Differentiation

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5.5 Differentiation of trigonometric functions
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5.1 Kick off with CAS
Gradients and tangents of a curve

1 a Using the graph application on CAS, sketch the graph of \( f(x) = \frac{1}{2}x^3 - 2x^2 \).

b Using the derivative template, sketch the graph of the gradient function, \( f'(x) \), on the same set of axes as \( f(x) \).

c Repeat this process for \( f(x) = -(x - 1)^2 + 3 \) and \( f(x) = x^4 + x^3 - 7x^2 - x + 6 \). Use a different set of axes for each \( f(x) \) and \( f'(x) \) pair.

d What do you notice about the positions of the turning points and \( x \)-intercepts? What do you notice about the shapes of the graphs?

2 a On a new set of axes, sketch the graph of \( f(x) = (x - 1)^2 - 2 \).

b Draw a tangent to the curve at \( x = 2 \).

c What is the equation of this tangent, and hence, what is the gradient of the parabolic curve at \( x = 2 \)?

d Draw a tangent to the curve at \( x = -1 \). What is the gradient of the curve at this point?

3 a Open the calculation application on CAS and define \( f(x) = (x - 1)^2 - 2 \).

b Determine the derivative function, \( f'(x) \), and calculate \( f'(2) \) and \( f'(-1) \).

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
Review of differentiation

The derivative of a function

The gradient of a curve is the rule for the instantaneous rate of change of the function at any point. The gradient at any point \((x, y)\) can be found using differentiation of first principles.

Consider the secant \(PQ\) on the curve \(y = f(x)\). The coordinates of \(P\) are \((x, f(x))\), and the coordinates of \(Q\) are \((x + h, f(x + h))\).

The gradient of the secant, otherwise known as the average rate of change of the function, is found in the following way.

\[
\text{Average rate of change} = \frac{\text{rise}}{\text{run}} = \frac{f(x + h) - f(x)}{x + h - x} = \frac{f(x + h) - f(x)}{h}
\]

As \(Q\) gets closer and closer to \(P\), \(h\) gets smaller and smaller and in fact is approaching zero. When \(Q\) is effectively the same point as \(P\), the secant becomes the tangent to the curve at \(P\). This is called the limiting situation.

\[
\text{Gradient of the tangent at } P = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

or \(f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\)

In this notation, \(f'(x)\) is the derivative of the function, or the gradient of the tangent to the curve at the point \((x, f(x))\). \(f'(x)\) is also the gradient function of \(f(x)\), and \(\frac{dy}{dx}\) is the gradient equation for \(y\) with respect to \(x\).
Consider the function $f(x) = (x + 2)(1 - x)$.

a Sketch the graph of the parabolic function, showing axis intercepts and the coordinates of any turning points.

b If P is the point $(x, f(x))$ and Q is the point $(x + h, f(x + h))$, find the gradient of the secant PQ using first principles.

c If P is the point $(-2, 0)$, determine the gradient of the tangent to the curve at $x = -2$.

**THINK**

a 1 Find the axis intercepts and the coordinates of the turning points.

b 1 Expand the quadratic.

**WRITE/DRAW**

a y-intercept: $x = 0$

$\therefore y = 2$

x-intercept: $y = 0$

$(x + 2)(1 - x) = 0$

$x + 2 = 0$ or $1 - x = 0$

$x = -2, 1$

Turning point:

$x = \frac{-2 + 1}{2} = -\frac{1}{2}$

$y = \left(-\frac{1}{2} + 2\right)\left(1 + \frac{1}{2}\right)$

$= \frac{3}{2} \times \frac{3}{2}$

$= \frac{9}{4}$

The turning point is $\left(-\frac{1}{2}, \frac{9}{4}\right)$.

b 1 Expand the quadratic.

$f(x) = (x + 2)(1 - x)$

$= -x^2 - x + 2$
Find the gradient of the secant 
PQ by applying the rule 
gradient of secant = \( \frac{f(x + h) - f(x)}{h} \)
and simplifying.

\[ \text{Gradient of secant:} \]
\[ \text{Gradient} = \frac{f(x + h) - f(x)}{h} \]
\[ = \frac{-2xh - h^2 - h}{h} \]
\[ = -2x - h - 1 \]

1 Find the rule for the gradient 
of the tangent at P by 
applying \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

\[ \text{The gradient of the tangent at P is given by} \]
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} (-2x - h - 1) \]
\[ = -2x - 1 \]

Substitute \( x = -2 \) into the formula 
for the gradient of the tangent.

The derivative of \( x^n \)
Differentiating from first principles is quite a tedious method, but there are rules 
to shortcut the process, depending on the function. Units 1 and 2 of Mathematical 
Methods cover differentiation when \( f(x) = x^n \).

\[ \text{If} \quad f(x) = ax^n, f'(x) = nax^{n-1} \quad \text{where} \quad n \in \mathbb{R} \quad \text{and} \quad a \in \mathbb{R} \]
\[ \text{and} \]
\[ \text{if} \quad f(x) = g(x) \pm h(x), f'(x) = g'(x) \pm h'(x) \]

**WORKED EXAMPLE 2**

**Differentiate:**
- \( a \) \( f(x) = x^3 - \frac{1}{2x} + 4 \)
- \( b \) \( y = \frac{\sqrt{x} - 3x^3}{4x^2} \)

**THINK**

- \( a \) 1 Rewrite the equation with negative indices.
  \[ f(x) = x^3 - \frac{1}{2x} + 4 \]
  \[ = x^3 - \frac{1}{2}x^{-1} + 4 \]

- 2 Differentiate each term separately.
  \[ f'(x) = 3x^2 + \frac{1}{2}x^{-2} \]

- 3 Write the answer with positive indices.
  \[ f'(x) = 3x^2 + \frac{1}{2x^2} \]
a If \( f(x) = 3x - x^2 \), what is the gradient of the curve when \( x = -2 \)?

b If \( f(x) = 2\sqrt{x} - 4 \), determine the coordinates of the point where the gradient is 2.

**THINK**

a 1 Rewrite the equation with negative indices and differentiate each term.

\[
f(x) = \frac{3}{x} - x^2 = 3x^{-1} - x^2
\]

\[
f'(x) = -3x^{-2} - 2x
\]

b 1 Rewrite \( \sqrt{x} \) with a fractional index and differentiate each term.

\[
y = \frac{\sqrt{x} - 3x^3}{4x^2}
\]

\[
y = \frac{x^{\frac{1}{2}} - 3x^3}{4x^2}
\]

\[
y = \frac{x^{-\frac{1}{2}} - 3x^3}{4}
\]

**WRITE**

a

\[
f(x) = \frac{3}{x} - x^2
\]

\[
f'(x) = -3x^{-2} - 2x
\]

The gradient of the curve when \( x = -2 \) is \( f'(-2) \).

\[
f'(-2) = \frac{-3}{(-2)^{-2}} - 2 \times (-2)
\]

\[
f'(-2) = \frac{-3}{4} + 4
\]

\[
f'(-2) = \frac{13}{4}
\]

b

\[
f(x) = 2\sqrt{x} - 4
\]

\[
f(x) = 2x^{\frac{1}{2}} - 4
\]

\[
f'(x) = \frac{1}{\sqrt{x}}
\]

Finding where the gradient is 2 means solving \( f'(x) = 2 \).

\[
\frac{1}{\sqrt{x}} = 2
\]

\[
\frac{1}{2} = \sqrt{x}
\]

\[
x = \frac{1}{4}
\]
3 Find \( f\left(\frac{1}{4}\right) \) to determine the \( y \)-value where the gradient is 2.

\[
f\left(\frac{1}{4}\right) = 2\sqrt{\frac{1}{4}} - 4
= 2 \times \frac{1}{2} - 4
= -3
\]

4 Write the answer.

The gradient is 2 at the point \( \left(\frac{1}{4}, -3\right) \).

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### Graphs of the gradient function

The previous section shows that the derivative of the function \( f(x) = x^n \) is one degree lower: \( f'(x) = nx^{n-1} \). This also applies to the gradient graphs of these functions. For example, if \( f(x) \) is a quadratic graph, \( f'(x) \) will be a linear graph; if \( f(x) \) is a cubic graph, \( f'(x) \) will be a quadratic graph, and so on.

<table>
<thead>
<tr>
<th>Given function ( f(x) )</th>
<th>Gradient function ( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A line of the form ( y = mx + c ) is degree one, and the gradient is ( m ). Example:</td>
<td>The gradient is a constant value, so the gradient graph is a line parallel to the ( x )-axis, ( y = m ), degree zero.</td>
</tr>
<tr>
<td>( f(x) = 2x + 1 )</td>
<td>( f'(x) = 2 )</td>
</tr>
<tr>
<td>( (0, 1) )</td>
<td>( (0, 2) )</td>
</tr>
<tr>
<td>A quadratic of the form ( y = ax^2 + bx + c ) is degree two. Example:</td>
<td>A line of the form ( y = mx + c ) is degree one. The line shown has an ( x )-intercept at ( x = -\frac{1}{2} ).</td>
</tr>
<tr>
<td>The function shown has a local minimum at ( x = -\frac{1}{2} ). ( f(x) = (x+3)(x-2) )</td>
<td>( f'(x) = 2x + 1 )</td>
</tr>
<tr>
<td>( (0, -6\frac{1}{4}) )</td>
<td>( (-\frac{1}{2}, 0) )</td>
</tr>
</tbody>
</table>
Given function $f(x)$

A cubic of the form $y = ax^3 + bx^2 + cx + d$ is degree three.

Example:
The function shown has turning points at $x \approx -1.8$ and $x \approx 1.1$.

Gradient function $f'(x)$

A quadratic of the form $y = ax^2 + bx + c$ is degree two. The curve shown has $x$-intercepts at $x \approx -1.8$ and $x \approx 1.1$.

Sometimes the graph of $f(x)$ may not be a known function, so the features of $f(x)$ need to be studied carefully in order to sketch the gradient graph.

- Turning points on the graph of $f(x)$ represent $x$-intercepts on the graph of $f'(x)$.
- Where the graph of $f(x)$ has a positive gradient, the graph of $f'(x)$ is above the $x$-axis.
- Where the graph of $f(x)$ has a negative gradient, the graph of $f'(x)$ is below the $x$-axis.

