \sqrt[3]{f(x)} \) and \( f(x) = x^2(x + 4)(x - 2) \)?

Non-algebraic functions

Although polynomial functions of high degrees can be used to approximate exponential and trigonometric functions, these functions have no exact algebraic form. Such functions are called transcendental functions.

Exponential functions

Exponential functions are those of the form \( f: \mathbb{R} \to \mathbb{R}, f(x) = a^x \) where the base \( a \in \mathbb{R} \setminus \{1\} \).

The index law \( a^0 = 1 \) explains why the graph of \( y = a^x \) must contain the point \((0, 1)\). The graph of \( y = 2^x \) would also contain the point \((1, 2)\), while the graph of \( y = 3^x \) would contain the point \((1, 3)\).

As the base becomes larger, exponential functions increase more quickly. This can be seen in the diagram comparing the graphs of \( y = 2^x \) and \( y = 3^x \).

The index law \( a^{-x} = \frac{1}{a^x} \) explains why for negative values of \( x \) the graphs of \( y = 2^x \) and \( y = 3^x \) approach the \( x \)-axis but always lie above the \( x \)-axis. The \( x \)-axis is a horizontal asymptote for both of their graphs and for any graph of the form \( y = a^x \).

Exponential functions of the form \( f: \mathbb{R} \to \mathbb{R}, f(x) = a^{-x} \) where \( a \in \mathbb{R} \setminus \{1\} \) have base \( \frac{1}{a} \). This is again explained by index laws,
\[
 a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x .
\]
However, it is often preferable to write \( y = 2^{-x} \) rather than \( y = \left(\frac{1}{2}\right)^x \).

The graph of \( y = 2^{-x} \) or \( \left(\frac{1}{2}\right)^x \) must contain the point \((0, 1)\), and other points on this graph include \((-1, 2)\) and \((1, \frac{1}{2})\).
The graph of \( y = 2^x \) illustrates a ‘growth’ form, whereas the graph of \( y = 2^{-x} \) takes a ‘decay’ form. The two graphs are reflections of each other in the \( y \)-axis.

**WORKED EXAMPLE 14**

Consider the function \( f(x) = -5^x \).

**a** Evaluate \( f(2) \).

**b** On the same set of axes sketch the graphs of \( y = 5^x \), \( y = -5^x \) and \( y = 5^{-x} \).

**c** Express \( y = 5^{-x} \) in an equivalent form.

**THINK**

**a** Calculate the required value.

*Note: \(-5^2 \neq (-5)^2\)*

**b** 1 Identify points on each curve.

2 Sketch the graphs on the same axes.

**WRITE/DRAW**

**a** \( f(x) = -5^x \)

\( f(2) = -5^2 = -25 \)

**b** \( y = 5^x \) contains the points (0, 1) and (1, 5).

\( y = -5^x \) contains the points (0, -1) and (1, -5).

\( y = 5^{-x} \) contains the points (0, 1) and (-1, 5).

**c** Write an equivalent form for the given rule.

\( y = -5^x \)

Since \( 5^{-x} = \left(\frac{1}{5}\right)^x \), an alternative form for the rule is \( y = \left(\frac{1}{5}\right)^x \) or \( y = 0.2^x \).

The exponential function \( y = e^x \)

The number \( e \) is known as Euler’s number after the eminent Swiss mathematician Leonhard Euler, who first used the symbol. Its value is 2.718 281 828 45…

Like \( \pi \), \( e \) is an irrational number of great importance in mathematics. It appears in later topic on calculus. Most calculators have keys for both \( \pi \) and \( e \).

As \( 2 < e < 3 \), the graph of \( y = e^x \) lies between those of \( y = 2^x \) and \( y = 3^x \), and has much the same shape.
The graph of \( y = e^x \) has the following key features.
- The points \((0, 1)\) and \((1, e)\) lie on the graph.
- There is a horizontal asymptote at \( y = 0 \).
- The domain is \( \mathbb{R} \).
- The range is \( \mathbb{R}^+ \).
- The function has one-to-one correspondence.
- As \( x \to \infty \), \( y \to \infty \), and as \( x \to -\infty \), \( y \to 0^+ \).

The graph shows an ‘exponential growth’ shape. Mathematical models of such phenomena, for example population growth, usually involve the exponential function \( y = e^x \).

Exponential decay models usually involve the function \( y = e^{-x} \).

The graph of \( y = e^{-x} \) is shown.

### Sketching the graph of \( y = ae^{nx} + k \)

A vertical translation affects the position of the horizontal asymptote of an exponential graph in the same way it does for a hyperbola or truncus. The graph of \( y = e^x + k \) has a horizontal asymptote with equation \( y = k \). If \( k < 0 \), then \( y = e^x + k \) will cut through the \( x \)-axis and its \( x \)-intercept will need to be calculated.

To sketch the graph of an exponential function:
- identify the equation of its asymptote
- calculate its \( y \)-intercept
- calculate its \( x \)-intercept if there is one.

If the function has no \( x \)-intercept, it may be necessary to obtain the coordinates of another point on its graph.

The graph of \( y = ae^{nx} + k \) has the following key features.
- There is a horizontal asymptote at \( y = k \).
- There is one \( y \)-intercept, obtained by letting \( x = 0 \).
- There is either one or no \( x \)-intercept. The relative position of the asymptote and \( y \)-intercept will determine whether there is an \( x \)-intercept.
- If \( a > 0 \), the range is \((k, \infty)\).
- If \( a < 0 \), the range is \((-\infty, k)\).
The values of $a$ and $n$ in the equation $y = ae^{nx} + k$ are related to dilation factors, and their signs will affect the orientation of the graph. The possibilities are shown in the following diagrams and table.

<table>
<thead>
<tr>
<th>Dilation factors</th>
<th>Graph behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a &gt; 0, n &gt; 0$</td>
<td>As $x \to -\infty$, $y \to k^+$</td>
</tr>
<tr>
<td>$a &gt; 0, n &lt; 0$</td>
<td>As $x \to \infty$, $y \to k^+$</td>
</tr>
<tr>
<td>$a &lt; 0, n &gt; 0$</td>
<td>As $x \to -\infty$, $y \to k^-$</td>
</tr>
<tr>
<td>$a &lt; 0, n &lt; 0$</td>
<td>As $x \to \infty$, $y \to k^-$</td>
</tr>
</tbody>
</table>

Calculating the $x$-intercept

To calculate the $x$-intercept of $y = e^x - 2$, the exponential equation $e^x - 2 = 0$ needs to be solved. This can be solved using technology. Alternatively, we can switch $e^x = 2$ to its logarithm form to obtain the exact value of the $x$-intercept as $x = \log_e(2)$.

In decimal form, $\log_e(2) \approx 0.69$, with the value obtained by using the ln key on a calculator. The symbol ‘ln’ stands for the natural logarithm, $\log_e$. Base $e$ logarithms as functions are studied in Topic 4.

Sketching the graph of $y = ae^{n(x-h)} + k$

Under a horizontal translation of $h$ units, the point (0, 1) on $y = e^x$ is translated to the point $(h, 1)$ on the graph of $y = e^{x-h}$. The $y$-intercept is no longer (0, 1), so it will need to be calculated.

By letting $x = h$ for the horizontal translation, the index for the exponential will be zero. This simplifies the calculation to obtain another point on the graph. For the graph of $y = ae^{n(x-h)} + k$ when $x = h$, $y = ae^0 + k \Rightarrow y = a + k$.

WORKED EXAMPLE 15

Sketch the following graphs and state the domain and range of each graph.

- **a** $y = 2e^x + 1$
- **b** $y = 3 - 3e^{-\frac{x}{2}}$
- **c** $y = \frac{1}{4}e^{x+1}$

**THINK**

- **1** State the equation of the asymptote.
- **2** Calculate the $y$-intercept.
- **3** Calculate any $x$-intercepts.

**WRITE/DRAW**

- **a** $y = 2e^x + 1$
  
The asymptote is $y = 1$.
  
- **b** $y = 3 - 3e^{-\frac{x}{2}}$
  
  $y$-intercept: Let $x = 0$.
  
  $y = 2e^0 + 1$
  
  $y = 2 + 1$
  
  $y = 3$
  
  The $y$-intercept is (0, 3).

- **c** $y = \frac{1}{4}e^{x+1}$
  
  As the $y$-intercept is above the positive asymptote, there is no $x$-intercept.
4 Locate another point if necessary and sketch the graph.

Growth shape

The domain is $R$ and the range is $(1, \infty)$.

5 State the domain and range.

b 1 State the equation of the asymptote.

$y = 3 - 3e^{-\frac{x}{2}}$

The asymptote is $y = 3$.

2 Calculate the $y$-intercept.

$y = 3 - 3e^{0}$

$y = 0$

The $y$-intercept is $(0, 0)$.

(0, 0) is also the $x$-intercept.

If $x = -2$, then $y = 3 - 3e < 0$.

3 Calculate any $x$-intercepts.

There are no $x$-intercepts as the $x$-axis is an asymptote.

5 State the domain and range.

c 1 State the equation of the asymptote.

$y = \frac{1}{4}e^{x} + 1$

The asymptote is $y = 0$.

2 Calculate the $y$-intercept.

$y = \frac{1}{4}e^{0} + 1$

$= \frac{1}{4}e$

The $y$-intercept is $\left(0, -\frac{e}{4}\right)$.
4 Locate another point if necessary and sketch the graph.

Let \( x = -1 \).

\[
y = -\frac{1}{4}e^0 = -\frac{1}{4}
\]

Another point on the graph is \((-1, -\frac{1}{4})\).

5 State the domain and range.

The domain is \(R\) and the range is \(R^-\).

