Undirected graphs and networks

9.1 Kick off with CAS
9.2 Basic concepts of a network
9.3 Planar graphs and Euler’s formula
9.4 Walks, trails, paths, cycles and circuits
9.5 Trees and their applications
9.6 Review
9.1 Kick off with CAS

Planar graphs and Euler’s formula

In graph theory, graphs are made up of vertices, with edges connecting the vertices. Planar graphs are graphs in which there are no intersecting edges. The edges and vertices in a planar graph divide the graph into a number of faces, as shown in the following diagram. When counting the number of faces remember to include the infinite face — the region surrounding the graph.

Euler’s formula links the number of vertices \(v\), edges \(e\) and faces \(f\) in a planar graph with the rule \(v - e + f = 2\).

1. Using CAS, define and save Euler’s formula for planar graphs.

2. Use CAS to solve Euler’s formula for
   a. the number of faces \(f\)
   b. the number of edges \(e\)
   c. the number of vertices \(v\).

3. Use your answer from 2a to calculate the number of faces in a graph with:
   a. 6 vertices and 8 edges
   b. 5 vertices and 5 edges.

4. Use your answer from 2b to calculate the number of edges in a graph with:
   a. 5 vertices and 4 faces
   b. 7 vertices and 5 faces.

5. Use your answer from 2c to calculate the number of vertices in a graph with:
   a. 6 edges and 4 faces
   b. 9 edges and 5 faces.
Basic concepts of a network

Definition of a network

What do the telephone system, the Australian Army, your family tree and the internet have in common? The answer is that they can all be considered networks. The simplest possible definition of a network, which will suit our purposes throughout this topic, is:

A network is a collection of objects connected to each other in some specific way.

In the case of the telephone system, the objects are telephones (and exchanges, satellites, ...). In the case of the Australian Army, the objects are units (platoons, companies, regiments, divisions, ...). In the case of the internet, the objects are computers; while your family tree is made up of parents, grandparents, cousins, aunts, ...

The mathematical term for these objects is a vertex. Consider the network represented in the diagram. This is perhaps the simplest possible network. It consists of two vertices (circles labelled 1 and 2) and one connection between them. This connection is called an edge.

In the case of the telephone system, the edges are the cables connecting homes and exchanges; in the Australian Army they are the commanding officers of various ranks; while in the family tree the links between the generations and between husband and wife can be considered as edges.

The first distinguishing features of a network are the total number of vertices and total number of edges.

WORKED EXAMPLE 1

Count the number of vertices and edges in the network shown.

THINK
1 Count the vertices by labelling them with numbers.
2 Count the edges by labelling them with letters.

WRITE/DRAW

Thus, there are 5 vertices.
Thus, there are 6 edges.
There are two things worth noting about this classification of a network:
1. the vertices and edges can be labelled in any order, using any suitable
   labelling system
2. vertices may have different numbers of edges connected to them. How many edges
   are connected to vertex 2 in Worked example 1?

**The degree of a vertex**

Each vertex may have a number of edges connecting it with the
rest of the network. This number is
called the **degree**. To determine the
degree of a vertex, simply count its
edges. The following table shows the
degree of each vertex in Worked
example 1.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

A vertex with degree 0 is *not connected* to any other vertex, and is called an
isolated vertex.

An edge which connects a vertex to itself is called a **loop** and contributes 2 towards
the degree.

If two (or more) edges connect the *same pair* of vertices they are called **parallel edges**
(or **multiple edges**) and all count towards the degree. Otherwise, if there is only
one connection between two vertices, the connection is called a **simple**, or **single**,
connection.

**WORKED EXAMPLE 2**

Determine the degree of each vertex in the figure shown.

- **THINK**
  1. Node 1: Has 2 simple edges.
  2. Node 2: Has 3 simple edges and 1 loop.
  5. Node 5: Has 1 simple edge and 3 parallel edges.

- **WRITE**
  
  Degree of node 1 = 2
  Degree of node 2 = 3 + 2
  = 5
  Degree of node 3 = 0
  Degree of node 4 = 2 + 3
  = 5
  Degree of node 5 = 1 + 3
  = 4
Representations of networks

So far we have seen the graphical representation of a network as a two-dimensional collection of vertices and edges. Hence, networks are sometimes called graphs.

There are other ways to represent the network without losing any of its essential features:

1. labelling vertices and labelling edges according to their vertices
2. matrix representation.

To label vertices, simply list them. If there are three vertices labelled A, B, and C write \( V = \{ A, B, C \} \). To label edges according to their vertices, identify the vertices that the edge connects. If an edge connects vertex 1 with vertex 3, we represent the edge as \( (1, 3) \). If there is a loop at vertex 4, its edge is \( (4, 4) \). If there are 2 parallel edges between vertices 2 and 4, we write \( (2, 4), (2, 4) \).

WORKED EXAMPLE 3
Label the vertices and edges for the figure shown, as in Worked example 2.

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Label the vertices.</td>
<td>( V = { 1, 2, 3, 4, 5 } )</td>
</tr>
<tr>
<td>2 Examine each edge, in turn.</td>
<td>( E = { (1, 4), (1, 2), (2, 2), (2, 4), (4, 5), (4, 5), (4, 5) } )</td>
</tr>
<tr>
<td>Vertex 1–vertex 4</td>
<td>(1, 4)</td>
</tr>
<tr>
<td>Vertex 1–vertex 2</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>Vertex 2–vertex 2 (loop)</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>Vertex 2–vertex 4</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>Vertex 2–vertex 5</td>
<td>(2, 5)</td>
</tr>
<tr>
<td>Vertex 4–vertex 5 (3 parallel edges)</td>
<td>(4, 5), (4, 5), (4, 5)</td>
</tr>
<tr>
<td>3 Combine vertices and edges into a list</td>
<td>( V = { 1, 2, 3, 4, 5 } )</td>
</tr>
<tr>
<td></td>
<td>( E = { (1, 4), (1, 2), (2, 2), (2, 4), (4, 5), (4, 5), (4, 5) } )</td>
</tr>
</tbody>
</table>

There are several points to note about this representation:

1. there is no ‘3’ in the list of edges \( E \). This implies it is an isolated vertex.
2. the number of pairs in \( E \) \( \{(1, 4), (1, 2), \ldots\} = 8 \) which must be the same as the number of edges.
3. the number of times a vertex appears anywhere inside \( E \) equals the degree of the vertex. For example, the digit 4 appears 5 times, so the degree of vertex 4 = 5.
4. from this representation of \( V \) and \( E \) we can construct (or reconstruct) the original graph.