**WORKED EXAMPLE 4**

a. The graph of the cubic function $f(x)$ is shown. Sketch the derivative function $f'(x)$ on the same set of axes as $f(x)$.

b. State the domain of the gradient function, $f'(x)$, for the function shown.
The gradient of a function only exists where the graph is smooth and continuous. That is, a single tangent must be able to be drawn at \( x = a \) for \( f'(a) \) to exist.

Equations of tangents and perpendicular lines

A tangent is a straight line. Therefore, to form the equation of the tangent, its gradient and a point on the line are needed. The equation can then be formed using \( y - y_1 = m(x - x_1) \).

For the tangent to a curve \( y = f(x) \) at a point \( P \), the gradient \( m \) is found by evaluating the curve’s derivative, \( f'(x) \), at \( P \), the point of contact or point of tangency. The coordinates of \( P \) provide the point \( (x_1, y_1) \) on the line.

Other important features of the tangent include:

- The angle of inclination of the tangent to the horizontal can be calculated using \( m = \tan(\theta) \).
- Tangents that are parallel to each other have the same gradient.
- The gradient of a line perpendicular to the tangent is found using \( m_Tm_P = -1 \). That is, if the gradient of a tangent is \( m_T \), then the gradient of a perpendicular line is \( -\frac{1}{m_P} \).
- The gradient of a horizontal tangent is zero.
- The gradient of a vertical tangent is undefined.
Consider the function \( f(x) = (1 - x)(x - 3)(x - 6) \). The graph of this function is shown.

a  Find the equation of the tangent to the curve at the point (4, 6).

b  Find the equation of the line perpendicular to the tangent at the point (4, 6).

**THINK**

- 1 Expand \( f(x) \).

  - 2 Find the derivative of \( f(x) \).

  - 3 Find the gradient at \( x = 4 \).

  - 4 Substitute the appropriate values into the formula \( y - y_1 = m(x - x_1) \).

**WRITE**

\[
\begin{align*}
f'(x) &= -3x^2 + 20x - 27 \\
f'(4) &= -3(4)^2 + 20(4) - 27 \\
&= -48 + 80 - 27 \\
&= 5
\end{align*}
\]

- \( m = 5 \) and \((x_1, y_1) = (4, 6)\)

  - \( y - y_1 = m(x - x_1) \)

  - \( y - 6 = 5(x - 4) \)

  - \( y - 6 = 5x - 20 \)

  - \( y = 5x - 14 \)

- \( m_p = -\frac{1}{m_T} \)

  - \( m = \frac{1}{5} \) and \((x_1, y_1) = (4, 6)\)

  - \( y - y_1 = m(x - x_1) \)

  - \( y - 6 = \frac{1}{5}(x - 4) \)

  - \( y - 6 = \frac{1}{5}x + \frac{4}{5} \)

  - \( y = \frac{1}{5}x + \frac{14}{5} + \frac{30}{5} \)

  - \( y = \frac{1}{5}x + \frac{44}{5} \)

  - or

  - \( 5y = -x + 34 \)

  - \( x + 5y = 34 \)
**EXERCISE 5.2**  

**Review of differentiation**

1. **WE1** The graph of \( y = (2 - x)^3 + 1 \) is shown.  
   a. Using first principles, find the equation for the gradient of the tangent to the curve at any point along the curve.  
   b. Hence, find the gradient of the tangent to the curve at the point (1, 2).

2. **a** Sketch the graphs of \( y = (x + 2)(2 - x) \) and \( y = x^2(4 - x) \) on the one set of axes.  
   b. Find the point(s) of intersection of the two curves, giving coordinates correct to 2 decimal places where appropriate.  
   c. If \( P \) is the point of intersection where \( x \in \mathbb{Z} \), use first principles to find the gradient of the tangents to each of the curves at this point.

3. **WE2** Differentiate:  
   a. \( f(x) = 4x^3 + \frac{1}{3x^2} + \frac{1}{2} \)  
   b. \( y = \frac{2\sqrt{x} - x^4}{5x^3} \)

4. Differentiate:  
   a. \( f(x) = (x + 3)(x^2 + 1) \)  
   b. \( y = \frac{4 - \sqrt{x}}{\sqrt{x^3}} \)

5. **WE3** a. If \( f(x) = \frac{1}{x^2} + 2x \), what is the gradient of the curve when \( x = -\frac{1}{2} \)?  
   b. If \( f(x) = \frac{2x - 4}{x} \), determine the coordinates of the point where the gradient is 1.  
   c. If \( y = (x - a)(x^2 - 1) \), find the gradient of the curve when \( x = -2 \) in terms of \( a \).

6. **WE4** a. The graph of \( f(x) \) is shown. Analyse this function and sketch the graph of \( f'(x) \).

   b. State the domain of the gradient function, \( f'(x) \), for the function shown.
8 The graph of \( f(x) \) is shown. Analyse this function and sketch the graph of \( f'(x) \).

9 \textbf{WE5} a Find the equation of the tangent to the curve with equation \( y = x(x - 2)^2(x - 4) \) at the point \((3, -3)\).
b Find the equation of the line perpendicular to the tangent at the point \((3, -3)\).

10 The equation of a tangent to a given parabola is \( y = -2x + 5 \). The equation of the line perpendicular to this tangent is \( y = \frac{1}{2}x + \frac{5}{2} \). The parabola also has a stationary point at \((0, 4)\). Determine the equation of the parabola and hence sketch the parabola, the tangent and the line perpendicular to the tangent, on the one set of axes.

11 Differentiate:
   a \( y = \frac{3}{4x^5} - \frac{1}{2x} + 4 \)
   b \( f(x) = \frac{10x - 2x^3 + 1}{x^4} \)
   c \( y = \sqrt{x} - \frac{1}{2 \sqrt{x}} \)
   d \( f(x) = \frac{(3 - x)^3}{2x} \)

12 For the following graphs:
   i state the domain of the gradient function, \( f'(x) \)
   ii sketch the graph of \( f'(x) \).

   a
   b

---

\textbf{CONSOLIDATE}

Apply the most appropriate mathematical processes and tools
13 Find the gradient of the secant in each of the following cases.
   a \( f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 5x + 6 \) between \( x = -1 \) and \( x = 0 \)
   b \( f: \mathbb{R} \to \mathbb{R}, f(x) = x^3(x - 3) \) between \( x = 2 \) and \( x = 4 \)
   c \( f: \mathbb{R} \to \mathbb{R}, f(x) = -x^3 \) between \( x = -3 \) and \( x = 0 \)

14 Use first principles to find the rule for the derived function where the function is defined by:
   a \( f(x) = 12 - x \)
   b \( f(x) = 3x^2 - 2x - 21 \)

15 Find the gradient of the tangent to each of the following curves at the specified point.
   a \( f(x) = x^2 - 3 \) at \( x = 2 \)
   b \( f(x) = (3 - x)(x - 4) \) at \( x = 1 \)
   c \( f(x) = (x - 2)^3 \) at \( x = 4 \)
   d \( f(x) = \sqrt{x} - \frac{3}{x} + 2x \) at \( x = 4 \)

16 Find the equations of the tangents to the following curves at the specified points.
   a \( f(x) = (x + 1)(x + 3) \) at \( x = -5 \)
   b \( f(x) = 8 - x^2 \) at \( x = a \)
   c \( f(x) = 2x/3 - 5 \) at \( x = 3 \)
   d \( f(x) = -2x/3 - 4x \) at \( x = -2 \)

17 Find the equation of the line perpendicular to the tangent for each of the functions defined in question 16.
   a Find the equation of the tangent to the curve \( f(x) = -(x - 2)^2 + 3 \) that is parallel to the line \( y = 3x + 4 \).
   b Find the equation of the tangent to the curve \( f(x) = -\frac{2}{x^2} + 1 \) that is perpendicular to the line \( 2y - 2 = -4x \).

19 The tangent to a parabolic curve at \( x = 4 \) has the equation \( y = -x + 6 \). The curve also passes through the points \((0, -10)\) and \((2, 0)\). Find the equation of the curve.

20 The tangent to a cubic function at the point \( x = 2 \) has a rule defined by \( y = 11x - 16 \). The cubic passes through the origin as well as the point \((-1, 0)\). Find the equation of the cubic function.

21 A line perpendicular to the graph of \( y = 2 \sqrt{x} \) has the equation \( y = -2x + m \), where \( m \) is a real constant. Determine the value of \( m \).
22 a Use CAS technology to sketch \( y = x(x - 2)(x + 3) \) and \( y = (2 - x)(x + 3)(x - 3) \) on the same set of axes.

b Find the coordinates of the point of intersection between the graphs where \( 1 < x < 2 \).

c Find the equation of the tangent and the line perpendicular to the tangent at the point defined in part b for the cubic function defined by \( y = x(x - 2)(x + 3) \).

5.3 Differentiation of exponential functions
The derivative of the exponential function from first principles

We can find the derivative of the exponential function as follows.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
f(x) = e^x
\]

\[
f(x + h) = e^{x+h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}
\]

\[
f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}
\]

We don’t know the value of \( \lim_{h \to 0} \frac{e^h - 1}{h} \), but we can investigate by substituting different values for \( h \) and looking at what happens to the limit as the value of \( h \) approaches zero.

If \( h = 1 \), \( \frac{e^h - 1}{h} = 1.7183 \)

\( h = 0.1, \) \( \frac{e^h - 1}{h} = 1.0517 \)

\( h = 0.01, \) \( \frac{e^h - 1}{h} = 1.0050 \)

\( h = 0.001, \) \( \frac{e^h - 1}{h} = 1.0005 \)

\( h = 0.0001, \) \( \frac{e^h - 1}{h} = 1.00005 \)

From these results, we can see that as the value of \( h \) gets smaller and approaches zero, the value of \( \frac{e^h - 1}{h} \) approaches 1:

\[
f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}
\]

As \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \), therefore

\[
f'(x) = e^x
\]
So the derivative of the exponential function, \( f(x) = e^x \), is itself. 

*Note:* This rule only applies to exponential functions of base \( e \).