Determined the equation of an exponential function

The form of the equation is usually specified along with the given information needed to determine the equation. This is necessary because it could be difficult to decide whether the base is \(e\) or some other value. The number of pieces of information given will also need to match the number of parameters or unknown constants in the equation.

The asymptote is a key piece of information to obtain. If a graph is given, the equation of the asymptote will be apparent. Insert this value into the equation and then substitute coordinates of known points on the graph. Simultaneous equations may be required to calculate all the parameters in the equation.

### WORKED EXAMPLE 16

a The diagram shows the graph of \(y = ae^x + b\). Determine the values of \(a\) and \(b\).

b The graph of \(y = a \times 10^{kx}\) contains the points \((2, 30)\) and \((4, 300)\). Form its equation.

#### THINK

a 1 Insert the equation of the asymptote into the equation of the graph.

#### WRITE

a \(y = ae^x + b\)

The asymptote is \(y = 3\). This means \(b = 3\). The equation becomes \(y = ae^x + 3\).
2 Use a known point on the graph to fully determine the equation.

The graph passes through the point \((-1, 3 - \frac{4}{e})\).
Substitute this point into the equation.
\[3 - \frac{4}{e} = ae^{-1} + 3\]
\[-\frac{4}{e} = a\]
\[a = -4\]
The equation is \(y = -4e^x + 3\).

3 State the values required.

b 1 Substitute the given points in the equation.

\[y = a \times 10^k\]

\((2, 30) \Rightarrow 30 = a \times 10^{2k}\]
\((4, 300) \Rightarrow 300 = a \times 10^{4k}\]

\[a \times 10^{2k} = 30 \quad [1]\]
\[a \times 10^{4k} = 300 \quad [2]\]

Divide equation [2] by equation [1]:
\[a \times 10^{4k} = \frac{300}{30} \quad 10^{2k} = 10\]
\[2k = 1\]
\[k = \frac{1}{2}\]

Substitute \(k = \frac{1}{2}\) in equation [1]:
\[a \times 10^1 = 30\]
\[a = 3\]
The equation is \(y = 3 \times 10^{\frac{1}{2}}\).

Circular functions

Circular functions, or trigonometric functions, are periodic functions such as \(y = \sin(x)\), \(y = \cos(x)\) and \(y = \tan(x)\).

The graph of the sine function has a wave shape that repeats itself every \(2\pi\) units. Its period is \(2\pi\) as shown in its graph.

\[y = \sin(x)\]
The graph oscillates about the line $y = 0$ (the $x$-axis), rising and falling by up to 1 unit. This gives the graph its range of $[-1, 1]$ with a mean, or equilibrium, position $y = 0$ and an amplitude of 1.

The graph of the cosine function has the same wave shape with period $2\pi$.

![Graph of $y = \cos(x)$]

The characteristics of both the sine function and the cosine function are:

- period $2\pi$
- amplitude 1
- mean position $y = 0$
- domain $\mathbb{R}$
- range $[-1, 1]$
- many-to-one correspondence.

Although the domain of both the sine and cosine functions is $\mathbb{R}$, they are usually sketched on a given restricted domain.

The two graphs of $y = \sin(x)$ and $y = \cos(x)$ are ‘out of phase’ by $\frac{\pi}{2}$, that is, 

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x).$$

In other words, a horizontal shift of the cosine graph by $\frac{\pi}{2}$ units to the right gives the sine graph. Likewise, a horizontal shift of the sine graph by $\frac{\pi}{2}$ units to the left gives the cosine graph; 

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x).$$

The periodicity of the functions is expressed by $f(x) = f(x + n2\pi)$, $n \in \mathbb{Z}$ where $f$ is sin or cos.

**Graphs of $y = a \sin(nx)$ and $y = a \cos(nx)$**

The value of $a$ affects the amplitude of the sine and cosine functions.

Because $-1 \leq \sin(nx) \leq 1$, $-a \leq a \sin(nx) \leq a$. This means the graphs of $y = a \sin(nx)$ and $y = a \cos(nx)$ have amplitude $|a|$.

As the amplitude measures a distance, the rise or fall from the mean position, it is always positive. If $a < 0$, the graphs will be inverted, or reflected in the $x$-axis.

The value of $n$ affects the period of the sine and cosine functions.
Since one cycle of \( y = \sin(x) \) is completed for \( 0 \leq x \leq 2\pi \), one cycle of \( y = \sin(nx) \) is completed for \( 0 \leq nx \leq 2\pi \). This means one cycle is covered over the interval \( 0 \leq x \leq \frac{2\pi}{n} \), assuming \( n > 0 \).

The graphs of \( y = a \sin(nx) \) and \( y = a \cos(nx) \) have:
- period \( \frac{2\pi}{n} \)
- amplitude \( |a| \)
- range \([-a, a]\).

The graphs of \( y = a \sin(nx) + k \) and \( y = a \cos(nx) + k \)
Any vertical translation affects the equilibrium or mean position about which the sine and cosine graphs oscillate.

The graphs of \( y = a \sin(nx) + k \) and \( y = a \cos(nx) + k \) have:
- mean position \( y = k \)
- range \([k - a, k + a]\).

Where the graph of \( y = f(x) \) crosses the \( x \)-axis, the intercepts are found by solving the trigonometric equation \( f(x) = 0 \).

**WORKED EXAMPLE 17**

a Sketch the graph of \( y = 3 \sin(2x) + 4 \), \( 0 \leq x \leq 2\pi \)
b The diagram shows the graph of a cosine function. State its mean position, amplitude and period, and give a possible equation for the function.

![Graph of y = 3 sin(2x) + 4, 0 ≤ x ≤ 2π](image)

**THINK**

a 1 State the period and amplitude of the graph.

2 State the mean position and the range.

**WRITE/DRAW**

a \( y = 3 \sin(2x) + 4 \), \( 0 \leq x \leq 2\pi \)

The period is \( \frac{2\pi}{2} = \pi \).

The amplitude is 3.

The mean position is \( y = 4 \).

The range of the graph is \([4 - 3, 4 + 3] = [1, 7]\).
3 Construct appropriate scales on the axes and sketch the graph.

b  1 Deduce the mean position.

b The minimum value is −8 and the maximum value is 4, so the mean position is

\[ y = \frac{-8 + 4}{2} = -2. \]

2 State the amplitude.

The amplitude is the distance from the mean position to either its maximum or minimum. The amplitude is 6.

3 State the period.

At \( x = \frac{\pi}{3} \), the graph is halfway through its cycle, so its period is \( \frac{2\pi}{3} \).

4 Determine a possible equation for the given graph.

Let the equation be \( y = a \cos(nx) + k \).

The graph is an inverted cosine shape, so \( a = -6 \).

The period is \( \frac{2\pi}{n} \).

\[
\frac{2\pi}{n} = \frac{2\pi}{3} \\
n = 3
\]

The mean position is \( y = -2 \), so \( k = -2 \).

The equation is \( y = -6 \cos(3x) - 2 \).

**Horizontal translations of the sine and cosine graphs**

Horizontal translations do not affect the period, amplitude or mean position of the graphs of sine or cosine functions. The presence of a horizontal translation of \( h \) units is recognised from the equation in the form given as \( y = a \sin(n(x - h)) + k \) in exactly the same way it is for any other type of function. The graph will have the same shape as \( y = a \sin(nx) + k \), but it will be translated to the right or to the left, depending on whether \( h \) is positive or negative respectively.

Translations affect the position of the maximum and minimum points and any \( x \)- and \( y \)-intercepts. However, successive maximum points would remain one period apart, as would successive minimum points.

The equation in the form \( y = a \sin(bx - c) + k \) must be rearranged into the form

\[ y = a \sin\left(b\left(x - \frac{c}{b}\right)\right) + k \]

to identify the horizontal translation \( h = \frac{c}{b} \).
**Combinations of the sine and cosine functions**

Trigonometric functions such as \( y = \sin(x) + \cos(x) \) can be sketched using addition of ordinates. In this example, both of the functions being combined under addition have the same period. If the functions have different periods, then to observe the periodic nature of the sum function the graphs should be sketched over a domain that allows both parts to complete at least one full cycle. For example, the function \( y = \sin(2x) + \cos(x) \) would be drawn over \([0, 2\pi]\), with two cycles of the sine function and one cycle of the cosine function being added together.

**Worked Example 18**

**a** Sketch the graph of the function \( f : [0, \frac{3\pi}{2}] \to R, f(x) = 4 \cos\left(2x + \frac{\pi}{3}\right)\).

**b** Sketch the graph of \( y = \cos(x) + \frac{1}{2} \sin(2x) \) for \( x \in [0, 2\pi] \).

**Think**

1. State the period, amplitude, mean position and horizontal translation.

2. Sketch the graph without the horizontal translation, \( y = 4 \cos(2x) \).

3. Calculate the coordinates of the end points of the domain of the given function.

**Write/Draw**

\[
f(x) = 4 \cos\left(2\left(x + \frac{\pi}{6}\right)\right)
\]

- The period is \( \frac{2\pi}{2} = \pi \).
- The amplitude is 4.
- The mean position is \( y = 0 \).
- The horizontal translation is \( \frac{\pi}{6} \) to the left.

\[
f(0) = 4 \cos\left(\frac{\pi}{3}\right)
= 4 \times \frac{1}{2}
= 2
\]

\[
f\left(\frac{3\pi}{2}\right) = 4 \cos\left(3\pi + \frac{\pi}{3}\right)
= 4 \times -\frac{1}{2}
= -2
\]

The end points of the graph are \((0, 2)\) and \(\left(\frac{3\pi}{2}, -2\right)\).
4 Calculate or deduce the positions of the $x$-intercepts. Each $x$-intercept on $y = 4 \cos(2x)$ is translated $\frac{\pi}{6}$ units to the left.