WORKED EXAMPLE 4
Construct a graph (network) from the following list of vertices and edges.

\[ V = \{ A, B, C, D, E \} \]
\[ E = \{ (A, B), (A, C), (A, D), (B, C), (B, D), (C, E), (D, E), (E, E) \} \]
1. Start with a single vertex, say vertex A, and list the vertices to which it is connected.

   Vertex A is connected to B, C and D.

2. Construct a graph showing these connections.

3. Take the next vertex, say B, and list the vertices to which it is connected.

   Vertex B is connected to A (already done), C and D (twice: parallel edge).

4. Add the edges from step 3.

   AB
   CD

5. Repeat steps 3 and 4 for vertex C.

   Vertex C is connected to A (already done), B (already done) and E.

6. Repeat steps 3 and 4 for vertex D and, finally, add the loop (E, E).

   Vertex D is connected to A (already done), B (already done) and E.
   Vertex E is connected to C (already done), D (already done) and E (loop).

As a check, count the edges in the list E (9) and compare it with the number of edges in your final graph.

There may be other geometric configurations which can be drawn from the same vertex and edge lists, but they would be isomorphic (or equivalent) to this one.

**Matrix representation of networks**

A method of representing a network in concise form is through the use of a matrix. Recall that a matrix is a rectangular collection, or ‘grid’ of numbers. To represent the network, write the names of the vertices above the columns of the matrix and to the left side of the rows of the matrix. The number of edges connecting vertices is placed at the intersection of the corresponding row and column. This is best shown with an example.

Represent the network shown (from Worked example 4) as a matrix.
THINK

1 Set up a blank matrix, putting the vertex names across the top and down the side. Thus there are 25 possible entries inside the matrix.

2 Consider vertex A. It is connected to vertices B, C, and D once each, so put a 1 in the corresponding columns of row 1 and in the corresponding row of column 1.

3 Consider vertex B. It is connected to C once and D twice. Put 1 and 2 in the corresponding columns of row 2 and in the corresponding rows of column 2.

4 Repeat for vertices C, D and E. Vertex C is connected to vertex E once, so put a 1 in the corresponding column of row 3 and in the corresponding row of column 3 (shown in red).
   Vertex D is connected to vertex E once, so put a 1 in the corresponding column of row 4 and in the corresponding row of column 4 (shown in black).
   Vertex E is connected to itself once (loop), so put a 1 in the corresponding column 5, row 5. Note: Only one entry is needed for loops (shown in green). A value of 1 in the leading diagonal denotes a loop in the network, connecting a vertex to itself. This is important to understand when calculating the degree.

5 Complete the matrix by placing a 0 in all unoccupied places.

WRITE

\[
\begin{array}{ccccc}
A & B & C & D & E \\
A & 1 & 1 & 1 & 0 \\
B & 1 & 0 & 1 & 2 & 0 \\
C & 1 & 1 & 0 & 0 & 1 \\
D & 1 & 2 & 0 & 0 & 1 \\
E & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]
6 Check your result by comparing the entries in the matrix with the original network representation. This is best done on a vertex-by-vertex basis.

In matrix representation:
1. the sum of a row (or a column) gives a degree of that vertex, except where a loop is present. Where a loop is present (denoted by a 1 in the leading diagonal), add 1 to the sum of the row or column.
2. if an entire row and its corresponding entire column has only 0s then that vertex is isolated
3. the matrix is diagonally symmetric.

**EXERCISE 9.2**

**Basic concepts of a network**

For questions 1 and 2, count the number of vertices and edges in the following networks.

1. 

2. 

For questions 3 and 4, determine the degree of the labelled vertices in each diagram.

3. 

4. 

For questions 5 and 6, list the vertices and label all the edges, according to their vertices, in each of these diagrams.

5. 

6. 

7. Construct a network from the following list of vertices and edges.

\[ V = \{1, 2, 3, 4, 5, 6\} \quad E = \{(1, 2), (1, 4), (1, 6), (2, 3), (2, 6), (3, 4), (4, 6)\} \]
8 Construct a network from the following list of vertices and edges.

\[ V = \{A, B, C, D, E, F, G\} \]

\[ E = \{(A, B), (A, C), (A, F), (B, F), (D, E), (D, F), (E, F), (F, G)\} \]

9 Represent the network from question 7 as a matrix.

10 Copy and complete the matrix representation of the network at right. The first few entries are shown.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 1 & 0 \\
2 & 1 & 1 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

11 Count the number of vertices and edges in the following networks.

12 The number of vertices and edges in the figure is:

A Vertices = 7, edges = 7
B Vertices = 7, edges = 10
C Vertices = 7, edges = 11
D Vertices = 11, edges = 11
E Vertices = 11, edges = 7

13 Determine the degree of the labelled vertices in each diagram.
14 The degree of vertex A in the figure is:
A 3
B 4
C 5
D 6
E 7

15 List the vertices and label all the edges, according to their vertices, in each of these diagrams.

a

b

16 Construct a network from the following list of vertices and edges.

a  $V = \{1, 2, 3, 4\}$  $E = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

b  $V = \{A, B, C, D, E\}$  $E = \{(A, B), (A, C), (A, C), (B, B), (B, C), (B, D), (C, D)\}$

17 Consider a network of 4 vertices, where each vertex is connected to each of the other 3 vertices with a single edge (no loops, isolated vertices or parallel edges).

a List the vertices and edges.
b Construct a diagram of the network.
c List the degree of each vertex.

18 Repeat question 17 for:

i a network of 5 vertices
ii a network of 8 vertices.

19 Using the results from questions 17 and 18, predict the number of edges for a similar network of:

a 10 vertices   b 20 vertices   c 100 vertices.

Note that the increase in the number of edges is one of the problems that had to be overcome in the design of computer networks.

20 Construct a network representing the following family tree. Use a single node to represent each married couple.

Allan and Betty had 3 children: Charles, Doris and Earl.
Charles married Frances and had 2 children, George and Harriet.
Doris married Ian and had 1 child, John.
Earl married Karen and had 3 children, Louise, Mary and Neil.
21 Represent the following networks by matrices.