It can be shown using the **chain rule**, which will be introduced in the next topic, that:

\[
\text{If } f(x) = e^{kx}, \text{ then } f'(x) = ke^{kx}
\]

and

\[
\text{if } f(x) = e^{g(x)}, \text{ then } f'(x) = g'(x)e^{g(x)}.
\]

WORKED EXAMPLE 6

Find the derivative of each of the following with respect to \( x \).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( y = e^{-\frac{1}{2}x} )</td>
</tr>
<tr>
<td>b</td>
<td>( y = \frac{1}{4}e^{2x} + e^{x^2} )</td>
</tr>
<tr>
<td>c</td>
<td>( y = \frac{e^{2x} + 3e^x - 1}{e^{2x}} )</td>
</tr>
<tr>
<td>d</td>
<td>( y = (e^x - 2)^2 )</td>
</tr>
</tbody>
</table>

**THINK**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1 Write the equation to be differentiated.</td>
</tr>
<tr>
<td></td>
<td>2 Apply the rule for ( \frac{d}{dx}(e^{kx}) ) with ( k = -\frac{1}{2} ).</td>
</tr>
<tr>
<td>b</td>
<td>1 Write the equation to be differentiated.</td>
</tr>
<tr>
<td></td>
<td>2 Apply the rule ( \frac{d}{dx}(e^{g(x)}) = g'(x)e^{g(x)} ), and differentiate each term separately.</td>
</tr>
<tr>
<td>c</td>
<td>1 Write the equation to be differentiated.</td>
</tr>
<tr>
<td></td>
<td>2 Split the right-hand side into three separate terms and divide through by ( e^{2x} ).</td>
</tr>
<tr>
<td></td>
<td>3 Apply the rule ( \frac{d}{dx}(e^{kx}) = ke^{kx} ) and differentiate each term separately.</td>
</tr>
<tr>
<td>d</td>
<td>1 Write the equation to be differentiated.</td>
</tr>
<tr>
<td></td>
<td>2 Expand the right side.</td>
</tr>
<tr>
<td></td>
<td>3 Differentiate each term separately.</td>
</tr>
</tbody>
</table>

**WRITE**

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>a</td>
<td>( y = e^{-\frac{1}{2}x} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x} )</td>
</tr>
<tr>
<td>b</td>
<td>( y = \frac{1}{4}e^{2x} + e^{x^2} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dx} = \frac{1}{4} \times 2e^{2x} + 2xe^{x^2} )</td>
</tr>
<tr>
<td></td>
<td>( = \frac{1}{2}e^{2x} + 2xe^{x^2} )</td>
</tr>
<tr>
<td>c</td>
<td>( y = \frac{e^{2x} + 3e^x - 1}{e^{2x}} )</td>
</tr>
<tr>
<td></td>
<td>( y = \frac{e^{2x}}{e^{2x}} + \frac{3e^x}{e^{2x}} - \frac{1}{e^{2x}} )</td>
</tr>
<tr>
<td></td>
<td>( = 1 + 3e^{-x} - e^{-2x} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dx} = -3e^{-x} + 2e^{-2x} )</td>
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<td></td>
<td>( = \frac{3}{e^x} + \frac{2}{e^{2x}} )</td>
</tr>
<tr>
<td>d</td>
<td>( y = (e^x - 2)^2 )</td>
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<tr>
<td></td>
<td>( y = e^{2x} - 4e^x + 4 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dx} = 2e^{2x} - 4e^x )</td>
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</table>
WORKED EXAMPLE 7

a Determine the gradient of the tangent to the curve with equation \( y = e^{-x} \) at the point where \( x = 1 \).

b Determine the equation of the tangent to the curve \( y = e^{-x} \) at the point where \( x = 1 \). What is the equation of the line perpendicular to this tangent?

THINK

a 1 The gradient of the tangent is given by \( \frac{dy}{dx} \).

2 Substitute \( x = 1 \).

b 1 We have the gradient but we need a point. Determine the corresponding \( y \)-value when \( x = 1 \).

2 Use \( y - y_1 = m(x - x_1) \) to find the equation of the tangent.

WRITE

a \[ y = e^{-x} \]
\[ \frac{dy}{dx} = -e^{-x} \]
\[ \frac{dy}{dx} = -e^{-1} \]
\[ = -\frac{1}{e} \]

The gradient of the curve is \( -\frac{1}{e} \).

b \[ x = 1: \]
\[ y = e^{-1} \]
\[ = \frac{1}{e} \]

If \( (x_1, y_1) = \left( 1, \frac{1}{e} \right) \) and \( m = -\frac{1}{e} \):
\[ y - \frac{1}{e} = -\frac{1}{e}(x - 1) \]
\[ y = -\frac{1}{e}x + \frac{1}{e} \]
\[ y = -\frac{1}{e} + \frac{2}{e} \]
\[ = -\frac{1}{e}(x - 2) \]

The equation of the tangent is \( y = -\frac{1}{e}(x - 2) \).

\[ m_p = -(-e) \]
\[ = e \]

If \( (x_1, y_1) = \left( 1, \frac{1}{e} \right) \) and \( m = e \):
\[ y - \frac{1}{e} = e(x - 1) \]
\[ y = ex - e \]
\[ y = ex - e + \frac{1}{e} \]

The equation of the perpendicular line is \( y = ex - e + \frac{1}{e} \).
Differentiation of exponential functions

1. **WE6** Find the derivative of each of the following functions.
   a. \( e^{-\frac{1}{3}x} \)
   b. \( 3x^4 - e^{-2x} \)
   c. \( y = \frac{4e^x - e^{-x} + 2}{3e^{3x}} \)
   d. \( y = (e^{2x} - 3)^2 \)

2. Consider the function defined by the rule
   \[ f(x) = \frac{1}{2}e^{3x} + e^{-x}. \]
   Find the gradient of the curve when \( x = 0 \).

3. **WE7** Find the equation of the tangent to the curve with equation \( y = e^{2x} \) at the point where \( x = 0 \).

4. Find the equations of the tangent and the line perpendicular to the curve with equation \( y = e^{-3x} + 4 \) at the point where \( x = 0 \).

5. Differentiate the following with respect to \( x \).
   a. \( 5e^{-4x} + 2e \)
   b. \( e^{-\frac{1}{2}x} + \frac{1}{3}x^3 \)
   c. \( 4e^{3x} - \frac{1}{2}e^{6\sqrt{x}} - 3e^{-3x+2} \)
   d. \( e^{5x} - e^{-x} + 2 \)
   e. \( \frac{e^x(2 - e^{-3x})}{e^{-x}} \)
   f. \( (e^{2x} + 3)(e^{-x} - 1) \)

6. Find the exact gradients of the tangents to the given functions at the specified points.
   a. \( y = 2e^{-x} \) at \( x = 0 \)
   b. \( y = \frac{4}{e^{2x}} \) at \( x = \frac{1}{2} \)
   c. \( y = \frac{1}{2}e^{3x} \) at \( x = \frac{1}{3} \)
   d. \( y = 2x - e^x \) at \( x = 0 \)

7. Determine the equation of the tangent to the curve with equation \( y = e^{-2x} \) at the point where \( x = -\frac{1}{2} \).

8. Determine the equations of the tangent and the line perpendicular to the curve \( y = e^{-3x} - 2 \) at the point where \( x = 0 \).

9. Determine the equations of the tangent and the line perpendicular to the curve \( y = e^{\sqrt{x}} + 1 \) at the point where \( x = 3 \).

10. Determine the derivative of the function \( f(x) = e^{-2x+3} - 4e \) and hence find:
    a. \( f'(-2) \) in exact form
    b. \( \{x : f'(x) = -2\} \).

11. Determine the derivative of the function \( f(x) = \frac{e^{3x} + 2}{e^x} \) and hence find:
    a. \( f'(1) \) in exact form
    b. \( \{x : f'(x) = 0\} \).

12. Find the equation of the tangent to the curve \( y = e^{x^2} + 3x - 4 \) at the point where \( x = 1 \).

13. For the function with the rule \( f(x) = Ae^x + Be^{-3x} \), where \( A \) and \( B \) are non-zero real constants, find \( f'(x) \) and show that \( f'(0) = 0 \) when \( e^{4x} = \frac{3B}{A} \).

14. The curve with the rule \( A = A_0e^{-0.69t} \) passes through the point \((0, 2)\).
    a. Find the value of \( A_0 \).
    b. Find \( \frac{dA}{dt} \) when \( t = 0 \).

15. Determine the exact value for \( f'(2) \) if \( f(x) = 3^{2x-4} \).

16. a. The graphs of the equations \( y = 2e^{-2x} + 1 \) and \( y = x^3 - 3x \) are shown. Find the coordinates of the point of intersection, giving your answer correct to 2 decimal places.
    b. Find the gradient of the tangent to the cubic at this point.
Applications of exponential functions

Exponential functions are commonly used to model a number of real world applications, including Newton’s Law of Cooling, population growth and decay, cell growth and decay, and radioactive decay.

A general equation to represent exponential growth and decay is given by

\[ A = A_0e^{kt} \]

where \( A_0 \) is the initial amount and \( k \) is a constant.

If the equation represents growth, then \( k \) is a positive value. If the equation represents decay, then \( k \) is a negative value.

WORKED EXAMPLE 8

The number of bacteria on a culture plate, \( N \), can be defined by the rule

\[ N(t) = 2000e^{0.3t}, \quad t \geq 0 \]

where \( t \) is the time, in seconds, the culture has been multiplying.

a How many bacteria are initially present?

b How many bacteria, to the nearest whole number, are present after 10 seconds?

c At what rate is the bacteria population multiplying after 10 seconds? Give your answer correct to the nearest whole number.

THINK

a Initially \( t = 0 \), so substitute this value into the rule.

b Substitute \( t = 10 \).

c \( \frac{dN}{dt} \) represents the required rate.

WRITE

a \( N(0) = 2000e^{0.3(0)} = 2000 \)

Initially there are 2000 bacteria present.

b \( N(10) = 2000e^{0.3(10)} = 2000e^3 = 40171 \)

After 10 seconds there are 40171 bacteria present.

c \( \frac{dN}{dt} = 600e^{0.3t} \)

\[ \frac{dN}{dt} = 600e^{0.3(10)} = 600e^3 = 12051 \]

After 10 seconds the bacteria are growing at a rate of 12051 per second.

EXERCISE 5.4

Applications of exponential functions

1 WEB The mass, \( M \) grams, of a radioactive substance is initially 20 grams; 30 years later its mass is 19.4 grams. If the mass in any year is given by

\[ M = M_0e^{-0.00152t} \]

where \( t \) is the time in years and \( M_0 \) is a constant, find:

a the value of \( M_0 \)

b the annual rate of decay

c the rate of decay after 30 years.
2 The intensity of light decreases as it passes through water. The phenomenon can be modelled by the equation

\[ I = I_0 e^{-0.0022d} \]

where \( I_0 \) is the intensity of light at the surface of the water and \( I \) is the intensity of light at a depth of \( d \) metres below the surface of the water.

a What is the intensity of light at a depth of 315 metres?
b What is the rate at which the intensity of light is decreasing at 315 metres?