Alternatively, let $y = 0$.

\[
4 \cos\left(2x + \frac{\pi}{3}\right) = 0
\]

\[
\cos\left(2x + \frac{\pi}{3}\right) = 0, \quad \frac{\pi}{3} \leq 2x + \frac{\pi}{3} \leq 3\pi + \frac{\pi}{3}
\]

\[
2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{13\pi}{2}
\]

\[
2x = \frac{\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{13\pi}{6}
\]

\[
x = \frac{\pi}{12}, \frac{\pi}{12}, \frac{13\pi}{12}, \frac{13\pi}{12}
\]

5 Apply the horizontal translation to key points on the graph already sketched and hence sketch the function over its given domain.

**b 1** Identify the two functions forming the sum function.

**b** $y = \cos(x) + \frac{1}{2} \sin(2x)$

$y = y_1 + y_2$ where $y_1 = \cos(x)$ and $y_2 = \frac{1}{2} \sin(2x)$.

- $y_1 = \cos(x)$ has period $2\pi$ and amplitude 1.
- $y_2 = \frac{1}{2} \sin(2x)$ has period $\pi$ and amplitude $\frac{1}{2}$.

3 Sketch the two functions on the same set of axes and add together $y$-values of known points.

The tangent function

The domain of the tangent function, $y = \tan(x)$, can be deduced from the relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Whenever $\cos(x) = 0$, the tangent function will be undefined and its graph will have vertical asymptotes. Because $\cos(x) = 0$ when $x$ is an odd multiple of $\frac{\pi}{2}$, the domain is $\mathbb{R}\setminus\{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbb{Z}\}$. 
If \( \sin(x) = 0 \), then \( \tan(x) = 0 \); therefore, its graph will have \( x \)-intercepts when \( x = n\pi, n \in \mathbb{Z} \).

The graph of \( y = \tan(x) \) is shown.

The key features of the graph of \( y = \tan(x) \) are:

- period \( \pi \)
- range \( \mathbb{R} \), which implies it is not meaningful to refer to an amplitude
- vertical asymptotes at \( x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \) i.e. \( x = (2k + 1)\frac{\pi}{2}, k \in \mathbb{Z} \)
- asymptotes spaced one period apart
- \( x \)-intercepts at \( x = 0, \pm \pi, \pm 2\pi, \ldots \) i.e. \( x = k\pi, k \in \mathbb{Z} \)
- mean position \( y = 0 \)
- domain \( \mathbb{R} \setminus \{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbb{Z}\} \).
- many-to-one correspondence.

The graphs of \( y = \tan(nx) \) and \( y = a\tan(x) \)

The period of \( y = \tan(x) \) is \( \pi \), so the period of \( y = \tan(nx) \) will be \( \frac{\pi}{n} \).

Altering the period alters the position of the vertical asymptotes, as these will now be \( \frac{\pi}{n} \) units apart. An asymptote occurs when \( nx = \frac{\pi}{2} \). Once one asymptote is found, others can be generated by adding or subtracting multiples of the period.

The mean position remains at \( y = 0 \), so the \( x \)-intercepts will remain midway between successive pairs of asymptotes.

The dilation factor \( a \) affects the steepness of the tangent graph \( y = a\tan(x) \). Its effect is illustrated by comparing the values of the functions \( f(x) = \tan(x) \) and \( g(x) = 2\tan(x) \) at the point where \( x = \frac{\pi}{4} \).

Because \( f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \) and \( g\left(\frac{\pi}{4}\right) = 2\tan\left(\frac{\pi}{4}\right) = 2 \), the point \( \left(\frac{\pi}{4}, 1\right) \) lies on the graph of \( y = f(x) \) but the point \( \left(\frac{\pi}{4}, 2\right) \) lies on the dilated graph \( y = g(x) \).
The graph of \( y = a \tan(nx) \) has:

- period \( \frac{\pi}{n} \)
- vertical asymptotes \( \frac{\pi}{n} \) units apart
- mean position \( y = 0 \) with \( x \)-intercepts on this line midway between pairs of successive asymptotes.

Also note that the graph has an inverted shape if \( a < 0 \).

The \( x \)-intercepts can be located using their symmetry with the asymptotes. Alternatively, they can be calculated by solving the equation \( a \tan(nx) = 0 \).

The graph of \( y = \tan(x - h) \)

A horizontal translation of \( h \) units will move the vertical asymptotes \( h \) units in the same direction, but the translation will not affect the period of the graph. An asymptote for the graph of \( y = \tan(x - h) \) occurs when \( x - h = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2} + h \).

Other asymptotes can be generated by adding or subtracting multiples of the period \( \pi \).

The \( x \)-intercepts will remain midway between successive pairs of asymptotes, as the mean position is unaffected at a horizontal translation. They may be found by this means or alternatively found by solving the equation \( \tan(x - h) = 0 \).

The graph of \( y = a \tan(nx - b) \) has:

- period \( \frac{\pi}{n} \)
- horizontal translation of \( h = \frac{b}{n} \), as the equation is \( y = a \tan\left(n\left(x - \frac{b}{n}\right)\right) \)
- mean position \( y = 0 \).

Sketch the graphs of:

\( a \) \( y = 2 \tan(3x) \) for \( x \in [0, \pi] \)

\( b \) \( y = -\tan\left(2x + \frac{\pi}{2}\right) \) for \( x \in (0, 2\pi) \).

**THINK**

1. State the period.

2. Calculate the positions of the asymptotes.

3. Calculate the positions of the \( x \)-intercepts.

**WRITE/DRAW**

\( a \) \( y = 2 \tan(3x) \)

The period is \( \frac{\pi}{3} \).

An asymptote occurs when \( 3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6} \).

Others are formed by adding multiples of the period.

Asymptotes occur at \( x = \frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{2} \) and \( x = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6} \) within the domain constraint \( x \in [0, \pi] \).

The mean position is \( y = 0 \), and the \( x \)-intercepts occur midway between the asymptotes. One occurs at \( x = \frac{1}{2}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \frac{\pi}{3} \). The next is a period apart at \( x = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{5\pi}{3} \), and the one after that is at \( x = \frac{2\pi}{3} + \frac{3\pi}{3} = \pi \).
4 Sketch the graph.

b 1 State the period.

\[ y = -\tan\left(2x + \frac{\pi}{2}\right) \]
\[ \therefore y = -\tan\left(2\left(x + \frac{\pi}{4}\right)\right) \]

The period is \( \frac{\pi}{2} \).

2 Calculate the positions of the asymptotes.

An asymptote occurs when \( 2x + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow x = 0 \).

Adding multiples of the period, another occurs at \( x = 0 + \frac{\pi}{2} = \frac{\pi}{2} \), another at \( x = \frac{\pi}{2} + \frac{\pi}{2} = \pi \), another at \( x = \pi + \frac{\pi}{2} = \frac{3\pi}{2} \), and another at \( x = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi \).

The asymptotes are \( x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi \).

3 Calculate the positions of the \( x \)-intercepts.

The mean position is \( y = 0 \), and the \( x \)-intercepts are midway between the asymptotes.

\( x \)-intercepts occur at \( x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4} \).

The graph is inverted due to the presence of the negative coefficient in its equation, \( y = -\tan\left(2x + \frac{\pi}{2}\right) \).
The graph of \( y = a \tan(n(x - h)) + k \)

Under a vertical translation of \( k \) units, the mean position becomes \( y = k \). The points that are midway between the asymptotes will now lie on this line \( y = k \), not on the \( x \)-axis, \( y = 0 \). The \( x \)-intercepts must be calculated by letting \( y = 0 \) and solving the ensuing trigonometric equation this creates.

The vertical translation does not affect either the asymptotes or the period.

The graph of \( y = a \tan(n(x - h)) + k \) has

- period \( \frac{\pi}{n} \)
- vertical asymptotes when \( n(x - h) = (2k + 1)\frac{\pi}{2}, \ k \in \mathbb{Z} \)
- mean position \( y = k \)
- \( x \)-intercepts where \( a \tan(n(x - h)) + k = 0 \).

---

**WORKED EXAMPLE 20** Sketch the graph of \( y = 3 \tan(2\pi x) + \sqrt{3} \) over the interval \( \frac{-7}{8} \leq x \leq \frac{7}{8} \).

**THINK**

1. State the period and mean position.

   - Period: \( \frac{\pi}{2} \)
   - Mean position: \( y = \sqrt{3} \).

2. Calculate the positions of the asymptotes.

   - An asymptote occurs when \( 2\pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{4} \).
   - Others are formed by adding and subtracting a period.
   - For the interval \( \frac{-7}{8} \leq x \leq \frac{7}{8} \), the asymptotes occur at:
     - \( x = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \)
     - \( x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \) and \( x = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4} \).
   - The asymptotes are \( x = \frac{3}{4}, x = -\frac{1}{4}, x = \frac{1}{4}, x = \frac{3}{4} \).

3. Calculate the positions of the \( x \)-intercepts.

   - \( x \)-intercepts: Let \( y = 0 \).
     - \( 3 \tan(2\pi x) + \sqrt{3} = 0, \frac{-7}{8} \leq x \leq \frac{7}{8} \)

     - \( \tan(2\pi x) = -\frac{\sqrt{3}}{3}, \frac{-7\pi}{4} \leq 2\pi x \leq \frac{7\pi}{4} \)

     - \( 2\pi x = \frac{-\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6} \)

     - \( 2\pi x = \frac{-5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{6}, x = \frac{1}{12}, \frac{7}{12}, \frac{5}{12} \)

---
4 Obtain the $y$-intercept. When $x = 0$, $y = 3 \tan(0) + \sqrt{3} = \sqrt{3}$. The point $(0, \sqrt{3})$ is on the mean position.