![Networks A, B, C](image)

22 Construct networks from the following matrix representations.

Note that the number of rows = number of columns = number of vertices. Watch out for loops.

![Matrices A, B, C](image)

### Planar graphs and Euler’s formula

A planar graph is a special kind of network or graph. The additional properties of planar graphs will allow us to map two-dimensional and even three-dimensional objects into graphs.

**Degenerate graph**

A graph with no edges is called a **degenerate graph** (or **null graph**).

**Complete graph**

A graph where all vertices are connected directly to all other vertices without parallel edges or loops is called a **complete graph**.

The figure on the left is degenerate; the one on the right is complete. How many edges would there be in a complete graph of 6 vertices?

![Degenerate and Complete Graphs](image)

For a complete graph, if \( E \) = number of edges and \( V \) = number of vertices then

\[
E = \frac{V(V - 1)}{2}.
\]

**Planar graphs**

A planar graph can be defined as follows:

If a graph has no edges which cross, then it is a planar graph.
Consider the following graphs.

Figure a is a planar graph because none of the paths \{A, B, C, D, E, F\} cross each other.

Figure b is apparently not a planar graph because the path (A, D) crosses the path (B, C).

Is figure c a planar graph?

**The regions of a planar graph**

Consider a simplified version of the graph in figure a, as shown at right. Note that the large circular vertices have been replaced by small black circles. Otherwise this is the identical network to figure a.

Now, observe how this planar graph can be divided into 3 regions: region I, region II and region III.

Note also that one of the regions (III) will always be infinite, because it continues beyond the bounds of the diagram.

All the other regions have a finite area. These regions are also called faces, for a reason which will soon become apparent.

The reason one region becomes infinite can be seen by considering the fact that when you look at three-dimensional objects, you can’t see all the faces at the same time, no matter from which angle you look.

**Converting non-planar graphs**

Although it may appear that a graph is not planar, by modifying the graph it may become clearly planar.

There is no specific method, but by trial and error it may be possible to remove all the crossing paths. (It may also help to imagine the nodes as nails in a board and the edges as flexible rubber bands.) Alternatively, it may be possible to move the vertices so that the connecting edges don’t cross. If there are no crossings left, the graph is planar.

**WORKED EXAMPLE 6**

Convert the graph below to a planar graph. Indicate the faces (regions) of the planar graph.

```
THINK
1 Confirm that the graph is non-planar.
   \[ E = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)\} \]

WRITE/DRAW
Edge (1, 3) crosses (2, 4).
Edge (3, 5) crosses (1, 4) and (2, 4).
```
Two crossings could be eliminated if vertex 2 were exchanged with vertex 3. Redraw the modified graph. Check that all the edges are connected to the same vertices.

Placing node 5 inside the rectangle is one way of eliminating all crossings. Note that this planar graph is only one of several possible answers.

Define the faces (regions).

The degree of each face is the number of edges defining that region. Consider the last figure in Worked example 6.

Face I is defined by edges (1, 3), (1, 4), (4, 5) and (5, 3), so its degree = 4.

Face II is defined by edges (3, 5), (5, 4), (4, 2) and (2, 3), so its degree = 4.

Face III is defined by edges (1, 3), (1, 4), (4, 2) and (2, 3), so its degree = 4.

In almost all cases, each region will have a degree of at least 3. Why? Can you think of exceptions?

WORKED EXAMPLE 7

Find the degree of each face of the graph shown in the figure.

THINK

Define the edges and faces of the graph.

WRITE/DRAW

Count the edges for each face.

For example:

Face I — edges (1, 2), (2, 7), (7, 1)
Face II — edges (1, 6), (6, 5), (5, 7), (7, 1).

E = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1), (1, 7), (5, 7), (4, 7), (2, 7)}
Converting three-dimensional solids to planar graphs

Another application of planar graphs is the conversion of the graph representing a three-dimensional solid (with flat faces) to a planar graph.

WORKED EXAMPLE 8

The figure at right shows a cube with vertices, \( V = \{A, B, C, D, E, F, G, H\} \).

Convert this to a planar graph.

THINK

1. List the edges (12 in all).

\[ E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D), (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\} \]

2. Imagine the three-dimensional cube ‘collapsing’ to a two-dimensional graph. Try collapsing the face A–B–C–D into the face E–F–G–H.

3. Check the edges to see that they are the same as in step 1. Note also the edges (A, E), (B, F), (C, G) and (D, H) which link the ‘collapsed’ faces.

There are some other interesting features of this planar graph:

1. the planar graph is, in a sense, a two-dimensional ‘projection’ of the original cube
2. the original ‘base’ of the cube (A–B–F–E) has become the infinite region of the planar graph.

Euler’s formula

By now it may be clear that there is a mathematical relationship between the vertices, edges and faces of planar graphs.

In fact, it is the same relationship, known as Euler’s formula, that you may have learned when studying solid geometry:

\[ V = E - F + 2 \]

WORKED EXAMPLE 9

Verify Euler’s formula for the ‘cube’ of the last figure in Worked example 8.

THINK

1. List the vertices.

\[ V = \{A, B, C, D, E, F, G, H\} \]
Note that the cube is a form of prism (an object with a uniform cross-section), and all prisms can be converted to planar graphs using the above technique of one face ‘collapsing’ into another.

### Planar graphs and Euler’s formula

1. **Convert the following graph to a planar graph.**

2. **Redraw the following network diagram so that it is a planar graph.**

3. **Find the degree of each face of the graph in question 2.**

4. **Find the degree of each face of the graph shown.**

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#### EXERCISE 9.3

**PRACTISE**

1. **Convert the following graph to a planar graph.**

2. **Redraw the following network diagram so that it is a planar graph.**

3. **Find the degree of each face of the graph in question 2.**

4. **Find the degree of each face of the graph shown.**

---

### PROOFS

**Note:**

- **2 Count the vertices.**
  
  \[ V = 8 \]

- **3 List the edges.**
  
  \[ E = \{(A, B), (A, D), (A, E), (B, C), (B, F), (C, D),
  
  (C, G), (D, H), (E, F), (E, H), (F, G), (G, H)\} \]

- **4 Count the edges.**

- **5 Define the faces (regions).**

  ![Diagram](image)

  There are 6 faces in all:
  
  \{I, II, III, IV, V, VI\}

  \[ F = 6 \]

- **6 Confirm Euler’s formula by substitution.**

  \[ V = E - F + 2 \]
  
  \[ 8 = 12 - 6 + 2 \]
  
  \[ 8 = 8 \]

  Therefore, Euler’s formula is verified.
5 Convert the three-dimensional triangular prism to a planar graph.