3 The graph shown is that of the function \( f: R \rightarrow R, f(x) = e^{2x} + qe^x + 3 \), where \( q \) is a constant.

a Find the value of \( q \).
b Find the exact value of \( m \), where \( m \) is a constant and \((m, 0)\) are the coordinates of the point where the function intersects the \( x \)-axis.
c Find the derivative function, \( f'(x) \).
d Find the gradient of the curve where it intersects the \( y \)-axis.

4 The graph shown is that of the function \( f: R \rightarrow R, f(x) = e^{-2x} + z e^{-x} + 2 \), where \( z \) is a constant.

a Find the value of \( z \).
b Find the exact value of \( n \), where \( n \) is a constant and \((n, 0)\) are the coordinates of the point where the graph intersects the \( x \)-axis.
c Find the derivative function, \( f'(x) \).
d Find the gradient of the curve where it passes through the origin.

5 An unstable gas decomposes in such a way that the amount present, \( A \) units, at time \( t \) minutes is given by the equation

\[ A = A_0 e^{-kt} \]

where \( k \) and \( A_0 \) are constants. It was known that initially there were 120 units of unstable gas.

a Find the value of \( A_0 \).
b Show that \( \frac{dA}{dt} \) is proportional to \( A \).
c After 2 minutes there were 90 units of the gas left. Find the value of \( k \).
d At what rate is the gas decomposing when \( t = 5 \)? Give your answer correct to 3 decimal places.
e Will there ever be no gas left? Explain your answer.
6 The bilby is an endangered species that can be found in the Kimberley in Western Australia as well as some parts of South Australia, the Northern Territory and Queensland. The gestation time for a bilby is 2–3 weeks and when they are born, they are only about 11 mm in length. The growth of a typical bilby can be modelled by the rule
\[ L = L_0 e^{0.599t} \]
where \( L_0 \) is its length in millimetres at birth and \( L \) is the length of the bilby in millimetres \( t \) months after its birth.

a. Determine the value of \( L_0 \).
b. What is the rate of change of length of the bilby at time \( t \) months?
c. At what rate is the bilby growing when it is 3 months old? Give your answer correct to 3 decimal places.

7 A body that is at a higher temperature than its surroundings cools according to Newton’s Law of Cooling, which states that
\[ T = T_0 e^{-zt} \]
where \( T_0 \) is the original excess of temperature, \( T \) is the excess of temperature in degrees centigrade after \( t \) minutes, and \( z \) is a constant.

a. The original temperature of the body was 95°C and the temperature of the surroundings was 20°C. Find the value of \( T_0 \).
b. At what rate is the temperature decreasing after a quarter of an hour if it is known that \( z = 0.034 \)? Give your answer correct to 3 decimal places.

8 The population of Australia since 1950 can be modelled by the rule
\[ P = P_0 e^{0.016t} \]
where \( P_0 \) is the population in millions at the beginning of 1950 and \( P \) is the population in millions \( t \) years after 1950. It is known that there were 8.2 million people in Australia at the beginning of 1950.

a. Calculate the population in millions at the beginning of 2015 correct to 1 decimal place.
b. During which year and month does the population reach 20 million?
c. Determine the rate of change of population at the turn of the century, namely the year 2000, correct to 2 decimal places.
d. In which year is the rate of increase of the population predicted to exceed 400 000 people per year?

9 The pressure of the atmosphere, \( P \) cm of mercury, decreases with the height, \( h \) km above sea level, according to the law
\[ P = P_0 e^{-kh} \]
where \( P_0 \) is the pressure of the atmosphere at sea level and \( k \) is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.

a. What are the values of \( P_0 \) and \( k \), correct to 2 decimal places?
b. Find the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places.
10 An entrance to a local suburban park has a series of posts connected with heavy chains as shown.

The chain between any two posts can be modelled by the curve defined by

\[ h = 0.295(e^x + e^{-x}), \quad -0.6 \leq x \leq 0.6 \]

where \( h \) metres is the height of the chain above the ground and \( x \) is the horizontal distance between the posts in metres. The \( x \)-axis represents the ground. The posts are positioned at \( x = -0.6 \) and \( x = 0.6 \).

a Calculate the amount of sag in the chain (i.e. the difference in height between the highest points of the chain and the lowest point of the chain). Give your answer in centimetres.

b Calculate the angle the chain makes with the post positioned on the right-hand side of the structure (i.e. at \( x = 0.6 \)). Give your answer correct to 1 decimal place.

11 The graph of the function

\[ f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = e^{-x} - 0.5e^{-2x} \]

is shown.

a Find the coordinates of the point at which the graph crosses the \( y \)-axis.

b Determine \( f'(x) \).

c Find the coordinates of the point at which the gradient is equal to zero.

d What is the angle, correct to 1 decimal place, which the graph makes with the positive direction of the \( x \)-axis if it is known that the graph cuts the \( x \)-axis at \((\log_2(e), 0)\)?

e What is the equation of the tangent to the curve when \( x = 1 \)?

f What is the equation of the line perpendicular to the curve when \( x = 1 \)?

12 The cane toad, originally from South America, is an invasive species in Australia. Cane toads were introduced to Australia from Hawaii in June 1935 in an attempt to control cane beetles, though this proved to be ineffective. The long-term effects of the toad’s introduction in Australia include the depletion of native species, because the cane toads produce a poison that kills most animals that try to eat them.
One study has found that cane toads are especially vulnerable to a common native predator, the meat ant. At waterholes in tropical Australia, interactions between cane toads and meat ants were observed. It was found that the ants are very effective in capturing and killing small young toads as they emerge from the water because the ants are not overpowered by the toad’s poison.

In a controlled experiment at a particular waterhole, it was observed that at the beginning of the experiment there were an estimated 30,000 tadpoles (future cane toads) in the water. The number of tadpoles increased by about 60,000 a day over the period of a week. This growth pattern can be defined by the equation

\[ T = T_0e^{kt} \]

where \( T_0 \) is the initial number of cane toad tadpoles (in thousands) at the waterhole during the time of the experiment, \( T \) is the number of cane toad tadpoles (in thousands) at the waterhole \( t \) days into the experiment, and \( k \) is a constant.

a Calculate the value of \( T_0 \).

b How many cane toad tadpoles are in the waterhole after a week if it is known that \( k = 0.387 \)? Give your answer to the nearest thousand.

c Find the rate at which the cane toad tadpole numbers are increasing after 3 days.

After a week, no more tadpoles could be supported by the habitat. In favourable conditions, tadpoles take about two weeks to develop into small cane toads, at which point they leave the water. Once the small cane toads emerged, meat ants were introduced into their environment. This caused 90% of the cane toads to be killed off over a period of a week. The growth and decline of the tadpoles/cane toads is shown.

The decline in the number of young cane toads can be defined by the equation

\[ C = C_0e^{mt} \]

where \( C_0 \) is the number of young cane toads (in thousands) just before the meat ants were introduced, \( C \) is the number of young cane toads (in thousands) \( t \) days after the meat ants were introduced and \( m \) is a constant.

d Determine the value of \( C_0 \).

e How many young cane toads still survived a week after the meat ants were introduced?

f Find \( m \) and the rate of decline in the number of cane toads after 4 days.
13 The graph of \( y = Ae^{-x^2} \), where \( A \) is a constant, is shown. Answer the following questions correct to 2 decimal places where appropriate.

a If the gradient of the graph is zero at the point \( (0, 5) \), determine the value of \( A \).

b Find \( \frac{dy}{dx} \).

c Determine the gradient of the tangent to the curve at the point where:
   i \( x = -0.5 \)
   ii \( x = 1 \)

14 Consider the curve with equation
   \( y = \frac{x^2 - 5}{2e^x} \). Using CAS:

a find \( \frac{dy}{dx} \)

b find the exact coordinates of the points on the curve where the gradient is equal to zero

c find the gradient of the tangent to the curve at \( x = \frac{1}{2} \), giving your answer correct to 3 decimal places.

5.5 Differentiation of trigonometric functions

The derivatives of \( \sin(x) \) and \( \cos(x) \)

The derivative of \( \sin(x) \) can be investigated using differentiation from first principles.

Consider \( f: \mathbb{R} \to \mathbb{R}, f(x) = \sin(x) \) where \( x \) is an angle measurement in radians.

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
 = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}
\]

To evaluate this limit, we must look at the unit circle.
\[ \angle NOM = x, \quad \angle QOM = x + h \]
\[ \angle PQO = \frac{\pi}{2} - (x + h) \]
\[ \angle RQS = \frac{\pi}{2} - \left( \frac{\pi}{2} - (x + h) \right) \]
\[ = x + h \]

By definition
\[ \sin(x) = MN \]
\[ \sin(x + h) = PQ \]
\[ \frac{\sin(x + h) - \sin(x)}{h} = \frac{PQ - MN}{h} = QR \]

From the diagram, it can be seen that \( \angle RQS = x + h \) and the arc \( QN \) has length \( h \).
As \( h \to 0 \), \( \angle RQS \) approaches \( \angle RQN \), which approaches \( x \). Furthermore, the arc \( QN \) approaches the chord \( QN \).
Consequently \( \frac{QR}{h} \to \frac{QR}{QN} \) but by definition, \( \frac{QR}{QN} = \cos(x) \).

Hence
\[ f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h} \]
\[ = \lim_{h \to 0} \frac{QR}{h} \]
\[ = \cos(x) \]

If \( f(x) = \sin(x) \), then \( f'(x) = \cos(x) \).

It can also be shown using the chain rule, which will be introduced in the next topic, that:

If \( f(x) = \sin(kx) \), then \( f'(x) = k \cos(kx) \), where \( k \) is a constant
and
if \( f(x) = \sin(g(x)) \), then \( f'(x) = g'(x)\cos(g(x)) \).

The derivative of \( \cos(x) \) can also be investigated geometrically, using the same method as shown for \( \sin(x) \) and yielding the following result.

If \( f(x) = \cos(x) \), then \( f'(x) = -\sin(x) \).

It can also be shown using the chain rule that:

If \( f(x) = \cos(kx) \), then \( f'(x) = -k \sin(kx) \), where \( k \) is a constant
and
if \( f(x) = \cos(g(x)) \), then \( f'(x) = -g'(x)\sin(g(x)) \).