5 Calculate the coordinates of the end points.

End points: Let $x = -\frac{7}{8}$.

$y = 3 \tan \left(2\pi \times -\frac{7}{8}\right) + \sqrt{3}$

$= 3 \tan \left(-\frac{7\pi}{4}\right) + \sqrt{3}$

$= 3 \times 1 + \sqrt{3}$

$= 3 + \sqrt{3}$

One end point is $\left(-\frac{7}{8}, 3 + \sqrt{3}\right)$.

Let $x = \frac{7}{8}$.

$y = 3 \tan \left(\frac{7\pi}{4}\right) + \sqrt{3}$

$= 3 \times -1 + \sqrt{3}$

$= -3 + \sqrt{3}$

The other end point is $\left(\frac{7}{8}, -3 + \sqrt{3}\right)$.

6 Sketch the graph.
Non-algebraic functions

1 WE14 Consider the function \( f(x) = -10^x \).
   a Evaluate \( f(2) \).
   b On the same set of axes, sketch the graphs of \( y = 10^x \), \( y = -10^x \) and \( y = 10^{-x} \).
   c Express \( y = 10^{-x} \) in an equivalent form.

2 For the functions defined by \( f(x) = 2^x \) and \( g(x) = 2^{-x} \), sketch the graph of the difference function \( y = (f - g)(x) \) and state its domain, range and rule.

3 WE15 Sketch the following graphs and state the domain and range of each.
   a \( y = -2e^x - 3 \) 
   b \( y = 4e^{-3x} - 4 \) 
   c \( y = 5e^{x-2} \)

4 a Sketch the graph of \( y = 2e^{1-3x} - 4 \), labelling any intercepts with the coordinate axes with their exact coordinates.
   b Sketch the graph of \( y = 3 \times 2^x - 24 \) and state its domain and range.

5 a WE16 The diagram shows the graph of \( y = ae^x + b \). Determine the values of \( a \) and \( b \).
   b The graph of \( y = a \times 10^{kx} \) contains the points \((4, -20)\) and \((8, -200)\). Form its equation.

6 The graph of \( y = a \times e^{kx} \) contains the points \((2, 36)\) and \((3, 108)\). Calculate the exact values of \( a \) and \( k \).

7 a WE17 Sketch the graph of \( y = 2 \cos(4x) - 3 \), \( 0 \leq x \leq 2\pi \).
   b The diagram shows the graph of a sine function. State its mean position, amplitude, and period, and give a possible equation for the function.

8 Sketch the graph of \( f: [0, 2\pi] \rightarrow \mathbb{R}, f(x) = 1 - 2 \sin \left( \frac{3x}{2} \right) \), locating any intercepts with the coordinate axes.

9 a WE18 Sketch the graph of the function \( f: \left[ 0, \frac{3\pi}{2} \right] \rightarrow \mathbb{R}, f(x) = -6 \sin \left( 3x - \frac{3\pi}{4} \right) \).
   b Sketch the graph of \( y = \cos(2x) - 3 \cos(x) \) for \( x \in [0, 2\pi] \).

10 Sketch the graph of \( y = (\sin(x))^2 = \sin^2(x) \) for \( x \in [-\pi, \pi] \).
11 \textbf{WE19} Sketch the graphs of:
   \begin{itemize}
   \item[a] \( y = 3 \tan \left( \frac{x}{2} \right) \) for \( x \in [-\pi, \pi] \)
   \item[b] \( y = -\tan(2x - \pi) \) for \( x \in [-\pi, \pi] \).
   \end{itemize}

12 The graph of \( y = a \tan(nx) \) has the domain \( \left( -\frac{\pi}{3}, \frac{\pi}{3} \right) \) with vertical asymptotes at \( x = -\frac{\pi}{3} \) and \( x = \frac{\pi}{3} \) only. The graph passes through the origin and the point \( \left( -\frac{\pi}{6}, -\frac{1}{2} \right) \). Determine its equation.

13 \textbf{WE20} Sketch the graph of \( y = 3 \tan(2\pi x) - \sqrt{3} \) over the interval \( -\frac{7}{8} \leq x \leq \frac{7}{8} \).

14 Sketch the graph of \( y = 1 - \tan \left( x + \frac{\pi}{6} \right) \) over the interval \( 0 \leq x \leq 2\pi \).

15 Sketch the graph of each of the following exponential functions and state their long-term behaviour as \( x \to \infty \).
   \begin{itemize}
   \item[a] \( y = \frac{4}{5} \times 10^x \)
   \item[b] \( y = 3 \times 4^{-x} \)
   \item[c] \( y = -5 \times 3^{-\frac{x}{2}} \)
   \item[d] \( y = -\left( \frac{2}{3} \right)^{-x} \)
   \end{itemize}

16 For each of the following functions, sketch the graph, state the range and identify the exact position of any intercepts the graph makes with the coordinate axes.
   \begin{itemize}
   \item[a] \( y = e^x - 3 \)
   \item[b] \( y = -2e^{2x} - 1 \)
   \item[c] \( y = \frac{1}{2}e^{-4x} + 3 \)
   \item[d] \( y = 4 - e^{2x} \)
   \item[e] \( y = 4e^{2x-6} + 2 \)
   \item[f] \( y = 1 - e^{-\frac{x+1}{2}} \)
   \end{itemize}

17 a The graph shown is of the function \( f(x) = ae^x + b \).
   Determine the values of \( a \) and \( b \) and write the function as a mapping.

b The graph shown has an equation of the form \( y = Ae^{\alpha x} + k \). Determine its equation.
c The graph of \( y = 2^x - b + c \) contains the points (0, 5) and (3, 9).
   i Calculate the values of \( b \) and \( c \).
   ii State the range of the graph.

d The graph of \( y = Ae^{x-2} + B \) contains the point (2, 10). As \( x \to -\infty \), \( y \to -2 \).
   i Calculate the values of \( A \) and \( B \).
   ii The graph passes through the point \( \left(a, 2\left(\frac{6}{e} - 1\right)\right) \). Find the value of \( a \).

18 State the period, amplitude and range of each of the following.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( y = 6 \sin(8x) )</td>
</tr>
<tr>
<td>b</td>
<td>( y = 2 - 3 \cos\left(\frac{x}{4}\right) )</td>
</tr>
<tr>
<td>c</td>
<td>( y = -\sin(3x - 6) )</td>
</tr>
<tr>
<td>d</td>
<td>( y = 3(5 + 2 \cos(6\pi x)) )</td>
</tr>
</tbody>
</table>

19 Sketch the following over the intervals specified.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( y = -7 \cos(4x), 0 \leq x \leq \pi )</td>
</tr>
<tr>
<td>b</td>
<td>( y = 5 - \sin(x), 0 \leq x \leq 2\pi )</td>
</tr>
<tr>
<td>c</td>
<td>( y = \frac{1}{2} \cos(2x) + 3, -\pi \leq x \leq 2\pi )</td>
</tr>
<tr>
<td>d</td>
<td>( y = 2 - 4 \sin(3x), 0 \leq x \leq 2\pi )</td>
</tr>
<tr>
<td>e</td>
<td>( y = 2 \sin\left(x + \frac{\pi}{4}\right), 0 \leq x \leq 2\pi )</td>
</tr>
<tr>
<td>f</td>
<td>( y = -4 \cos\left(3x - \frac{\pi}{2}\right) + 4, \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} )</td>
</tr>
</tbody>
</table>

20 a i Solve the equation \( 2\sin(2x) + \sqrt{3} = 0 \) for \( x \in [0, 2\pi] \).
   ii Sketch the graph of \( y = \sin(2x) \) for \( x \in [0, 2\pi] \).
   iii Hence find \( \{x: \sin(2x) < -\frac{\sqrt{3}}{2}, 0 \leq x \leq 2\pi\} \).

b State the maximum value of the function \( f(x) = 2 - 3 \cos\left(x + \frac{\pi}{12}\right) \) and give
   the first positive value of \( x \) for when this maximum occurs.

21 State the period and calculate the equation of the first positive asymptote for each of the following.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( y = \tan(4x) )</td>
</tr>
<tr>
<td>b</td>
<td>( y = 9 + 8 \tan\left(\frac{x}{7}\right) )</td>
</tr>
<tr>
<td>c</td>
<td>( y = -\frac{3}{2} \tan\left(\frac{4x}{5}\right) )</td>
</tr>
<tr>
<td>d</td>
<td>( y = 2 \tan(6\pi x + 3\pi) )</td>
</tr>
</tbody>
</table>
22. a Sketch the following graphs over the intervals specified.

i. \( y = -\tan(2x), \ x \in [0, \pi] \)

ii. \( y = 3 \tan\left(x + \frac{\pi}{4}\right), \ x \in [0, 2\pi] \)

iii. \( y = \tan\left(\frac{x}{3}\right) + \sqrt{3}, \ x \in [0, 6\pi] \)

iv. \( y = 5\sqrt{3}\tan\left(\pi x - \frac{\pi}{2}\right) - 5, \ x \in (-2, 3) \)

b. The graph of \( y = \tan(x) \) undergoes a set of transformations to form that of the graph shown.

i. Explain why there was no horizontal translation among the set of transformations applied to \( y = \tan(x) \) to obtain this graph.

ii. State the period of the graph shown.

iii. Form a possible equation for the graph.

23. A hybrid function is defined by the rule

\[
f(x) = \begin{cases} 
-\sin(x), & -2\pi \leq x \leq -\frac{\pi}{2} \\
\tan(x), & -\frac{\pi}{2} < x < \frac{\pi}{2} \\
\cos(x), & \frac{\pi}{2} \leq x \leq 2\pi 
\end{cases}
\]

a. Evaluate:

i. \( f\left(\frac{\pi}{3}\right) \)

ii. \( f(\pi) \)

iii. \( f\left(-\frac{\pi}{2}\right) \).

b. Sketch the graph of \( y = f(x) \).

c. Identify any points of the domain where the function is not continuous.

d. State the domain and range of the function.