6 Consider a ‘pyramid’ with an octagon for a base.

Convert the representation to a planar graph.

7 Verify Euler’s formula for the graph in question 5.

8 Convert a triangular pyramid to a planar graph. Verify Euler’s formula for your graph.

9 Modify the following graphs so that their representations are planar.

a

b

c

d

10 The graph represented in the figure is apparently not planar because:
   A edge (A, C) crosses edge (B, E)
   B edge (A, D) crosses edge (E, F)
   C edges (A, E), (F, E), (C, E) and (B, E) intersect
   D vertex E has a degree of 4
   E none of the above

11 A complete graph with 7 vertices would have:
   A 7 edges
   B 14 edges
   C 21 edges
   D 28 edges
   E 42 edges

12 a By moving vertex F only, modify the graph in question 10 so that it is clearly planar.
   b How many faces are there in your planar graph?
   c Find the degree of each face.
13 a By moving vertex 5 only, modify the graph so that it is clearly planar.

b How many faces are there in your planar graph?

(Hint: You may have to use curved edges to connect all the vertices.)

14 The degree of vertex E in the figure is:

A 1  B 2  C 3  D 4  E 5

15 Verify Euler’s formula for the figure in question 14.

16 Show that the sum of the degrees of all the vertices of any planar graph is always an even number. Also show that if \( S = \) sum of the degrees, and \( E = \) number of edges, that \( S = 2E \). (This is known as the handshaking lemma.)

17 In a planar graph, the number of edges = 5, the number of vertices = 4, therefore the number of faces is:

A 1  B 3  C 9  D 11  E unable to be determined from the given information

18 Convert the rectangular pyramid shown into a planar graph.

19 The diagram shown is a crude floor plan for a small house with 6 rooms, labelled A, B, . . ., F. Convert this plan to a planar graph where rooms are considered as vertices.

(Hint: What should the edges be?)

20 Eight people in a room shake hands with each other once.

a How many handshakes are there?

b Represent the handshakes as a complete graph.

c Represent the handshakes as a matrix.

### 9.4 Walks, trails, paths, cycles and circuits

In planar graphs we can define a walk as a sequence of edges and look at various sequences or pathways through the network. Sometimes you may wish to have a walk that goes through all nodes only once, for example, for a travelling salesperson who wishes to visit each town once. Sometimes you may wish to use all edges only once, such as for a road repair gang repairing all the roads in a shire.
Walks
There are different ways of naming a walk. For example, consider travelling from node 1 to node 3 in the figure. A walk could be specified via node 2, namely A–B, or by specifying the vertices, 1–2–3. Alternatively one could take the walk C–E–D, or C–F. Each of these routes is a walk.

Connected graphs
If there is a walk between all possible pairs of vertices, then it is a connected graph.

For example, in the figure on the left, there is no walk between vertices 1 and 2, nor between vertices 3 and 4, so it is not a connected graph. However, if we add a single edge, as in the figure on the right, between vertices 1 and 2, the entire graph becomes connected.

Euler trails
A trail is a walk in which no edges are repeated.

Consider a trail where every edge is used only once, as in our road repair gang example.

An Euler trail is one which uses every edge exactly once.

1. For an Euler trail to exist, all vertices must be of an even degree or there must be exactly two vertices of odd degree.
2. If the degrees of all the vertices are even numbers, start with any vertex. In this case the starting vertex and ending vertex are the same.
3. If there are two vertices whose degree is an odd number use either as a starting point. The other vertex of odd degree must be the ending point.

WORKED EXAMPLE 10 Using the following figure, identify an Euler trail.

THINK
1. Determine a starting vertex. Since there are vertices (3 and 5) whose degree is an odd number, use one of these to start.

WRITE
Use vertex 3 as the start.
Euler circuits

A circuit is a trail beginning and ending at the same vertex.

With our road repair gang example, it would be desirable that the Euler trail started and finished at the same point. This kind of Euler trail is called an Euler circuit.

An Euler circuit is an Euler trail where the starting and ending vertices are the same.

It is important to note that an Euler circuit cannot exist for planar graphs that have any vertices whose degree is odd. In such graphs, there is no Euler circuit. Therefore, the planar graph of Worked example 10 does not contain an Euler circuit because vertices 3 and 5 were of odd degree.

Find an Euler circuit for the planar graph shown.

THINK

1 Confirm that all vertices have even degree.
   Vertex 1 — degree = 2
   Vertex 2 — degree = 2
   Vertex 3 — degree = 2
   Vertex 4 — degree = 2
   Vertex 5 — degree = 2

2 Pick any vertex to start and determine a trail that uses each edge and ends at the same vertex.
   Start with vertex 1. Then the Euler circuit could be:
   A→B→C→D→E→C
   D→E→B→A→C are all Euler trails.

An Euler circuit algorithm

For some networks, it may be difficult to determine an Euler circuit, even after determining that all vertices have even degree. Here is an algorithm that ‘guarantees’ an Euler circuit.

Consider a network where all vertices are of even degree. Let \( V = \{1, 2, 3, \ldots\} \) be the list of vertices.

**Step 1.** Choose a starting vertex from the list \( V \). Call this vertex A.

**Step 2.** From vertex A, find the smallest possible path which returns to vertex A. This is a ‘subcircuit’ of the original network. Let \( S \) be the list of vertices in this subcircuit.
Step 3. For each vertex in $S$, choose a single vertex in turn as the starting vertex of a different subcircuit. It should also be as small as possible, and not use any previously used edge.

Step 4. For each of these new subcircuits (if there are any), add any new vertices to the list in $S$.

Step 5. Repeat steps 3 and 4 until there are no more new vertices, edges or subcircuits left; that is, the lists $S$ and $V$ are the same.