The derivative of \( \tan(x) \)

Consider the function \( f : R \to R, f(x) = \tan(x) \).

If \( f(x) = \tan(x) \), then \( f'(x) = \frac{1}{\cos^2(x)} = \sec^2(x) \).
In order to prove this differentiation, we would use the trigonometric identity 
\[ \tan(x) = \frac{\sin(x)}{\cos(x)} \] 
in conjunction with the quotient rule, which will also be introduced in the next topic.

It can also be shown using the chain rule that:

If \( f(x) = \tan(kx) \), then \( f'(x) = \frac{k}{\cos^2(kx)} = k \sec^2(kx), \) where \( k \) is a constant

and

if \( f(x) = \tan(g(x)) \) then \( f'(x) = \frac{g'(x)}{\cos^2(g(x))} = g'(x) \sec^2(g(x)). \)

Remember that these rules can only be applied if the angle \( x \) is measured in radians.

**WORKED EXAMPLE 9**

Determine the derivative of each of the following functions.

**a** \( \sin(8x) + x^4 \)

**b** \( \tan(5x) + 2\cos(x^2) \)

**c** \( \frac{1 - \sin^2(x)}{\cos(x)} \)

**d** \( \sin(6x^\circ) \)

**THINK**

**a** Apply the rule \( \frac{d}{dx}(\sin(kx)) = k \cos(kx) \) and differentiate each term separately.

**b** Apply the rules \( \frac{d}{dx}(\cos(g(x))) = -g'(x)\sin(g(x)) \) and \( \frac{d}{dx}(\tan(kx)) = \frac{k}{\cos^2(kx)} \).

**c** 1 Remember the trigonometric identity \( \sin^2(\theta) + \cos^2(\theta) = 1 \). Use this to simplify the equation.

2 Differentiate the simplified function.

**d** 1 The function \( \sin(6x^\circ) \) cannot be differentiated as the angle is not measured in radians. Convert the angle to radian measures by multiplying by \( \frac{\pi}{180} \), as \( 1^\circ = \frac{\pi}{180} \).

2 Differentiate the resultant function by applying the rule \( \frac{d}{dx}(\sin(kx)) = k \cos(kx) \).

**WRITE**

**a** \( y = \sin(8x) + x^4 \)

\[ \frac{dy}{dx} = 8\cos(8x) + 4x^3 \]

**b** \( y = \tan(5x) + 2\cos(x^2) \)

\[ \frac{dy}{dx} = \frac{5\sec^2(5x)}{\cos^2(5x)} - 2 \times 2x \sin(x^2) \]

\[ = \frac{5}{\cos^2(5x)} - 4x \sin(x^2) \]

**c** \( y = \frac{1 - \sin^2(x)}{\cos(x)} \)

\[ = \frac{\cos(x)}{\cos^2(x)} \]

\[ = \frac{\cos(x)}{\cos(x)} \]

\[ = \cos(x), \cos(x) \neq 0 \]

\[ \frac{dy}{dx} = -\sin(x) \]

**d** \( \sin(6x^\circ) = \sin\left(6 \times \frac{\pi}{180}\right) \)

\[ = \sin\left(\frac{\pi x}{30}\right) \]

\[ y = \sin\left(\frac{\pi x}{30}\right) \]

\[ \frac{dy}{dx} = \frac{\pi}{30} \cos\left(\frac{\pi x}{30}\right) \]
WORKED EXAMPLE 10

Find the equation of the tangent to the curve \( y = \sin(3x) + 1 \) at the point where \( x = \frac{\pi}{3} \).

THINK

1 First find the coordinates of the point; that is, determine the \( y \)-value when \( x = \frac{\pi}{3} \).

When \( x = \frac{\pi}{3} \),
\[
y = \sin\left(3 \times \frac{\pi}{3}\right) + 1
= \sin(\pi) + 1
= 0 + 1
= 1
\]
The point is \( \left(\frac{\pi}{3}, 1\right) \).

2 Find the derivative of the function.
\[
\frac{dy}{dx} = 3 \cos(3x)
\]

3 Determine the gradient at the point where \( x = \frac{\pi}{3} \).
\[
x = \frac{\pi}{3}, \quad \frac{dy}{dx} = 3 \cos\left(3 \times \frac{\pi}{3}\right)
= 3 \cos(\pi)
= 3(-1)
= -3
\]

4 Substitute the appropriate values into the rule \( y - y_1 = m(x - x_1) \) to find the equation of the tangent.
\[
m = -3, \quad (x_1, y_1) = \left(\frac{\pi}{3}, 1\right)
\]
\[
y - y_1 = m(x - x_1)
\]
\[
y - 1 = -3\left(x - \frac{\pi}{3}\right)
\]
\[
y - 1 = -3x + \pi
y = -3x + \pi + 1
\]
The equation of the tangent is \( y = 1 + \pi - 3x \).

EXERCISE 5.5

Differentiation of trigonometric functions

1 WE9 Differentiate each of the following functions with respect to \( x \).
   a \( 5x + 3 \cos(x) + 5 \sin(x) \)
   b \( \sin(3x + 2) - \cos(3x^2) \)
   c \( \frac{1}{3} \sin(9x) \)
   d \( 5 \tan(2x) - 2x^5 \)
   e \( 8 \tan\left(\frac{x}{4}\right) \)
   f \( \tan(9x^2) \)

2 Simplify and then differentiate \( \frac{\sin(x)\cos^2(2x) - \sin(x)}{\sin(x)\sin(2x)} \) with respect to \( x \).

3 WE10 Find the equation of the tangent to the curve \( y = -\cos(x) \) at the point where \( x = \frac{\pi}{2} \).

4 Find the equation to the tangent to the curve \( y = \tan(2x) \) at the point where \( x = \frac{-\pi}{8} \).
5 For each of the following functions, find \( \frac{dy}{dx} \).

\[ \text{a) } y = 2 \cos(3x) \quad \text{b) } y = \cos(x^o) \quad \text{c) } y = 3 \cos\left(\frac{\pi}{2} - x\right) \quad \text{d) } y = -4 \sin\left(\frac{x}{3}\right) \quad \text{e) } y = \sin(12x^o) \quad \text{f) } y = 2 \sin\left(\frac{\pi}{2} + 3x\right) \quad \text{g) } y = -\frac{1}{2} \tan(5x^2) \quad \text{h) } y = \tan(20x) \]

6 Determine the point on the curve with equation \( y = -2 \sin\left(\frac{x}{2}\right) \), \( x \in [0, 2\pi] \) where the gradient is equal to \( \frac{1}{2} \).

7 Find the equation of the tangent to the curve with equation \( y = 3 \cos(x) \) at the point where \( x = \frac{\pi}{6} \).

8 Find the equation of the tangent to the curve with equation \( y = 2 \tan(x) \) at the point where \( x = \frac{\pi}{4} \).

9 Find the angle that the curve with equation \( y = \sin(2x) \) makes with the positive direction of the \( x \)-axis the first time it intersects the \( x \)-axis when \( x > 0 \). Give your answer correct to 1 decimal place.

10 Find the equations of the tangent and the line perpendicular to the following graphs at the points indicated.

\[ \text{a) } y = \sin(3x) \text{ at } \left(\frac{2\pi}{3}, 0\right) \quad \text{b) } y = \cos\left(\frac{x}{2}\right) \text{ at } (\pi, 0) \]

11 Consider the function \( f: [0, 2\pi] \to \mathbb{R}, f(x) = \sin(x) - \cos(x) \). Find:

\[ \text{a) } f(0) \quad \text{b) } \{x : f(x) = 0\} \quad \text{c) } f'(x) \quad \text{d) } \{x : f'(x) = 0\} \]

12 Consider the function \( f: [-\pi, \pi] \to \mathbb{R}, f(x) = \sqrt{3}\cos(x) + \sin(x) \). Find:

\[ \text{a) } f(0) \quad \text{b) } \{x : f(x) = 0\} \quad \text{c) } f'(x) \quad \text{d) } \{x : f'(x) = 0\} \]

13 a) Use both or either of the trigonometric identities \( \sin^2(\theta) + \cos^2(\theta) = 1 \) and \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \) to simplify \( \frac{\sin(x)\cos(x) + \sin^2(x)}{\sin(x)\cos(x) + \cos^2(x)} \).

\[ \text{b) Hence, find } \frac{d}{dx} \left(\frac{\sin(x)\cos(x) + \sin^2(x)}{\sin(x)\cos(x) + \cos^2(x)}\right). \]

14 Determine the \( x \)-values over the domain \( x \in [-\pi, \pi] \) for which the gradients of the functions \( f(x) = \sin(2x) \) and \( f(x) = \cos(2x) \) are equal.

15 For the function \( f(x) = x - \sin(2x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \), find the point(s) where the gradient is zero. Give your answer correct to 3 decimal places.

16 For the function, \( f(x) = 2x + \cos(3x), 0 \leq x \leq \frac{\pi}{2} \), find the point(s) where the gradient is zero. Give your answer correct to 3 decimal places.
5.6 Applications of trigonometric functions

Trigonometric functions in real-life situations are usually used to model geometric scenarios or situations where cyclic phenomena are being investigated.

WORKED EXAMPLE 11

The circle shown has a radius of 1 unit.

The area of the triangle OQP is equal to

\[ A = \frac{1}{2} \cos(\theta), \]

where \( \angle QOX = \theta \) and \( \theta \) is in radian measure.

Find \( \frac{dA}{d\theta} \) when \( \theta = \frac{\pi}{6} \).

THINK

\( \theta = \frac{\pi}{6} \)

WRITE

As \( RQ \) is parallel to the \( x \)-axis, \( \angle RQO = \theta \) because it is alternate to \( \angle QOX \).

\[ \cos(\theta) = \frac{RQ}{OQ} = \frac{RQ}{1} \]

\[ \cos(\theta) = RQ \text{ and } OP = 1 \]

Area = \( \frac{1}{2} \times OP \times RQ \)

Area = \( \frac{1}{2} \times 1 \times \cos(\theta) \)

\[ = \frac{1}{2} \cos(\theta) \text{ (as required)} \]

\( \frac{dA}{d\theta} = -\frac{1}{2} \sin(\theta) \)

\[ \theta = \frac{\pi}{6}, \quad \frac{dA}{d\theta} = -\frac{1}{2} \sin\left(\frac{\pi}{6}\right) \]

\[ = -\frac{1}{2} \times \frac{1}{2} \]

\[ = -\frac{1}{4} \]

The previous example involved a geometric application question, but everyday application questions can also be solved using trigonometric functions.
The temperature on a particular day can be modelled by the function

\[ T(t) = -3 \cos \left( \frac{\pi t}{9} \right) + 18, \quad 0 \leq t \leq 18 \]

where \( t \) is the time in hours after 5.00 am and \( T \) is the temperature in degrees Celsius.