24. Use addition of ordinates to sketch the graphs of:

a. \( y = e^{-x} + e^{x} \)

b. \( y = \sin(2x) - 4\sin(x), \ 0 \leq x \leq 2\pi \)

c. \( y = x + \sin(x), \ 0 \leq x \leq 2\pi \)
25 a Use a graphical method to determine the number of the roots of the equation \( e^x = \cos(x) \).

b Calculate the values of the roots that lie in the interval \([-2\pi, 2\pi]\), giving their values correct to 3 decimal places.

26 a Use CAS technology to find the coordinates of the points of intersection of the graphs of \( y = \sin(2x) \) and \( y = \tan(x) \) for \(-2\pi \leq x \leq 2\pi\).

b Hence, or otherwise, give the general solution to the equation \( \sin(2x) = \tan(x) \), \( x \in \mathbb{R} \).

### 2.6 Modelling and applications

People in research occupations, such as scientists, engineers and economists, analyse data through the use of mathematical models in order to increase our understanding of natural phenomena and to draw inferences about future behaviour. In this section we consider some applications of the functions that are discussed in the previous sections of this topic.

#### Modelling with data

Consider the set of data shown in the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>2</th>
<th>4</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>15</td>
<td>23.5</td>
<td>21</td>
<td>8.5</td>
</tr>
</tbody>
</table>

In deciding what type of model this data might best fit, a linear model would be ruled out as the data is not steadily increasing or decreasing. The values increase and then decrease; there are no obvious signs that the data is oscillating or showing asymptotic behaviour. Observations such as these would rule out an exponential model, a trigonometric model and a hyperbola or truncus model.

The data is likely to be a polynomial model with a many-to-one correspondence. Plotting the points can help us recognise a possible model. If the variables \( t \) and \( h \) are time and height respectively, then we may suspect the polynomial would be a quadratic one. Three of the data points could be used to form the model in the form \( h = at^2 + bt + c \), or the entire set of data could be used to obtain the model through a quadratic regression function on CAS. The quadratic model \( h(t) = -0.9t^2 + 9.3t \) shows a good fit with the data.

#### Applications of mathematical models

The variables in a mathematical model are usually treated as continuous, even though they may represent a quantity that is discrete in reality, such as the number of foxes in a region. Values obtained using the model need to be considered in context and rounded to whole numbers where appropriate.

Domain restrictions must also be considered. A variable representing a physical quantity such as length must be positive. Similarly, a variable representing time...
The population of foxes on the outskirts of a city is starting to increase. Data collected suggests that a model for the number of foxes is given by

\[ N(t) = 480 - 320e^{-0.3t}, \ t \geq 0, \]

where \( N \) is the number of foxes \( t \) years after the observations began.

a How many foxes were present initially at the start of the observations?

b By how many had the population of foxes grown at the end of the first year of observations?

c After how many months does the model predict the number of foxes would double its initial population?

d Sketch the graph of \( N \) versus \( t \).

e Explain why this model does not predict the population of foxes will grow to 600.

**THINK**

a Calculate the initial number.

\[ N(0) = 480 - 320e^{0} \]
\[ = 480 - 320 \]
\[ = 160 \]
There were 160 foxes present initially.

b Calculate the number after 1 year.

\[ N(1) = 480 - 320e^{-0.3} \]
\[ \approx 242.94 \]
After the first year 243 foxes were present.

Express the change over the first year in context.

Over the first year the population grew from 160 to 243, an increase of 83 foxes.

2 Express the change over the first year in context.

2 Calculate the required value of \( t \).

Note: An algebraic method requiring logarithms has been used here. CAS technology could also be used to solve the equation.

\[ 320 = 480 - 320e^{-0.3t} \]
\[ 320e^{-0.3t} = 160 \]
\[ e^{-0.3t} = \frac{1}{2} \]
\[ -0.3t = \log_{e}\left(\frac{1}{2}\right) \]
\[ t = \frac{1}{0.3} \log_{e}\left(\frac{1}{2}\right) \]
\[ t \approx 2.31 \]
Modelling and applications

1 [WE21] The population of possums in an inner city suburb is starting to increase. Observations of the numbers present suggest a model for the number of possums in the suburb given by

$$P(t) = 83 - 65e^{-0.2t}, \quad t \geq 0,$$

where \(P\) is the number of possums observed and \(t\) is the time in months since observations began.

a How many possums were present at the start of the observations?
b By how many had the population of possums grown at the end of the first month of observations?
c When does the model predict the number of possums would double its initial population?
d Sketch the graph of \(P\) versus \(t\).
e Explain why this model does not predict the population of possums will grow to 100.

2 Consider the data points shown.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>4</td>
<td>2</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

a Discuss why neither a linear, trigonometric, exponential nor a power function of the form \(y = x^n\) is a likely fit for the data.
b Assuming the data set fits a hyperbola of the form \(y = \frac{a}{x - b} + k, \ x \in [0, \infty) \setminus \{2\}:

i use the data to determine the equation of the hyperbola

ii sketch the model, showing the data points.
3 The population, in thousands, of bees in a particular colony increases as shown in the table.

<table>
<thead>
<tr>
<th>Month ((t))</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population in thousands ((P))</td>
<td>36</td>
<td>38.75</td>
<td>42.5</td>
<td>45</td>
</tr>
</tbody>
</table>

a Plot the data points \(P\) against \(t\) and suggest a likely model for the data.

b Use the values when \(t = 2\) and \(t = 4\) to form a rule for the model expressing \(P\) in terms of \(t\).

c If the variable \(t\) measures the number of months since January, how many bees were in the colony in January, according to the model?

d What is the rate of increase in the population of bees according to the model?

4 Eric received a speeding ticket on his way home from work. If he pays the fine now, there will be no added penalty. If he delays the payment by one month his fine will be $435, and a delay of two months will result in a fine of $655. The relationship between the fine, \(F\), in dollars, and the number of months, \(n\), that payment is delayed is given by \(F = a(r)^n\) where \(a\) and \(r\) are constants.

a Calculate the value of \(a\) to the nearest integer and the value of \(r\) to 1 decimal place.

b If Eric pays the fine immediately, how much will it cost him?

c If Eric delays the payment for six months, how much can he expect to pay? Express the value to the nearest dollar.

5 A young girl and boy are lifted onto a seesaw in a playground. At this time the seesaw is horizontal with respect to the ground.

Initially the girl’s end of the seesaw rises. Her height above the ground, \(h\) metres, \(t\) seconds after the seesaw starts to move is modelled by \(h(t) = a \sin(nt) + k\).

The greatest height above the ground that the girl reaches is 1.7 metres, and the least distance above the ground that she reaches is 0.7 metres. It takes 2 seconds for her to seesaw between these heights.

a Find the values of \(a\), \(n\) and \(k\).

b Draw the graph showing the height of the girl above the ground for \(0 \leq t \leq 6\).

c For what length of time during the first 6 seconds of the motion of the seesaw is the girl’s height above the ground 1.45 metres or higher?

d Sketch the graph showing the height of the boy above the ground during the first 6 seconds and state its equation.
6 A parabolic skate ramp has been built at a local park. It is accessed by climbing a ladder to a platform as shown. The platform is 2 metres long. The horizontal distance from the origin is \( x \), and the vertical distance from the origin is \( y \).

The lowest point on the skate ramp is at \((5, 0)\) and the highest point is at \((9, \frac{32}{9})\).

a Find the value of \( a \) where \((0, a)\) is the point where the ladder connects with the platform.

b What are the coordinates of the point where the platform and the skate ramp meet?

c Find the equation of the parabolic section of the skate ramp.

d Write a hybrid function rule to define the complete skate ramp system for \( \{ x : -1 \leq x \leq 9 \} \).

e Determine the exact values of \( x \) when the skateboarder is 1.5 metres above the ground.

7 Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by \( T = 20 + 75e^{-0.062t} \), where \( T \) is the temperature of the coffee \( t \) minutes after it was initially poured into the mug.

a What was its initial temperature when it was first poured?

b To what temperature will the coffee cool if left unattended?

c How long does it take for the coffee to reach a temperature of 65°C? Give your answer correct to 2 decimal places.

d Manoj returns to the coffee when it has reached 65°C and decides to reheat the coffee in a microwave. The temperature of the coffee in this warming stage is \( T = A + Be^{-0.05t} \). Given that the temperature of the reheated coffee cannot exceed 85°C, calculate the values of \( A \) and \( B \).

e Sketch a graph showing the temperature of the coffee during its cooling and warming stage.

8 James is in a boat out at sea fishing. The weather makes a change for the worse and the water becomes very choppy. The depth of water above the sea bed can be modelled by the function with equation \( d = 1.5 \sin\left(\frac{\pi t}{12}\right) + 12.5 \), where \( d \) is the depth of water in metres and \( t \) is the time in hours since the change of weather began.

a How far from the sea bed was the boat when the change of weather began?

b What is the period of the function?

c What are the maximum and minimum heights of the boat above the sea bed?

d Sketch one cycle of the graph of the function.

e If the boat is \( h \) metres above the seabed for a continuous interval of 4 hours, calculate \( h \) correct to 1 decimal place.

f James has heard on the radio that the cycle of weather should have passed within 12 hours, and when the height of water above the sea bed is at a minimum after that, it will be safe to return to shore. If the weather change occurred at 9.30 am, when will he be able to return to shore?
9 A biologist conducts an experiment to determine conditions that affect the growth of bacteria. Her initial experiment finds the growth of the population of bacteria is modelled by the rule \( N = 22 \times 2^t \), where \( N \) is the number of bacteria present after \( t \) days.

a How long will it take for the number of bacteria to reach 2816?
b What will happen to the number of bacteria in the long term according to this model?
c The biologist changes the conditions of her experiment and starts with a new batch of bacteria. She finds that under the changed conditions the growth of the population of bacteria is modelled by the rule \( N = \frac{66}{1 + 2e^{-0.2t}} \).

i Show that in both of her experiments the biologist used the same initial number of bacteria.

ii What will happen to the number of bacteria in the long term according to her second model?