Step 6. Join the subcircuits at their intersection points.

WORKED EXAMPLE 12

Find one possible Euler circuit for the network shown using the Euler circuit algorithm.

THINK

1. Choose a starting vertex, and find its smallest subcircuit.
   The subcircuit is marked in pink.

2. Create the list $S$ from the first subcircuit. Find new subcircuits, not using any edges already used (apply step 3 of the algorithm).
   The new subcircuit is marked in green. Note that this subcircuit does not use any previously used edges.

   $S = \{1, 2, 3\}$
   From vertex 2, there is a subcircuit $2 \to 4 \to 5 \to 6 \to 2$.

3. Add to the list $S$ (apply step 4 of the algorithm).

   $S = \{1, 2, 3, 4, 5, 6\}$
   From vertex 3, there is a subcircuit $3 \to 9 \to 8 \to 7 \to 6 \to 10 \to 3$.

4. Find the new subcircuits (re-apply step 3).
   The new subcircuit is marked in orange.

5. Add to list $S$ (re-apply step 4).
   Check that all vertices are in the list (step 5).
   $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   $= V$, so stop.
6 Apply step 6.
Form the Euler trail, starting with the first subcircuit, and proceeding through all the other subcircuits at their intersections. Note that the second subcircuit is in the 1st set of square brackets [ ] and the next subcircuit is in the 2nd set of square brackets [ ].

7 List the Euler circuit.

---

6 Apply step 6.
Form the Euler trail, starting with the first subcircuit, and proceeding through all the other subcircuits at their intersections. Note that the second subcircuit is in the 1st set of square brackets [ ] and the next subcircuit is in the 2nd set of square brackets [ ].

7 List the Euler circuit.

---

### Paths and cycles

A **path** is a walk in which no vertices are repeated (except possibly the start and finish).

A **cycle** is a path beginning and ending at the same vertex.

In Euler trails and circuits each edge was used *exactly once*, while vertices could be re-used. Now, consider the case where it is desirable to use each vertex *exactly once*, as in our travelling salesman problem mentioned at the start of this section.

A **Hamiltonian path** uses every vertex exactly once.

It is important to note that not all edges need to be used. Furthermore, there can be only up to 2 vertices with degree 1 (dead ends). In this case these would be the start and/or the finishing vertices.

---

### WORKED EXAMPLE 13

Determine a Hamiltonian path in the planar graph shown.

**THINK**

1. Choose a starting node. If there is a vertex with degree = 1, then use it to start.

2. Attempt to visit each vertex. This will work for all feasible planar graphs.

The Hamiltonian path found is shown in red.

**WRITE**

Since there are no vertices with degree 1, choose any node to start. Choose vertex 1.

The path connecting nodes 1–2–3–4–5 was chosen as one of the Hamiltonian paths.

Note that there are several possible Hamiltonian paths for the planar graph in Worked example 13 and there are several paths which will not result in a Hamiltonian path. Can you find such a path?

### Hamiltonian cycles

When determining a Hamiltonian path, sometimes it is desirable to start and finish with the same vertex. For example, our travelling salesperson may live in one of the towns (vertices) she visits and would like to start and finish at her home...
town after visiting all the other towns once. This is similar to the concept of an Euler circuit.

A **Hamiltonian cycle** is a Hamiltonian path which starts and finishes at the same vertex.

**WORKED EXAMPLE 14** Determine a Hamiltonian cycle in the planar graph shown. (This is the same graph used in Worked example 13.)

**THINK**
1. Choose a starting (and finishing) vertex.
2. Attempt to visit each vertex and return to vertex 1. The Hamiltonian cycle found is shown in pink.

**WRITE**
Choose vertex 1.

The path connecting nodes 1–2–3–5–4–1 was chosen as one of the Hamiltonian cycles.

**EXERCISE 9.4**

**Walks, trails, paths, cycles and circuits**

1. **WE10** Using the figure shown at right and starting at vertex 1, identify an Euler trail.
2. Choosing the other vertex of degree 3 in the figure used for question 1, identify another Euler trail.
3. **WE11** Using the figure shown and starting at vertex 1, identify an Euler circuit.
4. What additional edge should be added to the planar graph at right so that it could be possible to define an Euler circuit?
   - A  CA
   - B  FA
   - C  FD
   - D  EB
   - E  None of the above
5. **WE12** Find an Euler circuit for the graph shown, using the Euler circuit algorithm.
6 In which of the following does an Euler circuit exist?

A

B

C

D

E

7 Starting at vertex 2, determine a Hamiltonian path for the graph shown.

8 The path shown in pink in the figure is:
   A an Euler trail
   B an Euler circuit
   C a Hamiltonian path
   D a Hamiltonian cycle
   E none of the above

9 Starting at vertex 2, determine a Hamiltonian cycle for the graph shown.

10 The network shown has:
    A an Euler circuit and a Hamiltonian path
    B a Hamiltonian path and cycle
    C an Euler trail and a Hamiltonian cycle
    D an Euler trail only
    E none of the above

11 Using the figure shown at right and starting at vertex 1, identify an Euler trail.

12 Starting with a vertex of degree 4 from the figure in question 11, identify another Euler trail.
13 Considering the networks as shown, which have Euler circuits?
   A Both  B Neither  C Figure a only
   D Figure b only  E None of the above

a

b

14 Using the graph shown at right, determine:
   a an Euler trail
   b an Euler circuit starting at vertex 1 (circled in pink).

15 Starting at vertex 7, determine a Hamiltonian path for the graph shown.

16 Which of the following paths is a Hamiltonian cycle for the figure at right?
   D 2–5–6–4–3–2  E 2–1–6–4–5–2

17 Using the network shown at right:
   a determine an Euler trail
   b determine an Euler circuit
   c determine a Hamiltonian path
   d determine a Hamiltonian cycle.

18 a Using the network shown at right, what two edges should be added to the network so that it has both an Euler circuit and a Hamiltonian cycle?
   b Determine the Euler circuit and Hamiltonian cycle.

19 The Police Commissioner wishes to give the impression of an increased police presence on the roads. The roads that the commissioner has to cover are depicted in the network diagram. A speed camera is set up once during the day on each of the roads.
a Determine a walk that the police officer could follow so that she does not travel more than once on any road.
b What type of walk is this?