For the remaining 6 hours of the 24-hour period, the temperature remains constant.

a Calculate the temperature at 8.00 am.

b At what time(s) of the day is the temperature 20°C? Give your answer correct to the nearest minute.

c Find \( \frac{dT}{dt} \).

d What is the rate of change of temperature at the time(s) found in part b, correct to 2 decimal places?

**THINK**

<table>
<thead>
<tr>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>a  At 8.00 am ( t = 3 ). Substitute this value into the equation.</td>
</tr>
<tr>
<td>a  ( T(3) = -3 \cos \left( \frac{3\pi}{9} \right) + 18 )</td>
</tr>
<tr>
<td>a  ( = -3 \cos \left( \frac{\pi}{3} \right) + 18 )</td>
</tr>
<tr>
<td>a  ( = -3 \times \frac{1}{2} + 18 )</td>
</tr>
<tr>
<td>a  ( = -1.5 + 18 )</td>
</tr>
<tr>
<td>a  ( = 16.5°C )</td>
</tr>
</tbody>
</table>

b 1 Substitute \( T = 20 \) into the equation.

b 20 = \(-3 \cos \left( \frac{\pi t}{9} \right) + 18 \)

2 Solve the equation for \( 0 \leq t \leq 18 \) using CAS.

20 = \(-3 \cos \left( \frac{\pi t}{9} \right) + 18 \)

\( t = 6.6, 11.4 \)

3 Interpret your answers and convert the \( t \) values to times of the day.

\( t = 6.6 \Rightarrow 11.36 \text{ am} \)

\( t = 11.4 \Rightarrow 4.24 \text{ pm} \)

4 Write the answer.

The temperature is 20°C at 11.36 am and 4.24 pm.

c  Determine \( \frac{dT}{dt} \).

c  \( \frac{dT}{dt} = 3 \times \frac{\pi}{9} \sin \left( \frac{\pi t}{9} \right) \)

\( = \frac{\pi}{3} \sin \left( \frac{\pi}{9} \right) \)

d  When \( t = 6.6 \) (11.36 am),

d  \( \frac{dT}{dt} = \frac{\pi}{3} \sin \left( \frac{6.6 \times \pi}{9} \right) \)

\( = 0.78 \)
Applications of trigonometric functions

1. Consider the following triangle.
   a. Show that the area, $A$ cm$^2$, is given by $A = 21 \sin \left( \frac{\theta}{3} \right)$.
   b. Find $\frac{dA}{d\theta}$.
   c. What is the rate of change of area with respect to $\theta$ when $\theta = \frac{\pi}{3}$?

2. The diagram shows a garden bed bordered by wooden sleepers. BDC is a triangular herb garden and ABDE is a rectangular garden for vegetables.
   a. Find BD and CD in terms of $a$ and $\theta$, where $a$ is a constant, $\theta$ is $\angle BCD$ as shown and $0 < \theta < \frac{\pi}{2}$.
   b. Find the total length, $L$ metres, of sleepers required to surround the garden bed. (This should include BD as well as the sleepers defining the perimeter.)
   c. Find $\frac{dL}{d\theta}$ in terms of $\theta$ and $a$.
   d. Let $a = 2$ and use CAS to sketch $\frac{dL}{d\theta}$ for $0 < \theta < \frac{\pi}{2}$.
      Hence, find when $\frac{dL}{d\theta} = 0$, correct to 1 decimal place.

3. A mass oscillates up and down at the end of a metal spring. The length of the spring, $L$ cm, after time $t$ seconds, is modelled by the function $L(t) = 2 \sin(\pi t) + 10$ for $t \geq 0$.
   a. What is the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P?
   b. Find $\frac{dL}{dt}$.
   c. Find the exact value of $\frac{dL}{dt}$ after 1 second.

4. Between 6 am and 6 pm on a given day the height, $H$ metres, of the tide in a harbour is given by
   $$ H(t) = 1.5 + 0.5 \sin \left( \frac{\pi t}{6} \right), 0 \leq t \leq 12. $$

When $t = 11.4$ (4.24 pm)

$$ \frac{dT}{dt} = \frac{\pi}{3} \sin \left( \frac{11.4 \times \pi}{9} \right) $$

$$ = -0.78 $$

e. Write the answer.

At 11.36 am the temperature is increasing at a rate of $0.78^\circ$C per hour.
At 4.24 pm the temperature is decreasing at a rate of $0.78^\circ$C per hour.

e. Write the answer.

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**EXERCISE 5.6**

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**PRACTISE**

Work without CAS

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**APPLICATIONS OF TRIGONOMETRIC FUNCTIONS**

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**1.** Consider the following triangle.

a. Show that the area, $A$ cm$^2$, is given by $A = 21 \sin (\theta)$.

b. Find $\frac{dA}{d\theta}$.

c. What is the rate of change of area with respect to $\theta$ when $\theta = \frac{\pi}{3}$?

---

**2.** The diagram shows a garden bed bordered by wooden sleepers. BDC is a triangular herb garden and ABDE is a rectangular garden for vegetables.

a. Find BD and CD in terms of $a$ and $\theta$, where $a$ is a constant, $\theta$ is $\angle BCD$ as shown and $0 < \theta < \frac{\pi}{2}$.

b. Find the total length, $L$ metres, of sleepers required to surround the garden bed. (This should include BD as well as the sleepers defining the perimeter.)

c. Find $\frac{dL}{d\theta}$ in terms of $\theta$ and $a$.

d. Let $a = 2$ and use CAS to sketch $\frac{dL}{d\theta}$ for $0 < \theta < \frac{\pi}{2}$.
   Hence, find when $\frac{dL}{d\theta} = 0$, correct to 1 decimal place.

---

**3.** A mass oscillates up and down at the end of a metal spring. The length of the spring, $L$ cm, after time $t$ seconds, is modelled by the function $L(t) = 2 \sin(\pi t) + 10$ for $t \geq 0$.

a. What is the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P?

b. Find $\frac{dL}{dt}$.

c. Find the exact value of $\frac{dL}{dt}$ after 1 second.

---

**4.** Between 6 am and 6 pm on a given day the height, $H$ metres, of the tide in a harbour is given by

$$ H(t) = 1.5 + 0.5 \sin \left( \frac{\pi t}{6} \right), 0 \leq t \leq 12. $$
a What is the period of the function?
b What is the value of $H$ at low tide and when does low tide occur?
c Find $\frac{dH}{dt}$.
d Find the exact value of $\frac{dH}{dt}$ at 7.30 am.
e Find the second time during the given time interval that $\frac{dH}{dt}$ equals the value found in part d.

5 Given that $f : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}, f(x) = 2 \sin(4x) + 1$, find:
a the values of $x$ for which $f(x) = 0.5$, giving your answer correct to 3 decimal places
b the coordinates where the gradient of the function is zero
c the value of $f'(x)$ when $x = \frac{\pi}{4}$
d the interval over which the gradient is positive.

6 A wire frame is shaped in the following way.
The diagonals shown are 100 cm long, and each diagonal makes an angle of $\theta$ with the horizontal.
a Show that the length of wire required to form the shape is given by
$$L = 300 \cos(\theta) + 400 \sin(\theta) + 200, \quad 0 \leq \theta \leq \frac{\pi}{2}$$
where $L$ is the total length of wire in centimetres and $\theta$ is the angle shown in radians.
b Find $\frac{dL}{d\theta}$.
c Use CAS to sketch the graph of $L$. Find the maximum length of the wire required and the value of $\theta$, correct to 2 decimal places, for which this occurs.

7 The triangle XYZ is inscribed by a circle with radius, $r$ cm. The actual placement of the triangle is dependent on the size of the angle XZY, $\theta$ radians, and the length of ZM, where M is the midpoint of XY.
a Show that $\angle XOM = \theta$.
b Show that the relationship between $\theta$, $r$ and $h$, where $h = d(ZM)$, is given by $\frac{h}{r} = \cos(\theta) + 1$.
c If the radius of the circle is 3 cm, find $\frac{dh}{d\theta}$
d Find the exact value of $\frac{dh}{d\theta}$ when $\theta = \frac{\pi}{6}$. 
8 The figure shows a circular running track with centre O. The track has a radius of 200 metres.

An athlete at a morning training session completes an obstacle course from N to P at a rate of 2 m/s and then a series of hurdles from P to M along the running track at a rate of 5 m/s.

a If $\angle MNP = \theta$ radians and the total time taken to complete the total course is $T$ seconds, show that

$$T = 40(5 \cos(\theta) + 2\theta), 0 < \theta \leq \pi/2.$$ 

b Find the value of $\theta$ when $\frac{dT}{d\theta} = 0$.

c What is the maximum time taken to complete to whole course? Give your answer in minutes and seconds.

9 A very young girl is learning to skip. The graph showing this skipping for one cycle is given.

The general equation for this graph is given by $h = a \cos(nt) + c$, where $h$ is the height in millimetres of the girl’s feet above the ground and $t$ is the time in seconds the girl has been skipping.

a Find the values of the constants $a$, $n$ and $c$, and hence restate the equation for one cycle of the skipping.

b Find $\frac{dh}{dt}$.

c What is the value of $\frac{dh}{dt}$ when $t = 0.25$ seconds?

10 The height, $h$ metres, above ground level of a chair on a rotating Ferris wheel is modelled by the function

$$h = 5 - 3.5 \cos\left(\frac{\pi t}{30}\right)$$

where $t$ is measured in seconds.

a People can enter a chair when it is at its lowest position, at the bottom of the rotation. They enter the chair from a platform. How high is the platform above ground level?

b What is the highest point reached by the chair?

c How long does 1 rotation of the wheel take?

d During a rotation, for how long is a chair higher than 7 m off the ground? Give your answer to 1 decimal place.
e Find \( \frac{dh}{dt} \).

f Find the first two times, correct to 2 decimal places, when a chair is descending at a rate of 0.2 m/sec.