10 ABCD is a square field of side length 40 metres. The points E and F are located on AD and DC respectively so that \( ED = DF = x \) m. A gardener wishes to plant an Australian native garden in the region that is shaded green in the diagram.

a Show that the area, \( A \) m\(^2\), to be used for the Australian native garden is given by \( A = 800 + 20x - \frac{1}{2}x^2 \).
b What restrictions must be placed on \( x \)?
c i Calculate the value of \( x \) for which the area of the Australian native garden is greatest.

ii Calculate the greatest possible area of the native garden.

11 The graph of \( y = g(x) \) is shown. The graph has a stationary point of inflection at the origin and also crosses the \( x \)-axis at the points where \( x = -\sqrt{5} \) and \( x = \sqrt{5} \). The coordinates of the maximum turning point and the minimum turning point are \((-\sqrt{3}, 12\sqrt{3})\) and \((\sqrt{3}, -12\sqrt{3})\) respectively.

a Use the above information to form the equation of the graph.
b Hence, show that \( g(x) = 2x^5 - 10x^3 \).
c A water slide is planned for a new theme park and its cross section shape is to be designed using a horizontal and vertical translation of the curve \( g(x) = 2x^5 - 10x^3 \).

The point A, the maximum turning point of the original curve, now lies on the y-axis. The point B, the minimum turning point of the original curve, now lies 1 unit above the x-axis. The point C is the image of the origin (0, 0) after the original curve is translated.

The water slide is modelled by the section of the curve from A to B with the x-axis as the water level.

i State the values of the horizontal and vertical translations required to achieve this model.

ii Give the height of A above the water level to 1 decimal place.

iii State the coordinates of the points C and B.

12 In an effort to understand more about the breeding habits of a species of quoll, 10 quolls were captured and relocated to a small reserve where their behaviour could be monitored. After 5 years the population size grew to 30 quolls.

A model for the size of the quoll population, \( N \), after \( t \) years on the reserve is thought to be defined by the function \( N: R^+ \cup \{0\} \rightarrow R \), \( N(t) = \frac{at + b}{t + 2} \).

a Calculate the values of \( a \) and \( b \).

b Sketch the graph of \( N \) against \( t \).

c Hence or otherwise, determine how large can the quoll population can grow to.

13 The water level in a harbor, \( h \) metres below a level jetty, at time \( t \) hours after 7 am is given by \( h = 3 - 2.5 \sin\left(\frac{1}{2}(t - 1)\right) \).

a How far below the jetty is the water level in the harbor at 7:30 am? Give your answer correct to 3 decimal places.

b What are the greatest and least distances below the jetty?

c Sketch the graph of \( h \) versus \( t \) and hence determine the values of \( t \) at which the low and high tides first occur. Give your answers correct to 2 decimal places.

d A boat ties up to the jetty at high tide. How much extra rope will have to be left so that the boat is still afloat at low tide?

14 A right circular cone is inscribed in a sphere of radius 4 cm, as shown in the cross section below.

a Express the radius, \( r \) cm, of the cone in terms of \( h \).

b Write an equation expressing the volume of the cone, \( V \) cm\(^3\), in terms of \( h \) and state any restrictions on \( h \).

c Sketch the graph of \( V \) versus \( h \).

d Use the graph to find the maximum volume for the cone to the nearest cm\(^3\).
The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology

**Extended-response** questions — providing you with the opportunity to practise exam-style questions. A summary of the key points covered in this topic is also available as a digital document.

**REVIEW QUESTIONS**
Download the Review questions document from the links found in the Resources section of your eBookPLUS.
EXERCISE 2.2

1 a Many-to-one correspondence  
b Domain \([-4, 2]\), range \([0, 16]\)  
c \(f: [-4, 2] \rightarrow R, f(x) = x^2\)  
d 12

2 a \(-\frac{32}{9}\)  
b \(4a^2 - 4\)  
c \(R\)

3 a \(\frac{36}{25} - \frac{48}{25}\)

4 a \(2x + 3y = -11\)  
b \(y = \frac{3x}{2} - 8\)  
c 56.3°

5 a \(y = 2(3x - 2)^2 - 8\)

6 a None  
b \(f(x) = 4(x - 1)^2 + 3\)

7 a \(y = -4(x + 2)^3 + 16\)

8 a \(4(x - 2)^2(x + 2)\)

9 a Maximum value 96, minimum value 0.

10 \(y = x^4 - 6x^3 + 1\) will make 2 intersections with the x-axis.

11 \(y = (x + 1)^6 + 10\)

12 \(y = (x + 4)(x + 2)^2(x - 2)^3(x - 5)\)
13 a i Many-to-one ii $[-3, 6], [-9, 7]$ iii Yes
b i One-to-many ii $[0, \infty), R$ iii No
c i Many-to-many ii $[-2, 2], [-2, 2]$ iii No
d i One-to-one ii $R, R$ iii Yes
e i Many-to-one ii $R, \{2\}$ iii Yes
f i One-to-one ii $R, R$ iii Yes

14 a

b

15 a $p = 6$
b $9y - 7x = 23$
c 8.3 units

16 a $-(x - 1)^2 - 4$
b $(1, -4)$
c

d

17 a $y = -\frac{3}{20}(x + 6)^2 + 12$
b $y = -\frac{4}{7}(x + 7)(2x + 5)$
c $y = \frac{1}{4}x^2 - 5$

18 a

b

c

Range $(-\infty, -4]$
19 \( y = \frac{1}{10}(x + 4)(4x - 5)^2 \)

20 a \( f(x) = -2x^3 + 9x^2 - 24x + 17 \)
\( f(1) = -2 + 9 - 24 + 17 = 0 \)
\( \therefore (x - 1) \) is a factor.

By inspection,
\(-2x^3 + 9x^2 - 24x + 17 = (x - 1)(-2x^2 + 7x - 17).\)
Consider the discriminant of the quadratic factor
\(-2x^2 + 7x - 17.\)
\( \Delta = 49 - 4(-2)(-17) \)
\( = 49 - 136 < 0 \)
As the discriminant is negative, the quadratic cannot be factorised into real linear factors; therefore, it has no real zeros.
For the cubic, this means there can only be one \( x \)-intercept, the one which comes from the only linear factor \((x - 1).\)

b For there to be a stationary point of inflection, the equation of the cubic function must be able to be written in the form \( y = a(x + b)^3 + c.\)
Let \(-2x^3 + 9x^2 - 24x + 17 = a(x + b)^3 + c\)
By inspection, the value of \( a \) must be \(-2.\)
\( \therefore -2x^3 + 9x^2 - 24x + 17 \)
\( = -2(x^3 + 3x^2b + 3xb^2 + b^3) + c \)
Equate coefficients of like terms:
\( x^3: 9 = -6b \Rightarrow b = -\frac{3}{2} \)
\( x^2: -24 = -6b^2 \Rightarrow b^2 = 4 \)
It is not possible for \( b \) to have different values.

Therefore, it is not possible to express the equation of the function in the form \( y = a(x + b)^3 + c.\)
There is no stationary point of inflection on the graph of the function.

c \( x \to \pm \infty, y \to \pm \infty \)

21 a \( y = -3(x + 5)^4 + 12 \)

b \( y = (2 + x)(1 - x)^3 \)
\( (-2, 0) (0, 2) (1, 0) \)

22 a i \( y = x^6 \)
\( (-1, 1) (1, 1) (0, 0) \)
ii \( \{x : x \leq 1\} \)

b \( y = 16 - (x + 2)^4 \)
\( (-2, 16) (-1, 15) (0, 0) \)

23 \( x = -1.75, x = 1.22 \)
Minimum turning points \((-1.31, -3.21)\) and \((1.20, -9.32)\), maximum turning point \((-0.636, -2.76)\).

b None

c Minimum turning point \((-2.17, -242)\), stationary point of inflection \((2, 20)\).

**EXERCISE 2.3**

1 a \(y = \frac{6}{x + 3} + 1\)

b i Maximal domain \(R \{1\}\)

ii Range \(R \{5\}\)

\[
y = \frac{5x - 2}{x - 1}
\]

\[
(0, 2) \\
(0.4, 0)
\]

\[
x = 1
\]

\[
y = \frac{4}{1 - 2x}
\]

\[
(1, -4)
\]

\[
y = 0
\]

\[
y = 5
\]

Domain \(R \{\frac{1}{2}\}\), range \(R \{0\}\).

2 \(y = \frac{8}{(x + 2)^2} - 2\)

\[
(-4, 0) \\
(0, 0)
\]

\[
x = -2
\]

\[
y = -2
\]

Domain \(R \{-2\}\), range \((-\infty, \infty)\).