20 The police officer knows that there is a dirt track linking the towns of Larebil and Yrnuoc. She wishes to meet the commissioner’s instructions from question 19 but also wishes to start and finish in her home town, Larebil.
a Determine a walk that would meet both the police officer’s and the commissioner’s requirements.
b What type of walk is this?

21 A security company, ‘Wotchemclose’, is responsible for patrolling stores in the towns from question 19. The company wants a patrol car to visit each town once each night without resorting to using the dirt road.
a If the security guard starts at Ruobal, determine a walk that will meet the company’s requirements.
b What type of walk is this?

22 A physical education teacher, I. M. Grate, wishes to plan an orienteering course through a forest following marked tracks. She has placed checkpoints at the points shown in the diagram. The object of any orienteer is to visit each of the checkpoints once to collect a mark.
a What walk could an efficient orienteer follow if the course starts at C and finishes at B?
b What walk should the orienteer follow if starting and finishing at C?
c What type of walks are these?

9.5 Trees and their applications

There are many applications where only part of the network is required as a solution to a problem. This section will look at such problems involving subgraphs and trees. To begin with we need a few more definitions.

Graphs and subgraphs

Until now we have used the term network to refer to a collection of vertices and edges. This network can also be called a graph. In practice, a graph should have at least 2 vertices and 1 edge. All or part of this graph can be considered as a subgraph.

For example, in the figure, the entire network can be considered as a graph, while the path in pink can be considered a subgraph.

Another subgraph could be defined by the path 1–2–3–4–1.

A ‘minimum’ subgraph could be defined by the path 1–2.

Often the edges in a graph are not just simply connectors, but could be assigned some quantity, such as distance, time or cost. For example, in the figure the distance between vertex 1 and vertex 2 could be assigned a distance of 40 metres. If the graph contains such quantities, then it is called a weighted graph.
Trees

A tree is a connected subgraph which cannot contain any:
1. loops
2. parallel (or multiple) edges
3. cycles.

WORKED EXAMPLE 15

Determine whether each of the figures is a tree, and if not, explain why not.

THINK
1. Examine each figure in turn, looking for loops, parallel edges or cycles.
2. Examine figure a.
3. Examine figure b.
4. Examine figure c.
5. Examine figure d.

WRITE
Figure a has parallel edges (at the top) so it is not a tree.
Figure b has no loops, parallel edges or cycles so it is a tree.
Figure c has a cycle (at the top) so it is not a tree.
Figure d has a loop (at the bottom) so it is not a tree.

The advantage of trees within a network is that the tree could determine an ‘efficient’ connection between vertices in the sense that there is a minimum distance, cost or time.

Shortest paths

Sometimes it may be useful to determine the shortest path between 2 selected vertices of a graph. For example, when going shopping, a person may leave his home and travel east via the playground, or north via the parking lot and still end up at the same shop. In one case the distance travelled may be the minimum.

WORKED EXAMPLE 16

Determine the shortest path between nodes A and F in the figure shown. Nodes are labelled A, ..., G and distances (in metres) between them are labelled in blue.
### WORKED EXAMPLE 17

Find the shortest path from vertex 1 to vertex 9. Vertices are labelled in black, distances in blue. (Note: Lines are not to scale.)

**THINK**

1. From the starting vertex (1), find the shortest path to each of the vertices directly connected to it.

**WRITE**

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>—</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>—</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4 + 3 = 7</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>—</td>
<td>6</td>
</tr>
</tbody>
</table>

2. Determine the set of vertices, $S$. 

$S = \{1, 2, 3, 4\}$

When choosing the possible paths in step 1, there is no point in finding paths that are not trees. There will always be a tree which covers the same vertices in less distance. Non-tree paths will include cycles and loops, which only add to the total distance.

### A shortest path algorithm

Sometimes it can be difficult to list all the paths between the starting and ending vertex. Here is an algorithm which "guarantees" the shortest path — assuming that the starting vertex is already chosen.

**Step 1.** From the starting vertex, find the shortest path to all other directly connected vertices. Include all such vertices, including the starting one in the list $S = \{A, B, \ldots\}$.

**Step 2.** Choose a vertex ($V$) directly connected to those in $S$ and find the shortest path to the starting vertex. Generally, there is one possible path for each degree of $V$, although some obvious paths can be eliminated immediately.

**Step 3.** Add the new vertex, $V$, to the list $S$.

**Step 4.** Repeat steps 2 and 3 until all vertices are in $S$. Find the shortest path to the vertex you want.
Apply step 2 of the algorithm for a vertex connected directly to one in $S$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
<th>Shortest path to 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>7 + 7 = 14</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>6 + 9 = 15</td>
<td></td>
</tr>
</tbody>
</table>

Apply step 3 of the algorithm and add to the set of vertices, $S$.

$S = \{1, 2, 3, 4, 5\}$

Re-apply step 2 of the algorithm for another vertex directly connected to one in $S$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
<th>Shortest path to 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>2</td>
<td>4 + 7 = 11</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>3</td>
<td>7 + 3 = 10</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>7</td>
<td>7 + 9 + 2 = 18</td>
<td></td>
</tr>
</tbody>
</table>

Re-apply step 3 of the algorithm and add to the set of vertices, $S$.

$S = \{1, 2, 3, 4, 5, 6\}$

Re-apply step 2 for another vertex directly connected to one in $S$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
<th>Shortest path to 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>6</td>
<td>10 + 2 = 12</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>3</td>
<td>7 + 9 = 16</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>5</td>
<td>14 + 4 = 18</td>
<td></td>
</tr>
</tbody>
</table>

Re-apply step 3 and add to the set of vertices, $S$.

$S = \{1, 2, 3, 4, 5, 6, 7\}$

Re-apply step 2 for another vertex directly connected to one in $S$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
<th>Shortest path to 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>6</td>
<td>10 + 5 = 15</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>7</td>
<td>12 + 4 = 16</td>
<td></td>
</tr>
</tbody>
</table>

Re-apply step 3 and add to the set of vertices, $S$.

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Re-apply step 2 for another vertex directly connected to one in $S$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Via</th>
<th>Distance</th>
<th>Shortest path to 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>8</td>
<td>15 + 3 = 18</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>7</td>
<td>12 + 2 = 14</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>5</td>
<td>14 + 12 = 26</td>
<td></td>
</tr>
</tbody>
</table>

Re-apply step 3 and add to the set of vertices, $S$.