11 A section of a water slide at a local aquatic complex is shown.

The water slide can be defined by the rule

\[
y = \frac{7}{2} \cos\left(\frac{\pi x}{20}\right) + \frac{5}{2}, \quad 0 \leq x \leq 20
\]

where \( y \) is the height in metres of the water slide above the water surface and \( x \) is the horizontal distance in metres between the start of the slide and the end of the slide. (Note: The \( x \)-axis represents the water surface.)

a How high must a person climb in order to reach the top of the water slide?

b Find \( \frac{dy}{dx} \).

c What is the exact gradient of the water slide:

i when \( x = 5 \)

ii when \( x = 10 \)?

d i How far, to the nearest whole metre, from the climbing tower does the slide come into contact with the water surface.

ii What angle does the slide make with the water surface at this point?

Give your answer correct to 2 decimal places.

12 The following represents the cross-section of a waterfall feature in an Australian native garden.

It consists of an undulating surface of corrugated plastic with vertical posts at each end. The relationship that defines this surface can be expressed by

\[
h(x) = 10 \cos\left(\frac{7x}{2}\right) - 5x + 90, \quad 0 \leq x \leq 4.5
\]

where \( h \) centimetres represents the vertical height of the water feature and \( x \) metres is the horizontal distance between the upright posts supporting the undulating surface. The posts supporting the undulating surface over which the water falls are situated at the points \( x = 0 \) and \( x = 4.5 \), as shown.

a Find the coordinates of the end points of the undulating surface. Give your answers correct to the nearest centimetre.
b Find the coordinates of point A, the first point in the interval [0, 4.5] where the gradient of the undulating surface is zero. Give your answer correct to 2 decimal places.

c What is the slope of the undulating surface at \( x = 0.4 \)? Give your answer correct to 1 decimal place.

13 A mechanism for crushing rock is shown. Rocks are placed on a steel platform, S, and a device raises and lowers a heavy mallet, H. The wheel, W, rotates, causing the upper block, B, to move up and down. The other wheel, \( V \), is attached to the block, B, and rotates independently, causing the mallet to move up and down. T is the top of the block B.

The distance, \( P \) metres, between T and the steel platform, S, is modelled by the equation \( P = -2 \cos(mt) + n \), where \( t \) is the time in minutes and \( m \) and \( n \) are constants. When \( t = 0 \), T is at its lowest point, 4 metres above the steel platform. The wheel, W, rotates at a rate of 1 revolution per 1.5 minutes.

a Show that \( n = 6 \) and \( m = \frac{4\pi}{3} \).

b Find \( \frac{dP}{dt} \).

c What is the exact rate of change of distance when \( t = 0.375 \) minutes?

14 At a skateboard park, a new skateboard ramp has been constructed. A cross-section of the ramp is shown. The equation that approximately defines this curve is given by

\[
h(x) = 2.5 - 2.5 \cos\left(\frac{x}{4}\right), -5 \leq x \leq 5
\]

where \( h \) is the height in metres above the ground level and \( x \) is the horizontal distance in metres from the lowest point of the ramp to each end of the ramp.

a Determine the maximum depth of the skateboard ramp, giving your answer correct to 1 decimal place.

b Find the gradient of the ramp, \( \frac{dh}{dx} \).

c Find \( \frac{dh}{dx} \) when \( x = 3 \), giving your answer correct to 3 decimal places.

d Find where \( \frac{dh}{dx} = 0.58 \), giving your answer correct to 3 decimal places.

15 An industrial process is known to cause the production of two separate toxic gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6.00 am until 6.00 pm.
The emission of the toxic gas $X$ can be modelled by the rule

$$x(t) = 1.5 \sin\left(\frac{\pi t}{3}\right) + 1.5, \quad 0 \leq t \leq 12$$

and the emission of the toxic gas $Y$ can be modelled by the rule

$$y(t) = 2.0 - 2.0 \cos\left(\frac{\pi t}{3}\right), \quad 0 \leq t \leq 12.$$ 

The graphs of these two functions are shown.

a. At what time of the day are the emissions the same for the first time, and how many units of each gas are emitted at that time? Give your answer correct to 2 decimal places, and remember to note whether the time is am or pm.

b. The Environment Protection Authority (EPA) has strict rules about the emissions of toxic gases. The total emission of toxic gases for this particular industrial process is given by

$$T(t) = x(t) + y(t)$$

i. Sketch the graph of the function $T(t)$.

ii. Find the maximum and minimum emissions in a 12-hour working day and the times at which these occur.

c. If the EPA rules state that all toxic emissions from any one company must lie within the range 0 to 7 units at any one time, indicate whether this company works within the guidelines.

16. At a suburban shopping centre, one of the stores sells electronic goods such as digital cameras, laptop computers and printers. The store had a one-day sale towards the end of the financial year. The doors opened at 7.55 am and the cash registers opened at 8.00 am. The store closed its doors at 11.00 pm. The total number of people queuing at the six cash registers at any time during the day once the cash registers opened could be modelled by the equation

$$N(t) = 45 \sin\left(\frac{\pi t}{5}\right) - 35 \cos\left(\frac{\pi t}{3}\right) + 68, \quad 0 \leq t \leq 15$$

where $N(t)$ is the total number of people queuing $t$ hours after the cash registers opened at 8.00 am.

a. Many people ran into the store and quickly grabbed bargain items. How many people were queuing when the cash registers opened?

b. When was the quietest time of the day and how many people were in the queue at this time?

c. How many people were in the queue at midday?

d. What was the maximum number of people in the queue between 3.00 pm and 7.00 pm?
The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

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  - **Differentiation**
  - **Sit topic test**
5 Answers

EXERCISE 5.2

1 a \( \frac{dy}{dx} = -12 + 12x - 3x^2 \)  
   b \(-3\)

2 a

\( y = (x + 2)(2 - x) \)

b \((-0.83, 3.31)\) and \((4.83, -19.31)\)

c The gradient of \( y = (x + 2)(2 - x) \) when \( x = 1 \) is \(-2\).

The gradient of \( y = x^2(4 - x) \) when \( x = 1 \) is 5.

3 a \( 12x^2 - \frac{2}{3x^3} \)

b \( \frac{-1}{x^2} - \frac{1}{5} \)

4 a \( 3x^2 + 6x + 1 \)

b \( \frac{6}{x^3} + \frac{1}{x^4} \)

5 a \(-14\)

6 \( 11 + 4a \)

7 a

\( x \in (-\infty, 2) \setminus \{-2\} \)

b \( x \in (-\infty, 2) \setminus \{-2\} \)

8

9 a \( y = -4x + 9 \)

b \( y = \frac{1}{x} - \frac{15}{x^2} \)

10 \( y = 4 - x^2 \)

11 a \( \frac{dy}{dx} = \frac{15}{4x^6} + \frac{1}{2x^2} \)

b \( f(x) = \frac{30}{x^4} + \frac{2}{x^2} - \frac{4}{x^5} \)

c \( \frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{4x^2} \)

d \( f(x) = -\frac{27}{2x^2} - x + \frac{9}{x^2} \)
12 a i Domain = $\mathbb{R}$
   ii
   \[ y = f(x) \]
   \[ (\frac{1}{2}, 0) \quad (3, 0) \]

b i Domain = $\mathbb{R}$
   ii
   \[ y = f(x) \]
   \[ (0, 0) \]

c i Domain = $(-\infty, 4) \setminus \{-2\}$
   ii
   \[ y = f(x) \]
   \[ (-2, 1) \quad (4, 1) \]

13 a 4
   b 36
   c -9
14 a \( f(x) = -1 \)
   b \( f(x) = 6x - 2 \)

15 a 4
   b \( \frac{39}{16} \)
   c 12
   d \( \frac{7}{2} \)

16 a \( y = -6x - 22 \)
   b \( y = -3a^2x + 2a^3 + 8 \)
   c \( y = \sqrt{3} + \sqrt{3} - 5 \)
   d \( y = -\frac{7x}{2} + 2 \)

17 a \( y = \frac{1}{6}x + \frac{53}{6} \)
   b \( y = -\frac{1}{3a^2}x^3 + a^3 + \frac{1}{3a} \)
   c \( y = -\sqrt{3}x + 5\sqrt{3} - 5 \)
   d \( y = \frac{2}{7}x + \frac{67}{7} \)

18 a \( y = 3x - \frac{3}{4} \)
   b \( y = \frac{3}{2}x - \frac{7}{2} \)
19 \( y = -(x^2 - 7x + 10) \)
20 \( y = x^3 - x \)
21 \( m = 12 \)
22 a

23 a

b \( (1 \frac{1}{2}, -3 \frac{3}{2}) \)

c Tangent: \( y = \frac{12}{x} - 9 \)
   Line perpendicular to tangent: \( y = \frac{4}{15}x - \frac{119}{40} \)

EXERCISE 5.3

1 a \( -\frac{1}{3}e^{-\frac{1}{3}x} \)
   b \( 12x^3 + 4xe^{-2x^2} \)
   c \( \frac{8}{3}e^{-2x} + \frac{2}{3}e^{-4x} - 2e^{-3x} \)
   d \( 4e^{4x} - 12e^{2x} \)
2 \( \frac{1}{2} \)
3 \( y = 2x + 1 \)
4 \( y_1 = -3x + 5, \quad y_p = \frac{1}{3}x + 5 \)
5 a \( -20e^{-4x} \)
   b \( -\frac{1}{2}e^{-\frac{1}{2}} + x^2 \)
   c \( 12e^{3x} - \frac{3e^{6\sqrt{7}}}{2\sqrt{7}} + 9e^{-3x} + 2 \)
   d \( 3e^{3x} + 3e^{-3x} - 4e^{-2x} \)
   e \( 4e^{2x} + e^{-x} \)
   f \( e^x - 2e^{3x} - 3e^{-x} \)
6 a $-2$
b $\frac{8}{e}$
c $\frac{3e}{2}$
d $1$

7 $y = -2ex$

8 $y_T = -3x - 1, \ y_P = \frac{1}{3}x - 1$

9 $y_T = \frac{e^{\sqrt{3}}}{2\sqrt{3}}x^2 + e^{\sqrt{3}} + 1 - \frac{3e^{\sqrt{3}}}{2\sqrt{3}}$

$c_1 = \frac{2\sqrt{3}}{e^{\sqrt{3}}}x^2 + e^{\sqrt{3}} + 1 + \frac{6\sqrt{3}}{e^{\sqrt{3}}}$