4 \(y = \frac{1}{4x^2} - 1\)

5 a i Maximal domain \([-9, \infty)\)

ii Range \((-\infty, 2]\)

\[
(-9, 2) \\
(-5, 0) \\
(0, -1)
\]

\[
y = -\sqrt{x + 9} + 2
\]

6 a Maximal domain \(\left[\frac{9}{4}, \infty\right)\), range \([-6, \infty)\)

\[
y = 2\sqrt{x - 1} + 3
\]

\[
(1, 3) \\
(0, 1)
\]

\[
x = \sqrt[3]{x - 1} + 3
\]

\[
(\frac{19}{8}, 0)
\]

\[
(0, 1)
\]

\[
y = 3\sqrt[4]{x - 9} - 6
\]

\[
(\frac{13}{4}, 0)
\]

\[
(0, 10)
\]

\[
(1 - 3, 0)
\]

\[
(0, 0)
\]

\[
y = \frac{1}{3}
\]

\[
(0, 10\frac{1}{3})
\]

\[
y = 3\sqrt[4]{x - 9} - 6
\]

\[
(\frac{13}{4}, 0)
\]

\[
(0, 0)
\]

\[
y = x
\]

\[
(1, 1)
\]

\[
y = x^3
\]

\[
(0, 0)
\]

\[
y = x^3
\]

\[
(1, 1)
\]

\[
y = x
\]

\[
(0, 0)
\]
8 a  Domain $R$

\[ y = x \]

\[ y = x^\frac{1}{2} \]

\[ (0, 0) \quad (1, 1) \]

\[ (\frac{1}{2}, 0) \quad y = x^\frac{1}{2} \]

b  Domain $R^+ \cup \{0\}$

\[ y = x \]

\[ y = x^\frac{1}{2} \]

\[ (0, 0) \quad (1, 1) \]

\[ (\frac{1}{2}, 0) \quad y = x^\frac{1}{2} \]

9 a  $R \setminus \{-9\}$

b  $(-\infty, \frac{1}{2}]$

c  $R \setminus \{-3\}$

d  $R$

10 a

\[ y = \frac{4}{x} + 5 \]

\[ y = 5 \]

\[ (\frac{4}{5}, 0) \quad y = 5 \]

\[ x = 0 \]

Domain $R \setminus \{0\}$, range $R \setminus \{5\}$

d  Domain $R \setminus \{0\}$, range $R \setminus \{5\}$

\[ y = 2 - \frac{3}{x + 1} \]

\[ y = 2 \]

\[ (\frac{1}{2}, 0) \quad y = 2 \]

\[ x = -1 \]

Domain $R \setminus \{-1\}$, range $R \setminus \{2\}$

d  Domain $R \setminus \{-1\}$, range $R \setminus \{2\}$

11 a  \[ y = \frac{-2}{x + 3} + 6 \]

b  \[ y = \frac{1}{2(x + 2)} - \frac{3}{2} \]

12 a

\[ y = \frac{2}{(3-x)^2 + 1} \]

\[ y = 1 \]

\[ (0, \frac{11}{9}) \quad (6, \frac{11}{9}) \]

Domain $R \setminus \{3\}$, range $(1, \infty)$
13 a \( y = \frac{-3}{(x - 4)^2} + 2 \)

b \( y = \frac{108}{(2x - 1)^2} - 4 \),

\( f : R \{\frac{1}{2}\} \rightarrow R, f(x) = \frac{108}{(2x - 1)^2} - 4 \).

d \( y = \frac{1}{(2x - 1)^2 - 1} \), range \((-\infty, -2)\)

domain \(R \{\frac{3}{2}\}, range (-1, \infty)\)

14 a i \( y = 2\sqrt{(x - 3)} + 2 \), domain \([3, \infty)\), range \([2, \infty)\);\n
\( y = -2\sqrt{(x - 3)} + 2 \), domain \([3, \infty)\), range \((-\infty, 2]\).

ii \( y = \sqrt{-2(x - 3)} - 1 \), domain \((-\infty, 3]\),

range \([-1, \infty)\); \( y = -\sqrt{-2(x - 3)} - 1 \), domain \((-\infty, 3]\), range \((-\infty, -1]\).

15 a \( a = 2, b = -5, c = -2 \)

b i \( a = -2, b = 4, c = -2 \) ii \( y = -\sqrt{-2x + 4} + 2 \)

16 a \( y = \sqrt{\frac{x + 2}{x} - 1} \)

Domain \(\{0\}, range (\infty, 5)\)
Domain $R$, range $R$.

$y = \sqrt[3]{x} + 5$

Domain $[-3, 6]$, range $[-1, 2]$.

$y = 2\sqrt{x} - 2$

$y = -\frac{7\sqrt{x} + 1}{2} - 2$

$y = 4\sqrt{x - 2} - 2$, $(2, -2)$

17 a Draw $y = x$ and construct its cube root.

b

Domain $R$, quadrants 1 and 3

18 a Domain $[0, \infty)$, quadrant 1

c $\{x : x < -1\} \cup \{x : 0 < x < 1\}$

19

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \frac{1}{x^2 - 4}$</td>
<td>$R{\pm 2}$</td>
<td>$R\left(\frac{1}{4}, 0\right)$</td>
<td>$x = \pm 2, y = 0$</td>
</tr>
<tr>
<td>$y = \frac{1}{x^3 + 4}$</td>
<td>$R$</td>
<td>$\left(0, \frac{1}{4}\right)$</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$y = \frac{1}{(x - 4)^2}$</td>
<td>$R{4}$</td>
<td>$(0, \infty)$</td>
<td>$x = 4, y = 0$</td>
</tr>
</tbody>
</table>

$y = \frac{1}{(x - 4)^2}$ is a truncus.

20 The maximal domain is $x \in [-3, 2]$.

EXERCISE 2.4

1 a $f(-8) = 2$, $f(-1) = -1$, $f(2) = 0$

b

c i $x = -1$

ii Domain $R$, range $R$.

$\begin{align*}
x + 4, & \quad x < 0 \\
x, & \quad 4 \leq x \leq 8
\end{align*}$
3 \[ y = -\sqrt{1+x} - \sqrt{1-x}, \text{ domain } [-1, 1] \]

b \[ y = \sqrt{1-x^2}, \text{ domain } [-1, 1], \text{ range } [0, 1]. \]

4 \((f-g)(x) = x^3 - x^2\)

5 The graphs of \(f\) and \(g\) intersect when \(x = 0, x = 1\), which gives the places where the difference function has \(x\)-intercepts.

6 Domain \(R\), range \([0, \infty)\)

7 a

b Domain \(R\), range \(R\), \(x = 0\)

c \((-\infty, 1]\), no point of discontinuity

8 a i \((-1, 0)\)

ii \((0, 1)\)

b \((2, 2)\)

c \(R \setminus \{-1, 2\}\)

9 a The branch to the left of \(x = 1\) has the rule \(f(x) = \sqrt{1-x}\), so \(f(1) = 0\).

The branch to the right of \(x = 1\) has the rule \(f(x) = x - 2\), so \(f(1) \rightarrow -1\) (open circle).

These branches do not join, so the hybrid function is not continuous at \(x = 1\).

b Many-to-one correspondence

c \[ x = 6 \]

10 \[ y = \begin{cases} 
\frac{4}{3}(x + 3)(x + 1), & x < 0 \\
4, & 0 \leq x \leq 2 \\
-2x + 8, & x \geq 3 
\end{cases} \]
11 a \( a = 8, b = 32 \)

\[
\begin{array}{l}
\text{b} \quad i \quad k > 4 \\
\text{ii} \quad k = 4 \text{ or } k \leq 0 \\
\text{iii} \quad 0 < k < 4 \\
\text{c} \quad \{-1, 32\}
\end{array}
\]

12 a \( y = 3 \), domain \( \mathbb{R} \), range \( \{3\} \), horizontal line through \((0, 3)\)

\[
\begin{array}{l}
\text{b} \quad y = 7 - 4x, \text{ domain } \mathbb{R}, \text{ range } \mathbb{R}, \text{ straight line through } (0, 7) \text{ and } (0, 3) \\
\text{c} \quad y = 2(5 - 2x)(x - 1), \text{ domain } \mathbb{R}, \text{ range } \left(-\infty, \frac{9}{4}\right], \text{ concave down parabola with turning point } (\frac{7}{4}, \frac{9}{4}) \text{ and passing through } (0, -10), (1, 0) \text{ and } (2.5, 0)
\end{array}
\]

13 a \( \text{i} \quad -6 \)

\[
\begin{array}{l}
\text{b} \quad [-1, \infty) \\
\text{c} \quad \text{i}
\end{array}
\]

14

15 a

16 a
EXERCISE 2.5

1 a $-100$

b $0, 0, (\pm \sqrt{2}, 0), (-1, 1), \left(\frac{1 \pm \sqrt{3}}{2}, 1\right)$

c $y = \left(\frac{1}{10}\right)^x$ or $y = 0.1^x$.

2 $y = 2^x - 2^{-x}$, domain $R$, range $R$

3 a $y = -2e^{3x} - 3$

b $-4e^{-3x} - 4$

c $y = 5e^{x^2}$

4 a $y = 2e^{1 - 3x} - 4$

Where the polynomial graph cuts the $x$-axis, the cube root graph has vertical points of inflection; where the polynomial touches the $x$-axis, the cube root graph also touches the $x$-axis but at a sharp point.

Wherever the polynomial graph has the values $y = 0, y = -1$ and $y = 1$, the cube root graph must have the same value. There are 9 points of intersection.
Domain $R$, range $(-24, \infty)$.

5 a $a = -2, b = 2$
b $y = -2 \times 10^5$

6 $a = 4, k = \log_e(3)$.