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ Stop, because all vertices are in the list.

Apply step 4 and determine the shortest path.

Path = 1–2–3–6–7–9
Distance = 4 + 3 + 3 + 2 + 2 = 14.
Spanning trees

In the network shown, the vertices represent school buildings and the edges represent footpaths. The numbers represent the distance, in metres, between the buildings. The school council has decided to cover some of the footpaths so that the students can access any building during rainy weather without getting wet. Three possible trees which would accomplish this are shown in the figures.

Note that each of these trees included all the vertices of the original network. These trees are called spanning trees because of this property. In practice, the school council would like to make the total distance of covered footpaths as small as possible, in order to minimise cost. In this case, they would have the minimum spanning tree. Can you determine which of the figures is the minimum spanning tree?

Minimum spanning trees and Prim's algorithm

One method of determining the minimum spanning tree is called Prim's algorithm. The steps are as follows:

Step 1. Choose the edge in the network which has the smallest value. If 2 or more edges are the smallest, choose any of these.

Step 2. Inspect the 2 vertices included so far and select the smallest edge leading from either vertex. Again, if there is a ‘tie’, arbitrarily choose any one.

Step 3. Inspect all vertices included so far and select the smallest edge leading from any included vertex. If there is a ‘tie’, choose any, arbitrarily.

Step 4. Repeat step 3 until all vertices in the graph are included in the tree.

WORKED EXAMPLE 18

Determine the minimum spanning tree for the network representing footpaths in a school campus.

THINK

1 Find the edge with the smallest distance. This can be done by listing all the edges and choosing the smallest.

WRITE

A–B = 32, by inspection
2 Inspect A and B and find the shortest edge connecting one of these to a third vertex.

A–E = 62 — choose this

A–C = 100

B–D = 75

3 Inspect A, B and E and find the shortest edge connecting one of these to another vertex.

A–C = 100

E–C = 45 — choose this

E–D = 50

B–D = 75

4 Continue until all vertices have been connected. In this case only vertex D remains.

B–D = 75

E–D = 50 — choose this

C–D = 78

5 Since all vertices have been connected, this is the minimum spanning tree. Calculate the total distance of the minimum spanning tree.

Total distance = 32 + 62 + 45 + 50 = 189 m

Maximum spanning tree

In some cases you may be required to find the maximum spanning tree instead of the minimum spanning tree. In this case, Prim’s algorithm works by finding the largest edges at each stage instead of the smallest edges.

WORKED EXAMPLE 19

The figure shown represents a telephone network connecting 6 towns, A, B, ..., F. The numbered edges represent the ‘capacity’ of the telephone connection between the towns connected, that is, the maximum number of calls that can be made at the same time along that edge.

A telephone engineer wishes to determine the maximum capacity of the system in terms of a tree connecting all the towns so that calls can be routed along that tree.

THINK

1 Because this is a maximum spanning tree, find the edge with the largest capacity.

WRITE

This is edge A–E (66).
Dijkstra’s algorithm

Another method for determining the shortest path between a given vertex and each of the other vertices is by using Dijkstra’s algorithm. An effective way to use this algorithm is using a tabular method, as shown in Worked example 20.

WORKED EXAMPLE 20

Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra’s algorithm in tabular form.

THINK

1 Construct a table representing the network. Where there is no value, you are unable to take that path.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X</td>
<td>16</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>X</td>
<td>9</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>9</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>10</td>
<td>6</td>
<td>X</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>15</td>
<td>14</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

WRITE

Maximum capacity

\[= 66 + 62 + 60 + 58 + 58\]

\[= 304\] telephone calls at the same time.
Starting at E, place a 0 above it. The options are B (15 km) and D (14 km). Write these above the letters.

Go to the shortest distance – D and go down vertically.

B: Add 14 and 10 (= 24); this is greater than the 15 already above B so not a choice.
C: Add 14 and 6 (= 20); since nothing above C write this above C.
E: We started at E so don’t want to go back there.

Now E and D are done, the next lowest is B; again work vertically.

A: Add 15 and 16 (= 31); since nothing above A write this above A.
C: Add 15 and 9 (= 34); this is greater than 20 so leave it as 20.
D: Already been to D.
E: Already been to E.

Now E, D and B are done, the next lowest is C; again work vertically.

A: Add 20 and 8 (= 28), since this is lower than the 31 that is currently there so replace it with the 28.
B: Already been to B.
D: Already been to D.

The shortest distance from A to E is then the number above E, in this case 28.
**Trees and their applications**

1. **WE15** Determine all the trees connecting vertices A and B, without going through vertex F.

2. Consider the following paths:
   - i. A–B–C–E–F–D–A
   - ii. A–D–F–E–C–B
   - iii. A–B–D–A

   Which (if any) are trees?
   - A All are trees
   - B i and iv only
   - C ii and iv only
   - D iv only
   - E None are trees

3. **WE16** Determine the shortest path from A to B, where the distances (in blue) are in kilometres.

4. The network shown at right represents the time (blue numerals in minutes) that it takes to walk along pathways connecting 8 features in a botanical garden. Vertices 1 and 8 are entrances. Find the minimum time to walk between the entrances, along pathways only.

5. **WE17** Referring to the network shown at right, where distances are in km, find:
   - a. the shortest path from B to F
   - b. the shortest path from A to C.

6. Referring to the network in question 5, where distances are in km, find the shortest path from E to C.

7. **WE18** Find the minimum spanning tree for the network shown at right.

8. The total length of the minimum spanning tree in question 7 is:
   - A 20
   - B 37
   - C 40
   - D 51
   - E 66
9. In the figure shown, the calculations using Prim’s algorithm for the minimum spanning tree would be:
   A \(2 + 4 + 4 + 5 + 7\)  
   B \(4 + 4 + 5 + 7\)  
   C \(4 + 4 + 6 + 7\)  
   D \(4 + 5 + 7 + 4\)  
   E 20

10. Draw the minimum spanning tree in each of the following graphs and calculate the total length:

   a
   ![Graph a]

   b
   ![Graph b]

11. Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra’s algorithm in tabular form.