10 a $-2e^2$
b $\frac{3}{2}$

11 a $2e^2 - \frac{2}{e}$
b $0$

12 $y = 5x - 4$

13 $f(x) = Ae^x - 3Be^{-3x}$
0 $= Ae^x - 3Be^{-3x}$
$e^{-3x} \neq 0$, no real solution
So, $0 = Ae^x - 3B$
$3B = Ae^x$
$e^{3x} = \frac{3B}{A}$

14 a $2$
b $-1.38$

15 $2 \log_e (3)$

16 a $(1.89, 1.05)$
b $7.66$

EXERCISE 5.4
1 a $20$
b $-0.0304e^{-0.00152t}$
c $0.0291$ g/year

2 a $0.5I_0$
b $0.0011I_0$

3 a $q = -4$
b $(m, 0) = (\log_e 3, 0)$

c $f(x) = 2e^{2x} - 4e^x$

d $f'(0) = -2$

4 a $z = -3$
b $(a, 0) = (-\log_e 2, 0)$

c $f(x) = -2e^{-2x} + 3e^{-x}$

d $1$

5 a $120$
b $\frac{dA}{dt} = -120ke^{-kt}$
c $\frac{dA}{dt} = -k \times 120e^{-kt}$

d $\frac{dA}{dt} = -k \times A$

e $\frac{dA}{dt} \propto A$

$c \: \frac{1}{2} \: \log_e \left( \frac{4}{3} \right)$

d $8.408$ units/min

e As $t \to \infty$, $A \to 0$. Technically, the graph approaches the line $A = 0$ (with asymptotic behaviour, so it never reaches $A = 0$ exactly). However, the value of $A$ would be so small that in effect, after a long period of time, there will be no gas left.

6 a $11$
b $\frac{dL}{dt} = 6.589e^{0.599t}$
c $39.742$ mm/month

7 a $T_0 = 75$
b $1.531^\circ$C/min

8 a $23.2$ million

b $2005$, September
c $0.29$ million/year

d $2019$

9 a $P_0 = 75.32$ cm of mercury, $k = 0.24$

b $5.37$ cm of mercury/km

10 a $10.94$ cm

b $20.6$

11 a $(0, 0.5)$

b $f(x) = -e^{-x} + e^{-2x}$

c $(0, 0.5)$

d $63.4^\circ$

e $y = -0.2325x + 0.5327$

f $y = 4.3011x - 4.0009$

12 a $30$
b $450000$

c $37072$ 2/day

d $450$

e $450000$

f $u = -\frac{1}{7} \log_e (10), 39711$ day

13 a $5$

c $-10^x$

d $3.89$

e $-3.68$

14 a $\frac{x^3 - 6x}{e^{3x}}$
b $\left( \pm \sqrt[3]{6}, \frac{1}{2e^{\theta}} \right)$
c $-0.593$

EXERCISE 5.5
1 a $5 - 3 \sin(x) + 5 \cos(x)$

b $3 \cos(3x + 2) + 6x \sin(3x^2)$

c $3 \cos(9x)$

d $10 \sec^2(2x) - 10x^4$

e $2 \sec^2\left( \frac{x}{4} \right)$

f $\frac{\pi}{20} \sec^2\left( \frac{\pi x}{20} \right)$

2 $-2 \cos(2x)$

3 $y = x - \frac{\pi}{2}$

4 $y = 4x + \frac{\pi}{2} - 1$

5 a $-6 \sin(3x)$

b $-\frac{\pi}{180} \sin\left( \frac{\pi x}{180} \right)$

c $3 \cos(x)$

d $-\frac{1}{3} \cos\left( \frac{x}{3} \right)$

e $\frac{\pi}{15} \cos\left( \frac{\pi x}{15} \right)$

f $-6 \sin(3x)$

g $-5x \sec^2(5x^2)$

h $20 \sec^2(20x)$
6 \left( \frac{4\pi}{3} - \sqrt{3} \right)

7 y = \frac{3}{2x} + \frac{\pi}{4} + \frac{3\sqrt{3}}{2}

8 y = 4x + 2 - \pi

9 116.6°

10 a \ y_T = 3x - 2\pi, \ y_p = -\frac{1}{3} + \frac{2\pi}{9}

b \ y_T = -\frac{1}{2} + \frac{\pi}{2}, \ y_p = 2x - 2\pi

11 a -1

b \ \frac{\pi}{4}, \frac{5\pi}{4}

c \ \cos(x) + \sin(x)

d \ \frac{3\pi}{4}, \frac{7\pi}{4}

12 a \ \sqrt{3}

b \ \frac{\pi}{3}, \frac{\pi}{3}

c \ -\sqrt{3} \sin(x) + \cos(x)

d \ \frac{\pi}{6}, \frac{\pi}{6}

13 a \ \tan(x)

b \ \sec^2(x)

14 x = -\frac{5\pi}{8} - \frac{\pi}{8} \cdot \frac{3\pi}{8} - \frac{7\pi}{8}

15 \ (-0.524, 0.342), (0.524, -0.342)

16 \ (0.243, 1.232), (0.804, 0.863)

EXERCISE 5.6

1 a \ A = \frac{1}{2}ab \sin(c)

\quad = \frac{1}{2} \times 6 \times 7 \cos(\theta)

\quad = 21 \sin(\theta)

b \ 21 \cos(\theta)

c \ 10.5 \text{ cm}^2/\text{radian}

d \ \theta = 1.1°

2 a \ \text{BD} = a \sin(\theta), \ \text{CD} = a \cos(\theta)

b \ L = a + 2a \sin(\theta) + a \cos(\theta) + 4

c \ 2a \cos(\theta) - a \sin(\theta)

d \ \theta = 1.1071

3 a \ 10

b \ 2\pi \cos(\pi t)

c \ -2\pi \text{ cm/s}

4 a \ 12 \text{ hours}

b \ \text{Low tide} \to \text{1 metre at 3.00 pm}

\quad = \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right)

\quad = \sqrt{\frac{2\pi}{24}}

\quad = 4.30 \text{ pm}

5 a \ 0.849, 1.508

b \ \left(\frac{\pi}{8}, \frac{3\pi}{8}\right), \left(\frac{3\pi}{8}, -1\right)

c \ -8

d \ \left[0, \frac{\pi}{8}\right] \cup \left(\frac{3\pi}{8}, \frac{\pi}{2}\right)

6 a \ L = 3 \times 100 \cos(\theta) + 4 \times 100 \sin(\theta) + 2 \times 100

\quad = 300 \cos(\theta) + 400 \sin(\theta) + 200 \text{ as required}

b \ -300 \sin(\theta) + 400 \cos(\theta)

c \ 700 \text{ cm, } \theta = 0.93°

7 a \ \angle XOM = 2\theta \text{ because the angle at the centre of the circle is twice the angle at the circumference.}

\angle XOM = 2YOM = \frac{1}{2} \times 2\theta

\angle XOM = \theta \text{ as required}

b \ XM = r \sin(\theta)

\quad = \frac{r \sin(\theta)}{r}

\quad = \frac{\sin(\theta)}{\cos(\theta)}

\quad = \frac{1}{\cos(\theta)}

\quad = \frac{\cos(\theta)}{\cos(\theta)}

\quad = \cos(\theta) + 1

\quad = \frac{h}{r}

\quad = \cos(\theta)

\quad = \frac{h}{r}

\quad = \cos(\theta)

c \ -3 \sin(\theta)

d \ -\frac{3\sqrt{3}}{2}

8 a \ \text{Distance} \div \text{time} = \text{velocity}

\text{Distance} = \text{velocity} \times \text{time}

\text{Distance} \div \text{velocity} = \text{time}

\text{So } d(PM) = 400 \cos(\theta)
\[T_{\text{obstacles}} = \frac{400 \cos(\theta)}{2}\]
\[T_{\text{obstacles}} = 200 \cos(\theta)\]
\[d(\text{PM}) = 200 \times 2 \theta\]
\[d(\text{PM}) = 400 \theta\]
\[T_{\text{hurdles}} = \frac{400 \theta}{5} = 80 \theta\]
\[T_{\text{total}} = T_{\text{obstacles}} + T_{\text{hurdles}}\]
\[T_{\text{total}} = 200 \cos(\theta) + 80 \theta\]
\[T_{\text{total}} = 40(5 \cos(\theta) + 2 \theta)\]

b 0.4115

c 9 a \(h = 50 \cos(2\pi t) + 50\)
b \(-100\pi \sin(2\pi t)\)
c \(-100\pi\) mm/s

10 a 1.5 metres

b 8.5 metres

c 60 seconds

d 18.4 seconds

e \(\frac{7\pi}{60} \sin\left(\frac{\pi t}{30}\right)\)
f 35.51 seconds, 54.49 seconds

11 a 6 metres

b \(-\frac{7\pi}{40} \sin\left(\frac{\pi t}{20}\right)\)

c i \(-\frac{7\sqrt{2}\pi}{80} = -0.3888\)

ii \(-\frac{7\pi}{40} = -0.5498\)

d i 15 metres

ii 159.27°

12 a (0, 100), (4.5, 57.5)

b (0.94, 75.41)

c \(-39.5\)

13 a \(P = -2 \cos(mt) + n\)

When \(t = 0\), \(P = 4:\)
\[4 = -2 \cos(0) + n\]
\[4 + 2 = n\]
\[6 = n\]
\[\frac{3}{2} = \frac{2\pi}{m}\]
\[3m = 4\pi\]
\[m = \frac{4\pi}{3}\]

b \(\frac{8\pi}{3} \sin\left(\frac{4\pi t}{3}\right)\)

c \(\frac{8\pi}{3}\) m/min

14 a 1.7 metres

b \(0.625 \sin\left(\frac{1}{4}\right)\)

c 0.426

d 4.756 metres

15 a 2.86 units at 7.55 am \((t = 1.92)\)

b i \(T(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 3.5 - 2.0\cos\left(\frac{\pi t}{3}\right)\)

ii Minimum 1 unit at 11.23 am \((t = 5.39)\) and 5.23 pm \((t = 11.39)\)

Maximum 6 units at 8.23 am \((t = 2.39)\) and 2.23 pm \((t = 8.39)\)

c Emissions of 1 unit and 6 units lie within the guidelines.

16 a 33

b 1 person at 2.28 pm \((t = 6, 46)\)

c 112

d 86