7 a $y = 3 \times 2^x - 24$

b Mean position $= 5$, amplitude $= 8$, period $= 2$
y $= -8 \sin(\pi x) + 5$

8 $f(x) = 1 - 2 \sin \left(\frac{3x}{2}\right)$

x-intercepts at $x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$. y-intercept $= (0, 1)$

9 a $y = 4 \cos(2x) - 3 \cos(x)$

b $y = \cos(2x) - 3 \cos(x)$

$y_1 = \cos(2x)$

$y_2 = -3 \cos(x)$

$y_3 = 5$

$y_4 = -3$

$y_5 = -5$

10 $y = \sin^2(x)$

11 a $y = 3 \tan \left(\frac{x}{2}\right)$

b $y = -\tan(2x - \pi)$

12 $y = \frac{1}{2} \tan \left(\frac{3x}{2}\right)$

13 $y = 3 \tan(2x) - \sqrt{3}$

14 $y = 1 - \tan(x + \frac{\pi}{6})$
15 a

\[ y = \frac{4}{5} 	imes 10^x \]

\( x \to \infty, y \to \infty \)

16 a

\[ y = e^{-x} - 3 \]

Range \((-\infty, \infty)\)

17 a

\[ a = -11, b = 11, f: R \to R, f(x) = -11e^x + 11 \]

b \[ y = e^{-2x} + 4 \]

c \[ i \ b = -1, c = -7 \]

ii \((-7, \infty)\)

d \[ i \ A = 12, B = -2 \]

ii \(a = 1\)

18 a

Period \(\frac{\pi}{4}\), amplitude 6, range \([-6, 6]\)

b Period \(8\pi\), amplitude 3, range \([-1, 5]\)

c Period \(\frac{2\pi}{3}\), amplitude 1, range \([-1, 1]\)

d Period \(\frac{1}{3}\), amplitude 6, range \([9, 21]\)
19 a

\[ y = -7 \cos(4x) \]

b

\[ y = 5 - \sin(x) \]

(0, 5) (2π, 5)

\[
\begin{array}{c}
\frac{\pi}{2} \\
\frac{3\pi}{2} \\
\frac{3\pi}{2} \\
\frac{5\pi}{2} \\
\frac{7\pi}{2} \\
\end{array}
\]

\[ (\frac{\pi}{2}, 0) \]

\[ (2\pi, 0) \]

(0, 7) (π, 7)

(π, -7) (2π, -7)

19 c

\[ y = \frac{1}{2} \cos(2x) + 3 \]

\[-\frac{\pi}{2} \]

\[-\frac{\pi}{2} \]

0

\[
\begin{array}{c}
\frac{\pi}{2} \\
\frac{3\pi}{2} \\
\frac{3\pi}{2} \\
\frac{5\pi}{2} \\
\frac{7\pi}{2} \\
\end{array}
\]

19 d

\[ y = 2 - 4 \sin(3x) \]

(0, 2) (2π, 2)

0

\[
\begin{array}{c}
\frac{\pi}{2} \\
\frac{3\pi}{2} \\
\frac{3\pi}{2} \\
\frac{5\pi}{2} \\
\frac{7\pi}{2} \\
\end{array}
\]

19 e

\[ y = 2 \sin(x + \frac{\pi}{4}) \]

(0, \sqrt{2}) (2π, \sqrt{2})

0

\[
\begin{array}{c}
\frac{\pi}{4} \\
\frac{3\pi}{4} \\
\frac{3\pi}{4} \\
\frac{5\pi}{4} \\
\frac{7\pi}{4} \\
\end{array}
\]

f

\[ y = -4 \cos\left(3x - \frac{\pi}{2}\right) + 4 \]

(0, 4) (\frac{\pi}{2}, 0) (\frac{3\pi}{2}, 0)

20 a

\[
\begin{array}{c}
\frac{\pi}{2} \\
\frac{3\pi}{2} \\
\frac{3\pi}{2} \\
\frac{5\pi}{2} \\
\frac{7\pi}{2} \\
\end{array}
\]

\[ x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{3} \]

\[
\begin{array}{c}
\frac{\pi}{4} \\
\frac{3\pi}{4} \\
\frac{3\pi}{4} \\
\frac{5\pi}{4} \\
\frac{7\pi}{4} \\
\end{array}
\]

\[ y = \sin(2x) \]

(0, 0) (2π, 0)

21 a

Period π/4, asymptote \( x = \frac{\pi}{8} \)

b

Period 7π/2, asymptote \( x = \frac{7\pi}{2} \)

c

Period 5π/4, asymptote \( x = \frac{5\pi}{8} \)

d

Period 1/6, asymptote \( x = \frac{1}{12} \)

22 a

\[ y = -\tan(2x) \]

(0, 0) (2π, 0)

(0, 3) (2π, 3)

(\frac{\pi}{4}, 0)

\[ x = \frac{\pi}{4}, \frac{3\pi}{4} \]

\[ y = 3 \tan(x + \frac{\pi}{4}) \]

\[ x = \frac{\pi}{4}, \frac{5\pi}{4} \]
### Exercise 2.6

**1 a** 18  
**b** 12  
**c** 1.62 months  
**d**  
**e** The population cannot exceed 83.
2 a The data points increase and decrease, so they cannot be modelled by a one-to-one function. Neither a linear model nor an exponential model is possible. The data is not oscillating, however, so it is unlikely to be trigonometric. The jump between \(x = 1\) and \(x = 3\) is a concern, but the data could be modelled by a polynomial such as a cubic with a turning point between \(x = 1\) and \(x = 3\). However, \(y = x^n\) requires the point \((0, 0)\) to be on it and that is not true for the data given.

b i \(y = -\frac{4}{x - 2} + 6\)

ii

\[\begin{array}{c}
(1, 2) \\
(3, 10) \\
(4, 8) \\
(0, 4) \\
\end{array}\]

\(x = 2\)

\(y = 6\)

\(y = \frac{4}{x - 2} + 6\)

3 a \(P (’000s)\)

\[\begin{array}{c}
\text{The data appears to be linear.} \\
P = 3.125t + 32.5 \\
32,500 \text{ bees} \\
3.125 \text{ thousand per month} \\
\end{array}\]

4 a \(a = 290, r = 1.5\)     b \(\$290\)

c \(\$3303\)

5 a \(a = 0.5, n = \frac{\pi}{2}, k = 1.2\)

b, d

\[h_{girl} = 0.5 \sin \left(\frac{\pi}{2} t\right) + 1.2\]

\[h_{boy} = -0.5 \sin \left(\frac{\pi}{2} t\right) + 1.2\]

c \(\frac{8}{3}\) seconds

d \(h_{boy} = -0.5 \sin \left(\frac{\pi}{2} t\right) + 1.2\)

6 a \(a = 2\)

b \((2, 2)\)

c \(y = \frac{2}{3}(x - 5)^2, 2 \leq x \leq 9\)

d \(y = \begin{cases} 2x + 2, & -1 < x < 0 \\ 2, & 0 \leq x \leq 2 \\ \frac{2}{3}(x - 5)^2, & 2 < x \leq 9 \end{cases}\)

e \(x = 5 \pm \frac{3\sqrt{3}}{2} \) or \(x = -\frac{1}{4}\)

7 a \(95^\circ C\)

b The temperature approaches \(20^\circ C\)

c \(8.24\) minutes

d \(A = 85, B \approx -30\)

e \(T = 20 + 75e^{-0.062t}\)

\[\begin{array}{c}
\text{If } t = 8.24, \text{ then } T = 85 \\
\text{If } t = 85 - 30e^{-0.05t}, \text{ then } T = 85.25 \\
\end{array}\]

8 a \(12.5\) metres

b \(24\) hours

c Maximum is 14 metres, minimum is 11 metres

d \(14\)

c \(11\)

\[\begin{array}{c}
(0, 12.5) \\
(6, 14) \\
(18, 11) \\
(24, 12.5) \\
\end{array}\]

\[\begin{array}{c}
\text{If } t = 1.5 \sin \left(\frac{\pi}{12} t\right) + 12.5 \\
\end{array}\]

e \(h = \frac{3\sqrt{3} + 50}{4} \approx 13.8\)

f \(3:30\) pm the following day

9 a \(7\) days

b As \(t \to \infty\), \(N \to \infty\).

c i \(N = 22 \times 2^t, t = 0\)

\[\begin{array}{c}
N = 22 \times 20 \\
= 22 \\
\end{array}\]

\[\begin{array}{c}
N = \frac{66}{1 + 2e^{-0.2t}}, t = 0 \\
= \frac{66}{1 + 2e^{-0.2 \times 0}} \\
= \frac{66}{3} \\
= 22 \\
\end{array}\]

Initially there are 22 bacteria in each model.

ii The population will never exceed 66.
10 a The garden area is the area of the entire square minus the area of the two right-angled triangles.
\[ A = 40 \times 40 - \frac{1}{2} x \times x - \frac{1}{2} \times (40 - x) \times 40 \]
\[ = 1600 - \frac{1}{2} x^2 - 20(40 - x) \]
\[ = 1600 - \frac{1}{2} x^2 - 800 + 20x \]
\[ = \frac{1}{2} x^2 + 20x + 800 \]

b \( 0 < x < 40 \)

c i 20
ii 1000 m²

11 a \( y = 2x^3(x - \sqrt{5})(x + \sqrt{5}) \)

b \( y = 2x^3(x - \sqrt{5})(x + \sqrt{5}) \)
\[ = 2x^3(x^2 - 5) \]
\[ = 2x^5 - 10x^3 \]

c i Horizontal translation of \( \sqrt{3} \) units to the right and vertical translation of \( 12\sqrt{3} + 1 \) units upward
ii \( (24\sqrt{3} + 1) \approx 42.6 \) metres
iii B \((2\sqrt{3}, 1), C (\sqrt{3}, 12\sqrt{3} + 1)\)

12 a \( a = 38, b = 20 \)

b

\[ N = 38 \]
\[ N = \frac{38t + 20}{t + 2} \]

(c) The population will never exceed 38.

13 a 3.619 m below the jetty
b 5.5 m and 0.5 m

d 79 cm³

14 a \( r = \sqrt{8h - h^2} \)

b \( V = \frac{1}{3} \pi h^2(8 - h), 0 < h < 8 \)

c \( V = \frac{1}{3} \pi h^2(8 - h) \)