12. Determine the shortest path from A to E, where the distances are in kilometres, by using Dijkstra’s algorithm in tabular form.

13. Consider the figure shown at right. Which of the paths marked in pink on the figures are trees?

   a
   ![Figure a]

   b
   ![Figure b]

   c
   ![Figure c]

   d
   ![Figure d]

14. Determine all the trees connecting vertices A and B, without going through vertex F.
15 Which of the following statements is true?
   A Hamiltonian cycle is a tree.
   B A tree can contain multiple edges or loops.
   C A Hamiltonian path is not a tree.
   D A tree can visit the same vertex more than once.
   E A tree can have one edge.

Questions 16 to 18 refer to the network shown in the figure. Vertices are labelled, A, B, ..., H and the time it takes to travel between them, in minutes, is given by the numbers in blue.

16 a List all possible trees connecting A and H, passing through B.

   b Use Dijkstra's algorithm to determine the shortest time for A–H.

17 The total number of possible trees connecting A and H is given by:
   A 8  B 10  C 12
   D 15  E 28

18 The shortest time it would take to travel between A and H is given by the tree:
   D A–C–F–G–H  E A–E–H

19 Use Dijkstra's algorithm to determine the shortest path from A to K, where the distances are in km.

20 The figure shows a network connecting vertices A, ..., H

   a How many different trees are there connecting A to G?
   b Find the shortest path connecting A to G.
   c How many different trees are there connecting D to F?
   d Find the shortest path connecting D to F.
21 Using Prim’s algorithm, determine the length of the minimum spanning tree in each of the graphs shown.

```
24
22
26
23
15
14
```

22 Draw the minimum spanning tree in each of the following graphs and calculate the total length:

```
14
7
6
5
2
1
3
7
5
13
2
12
6
```

23 Flyemsafe Airlines wish to service six cities. The directors have decided that it is too costly to have direct flights between all the cities. The airline needs to minimise the number of routes which they open yet maximise the total number of passengers that they can carry. The network diagram shown has edges representing routes and vertices representing cities. The numbers on the edges are projected capacities. Find:

a the maximum spanning tree that will meet the airline’s requirements
b the total carrying capacity of this tree.
24 A fairground has 5 main attractions which are joined by paths to the entrance/exit gate. The numbers show the distance along the paths in metres.

a Draw an undirected graph to represent the fairground and then write down:
   i the number of edges
   ii the number of vertices
   iii the degree of each vertex.

b What is the minimum distance a person would have to walk to visit every attraction, beginning and ending at the entrance/exit?

c If each attraction needs to be able to communicate via a phone line, draw the minimum possible tree to represent this.

d Complete a matrix for the graph shown.

e Following a Hamiltonian cycle would be an efficient way to visit every attraction in the fairground. Suggest a route a visitor could follow in order to create a Hamiltonian cycle, beginning and ending at the entrance/exit.
The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

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**Interactivities**

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.
9 Answers

EXERCISE 9.2

1 Vertices = 6, edges = 8
2 Vertices = 7, edges = 9
3 $\text{deg}(A) = 3$, $\text{deg}(B) = 4$, $\text{deg}(C) = 4$
4 $\text{deg}(A) = 3$, $\text{deg}(B) = 8$, $\text{deg}(C) = 3$
5 $V = \{A, B, C, D, E, F\}$  
   $E = \{(A, B), (A, D), (A, E), (B, D), (B, E), (C, D), (D, F)\}$
6 $V = \{A, B, C, D, E, F, G\}$  
   $E = \{(A, B), (B, C), (B, E), (B, F), (C, E), (C, F),
   (D, F)\}$
7 $V = \{A, B, C, D, E, F\}$  
   $E = \{(A, B), (A, D), (A, E), (B, D), (B, E), (C, D), (D, F)\}$
8 $V = \{A, B, C, D, E, F, G\}$  
   $E = \{(A, B), (B, C), (B, E), (B, F), (C, E), (C, F),
   (E, F), (E, G), (F, G)\}$
9 Vertices = 6, edges = 8
10 Vertices = 6, edges = 9
11 Vertices = 7, edges = 11
12 Vertices = 9, edges = 16
13 $\text{deg}(A) = 2$, $\text{deg}(B) = 2$, $\text{deg}(C) = 1$
14 $\text{deg}(A) = 4$, $\text{deg}(B) = 1$, $\text{deg}(C) = 0$
15 a $V = \{1, 2, 3, 4, 5\}$
   $E = \{(1, 2), (1, 3), (2, 3), (3, 4), (4, 5)\}$
   $b V = \{U, V, W, X, Y, Z\}$
   $E = \{(U, V), (U, W), (U, X), (V, W), (V, X), (V, Z),
   (W, X), (W, Y), (X, Z)\}$
   $c V = \{1, 2, 3, 4, 5, 6, 7\}$
   $E = \{(1, 2), (1, 3), (1, 5), (2, 4), (2, 6), (2, 7), (3, 6), (5, 7)\}$
   $d V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   $E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (2, 5),
   (2, 6), (2, 7), (3, 5), (3, 6), (4, 5), (4, 9), (5, 6), (5, 7), (5, 8),
   (5, 9), (6, 7)\}$
16 a $V = \{1, 2, 3\}$
   $E = \{(1, 2), (1, 3), (2, 3)\}$
   $b V = \{1, 2, 3, 4\}$
   $E = \{(1, 2), (1, 3), (2, 3), (1, 4)\}$
   $c$ Degree of each vertex = 3
17 a $V = \{1, 2, 3, 4\}$
   $E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$
   $b V = \{1, 2, 3, 4\}$
   $E = \{(1, 2), (1, 3), (1, 4)\}$
   $c$ Degree of each vertex = 4
18 i a $V = \{1, 2, 3, 4, 5\}$
   $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5),
   (3, 4), (3, 5), (4, 5)\}$
   $b$ Degree of each vertex = 4
ii a $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8),
   (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4),
   (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7),
   (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8)\}$
   $b$ Degree of each vertex = 7
19 a 45 b 190 c 4950
EXERCISE 9.3

1 FACE 1: degree = 3
FACE 2: degree = 3
FACE 3: degree = 3
FACE 4: degree = 3
FACE 5: degree = 3

2 FACE 1: degree = 4
FACE 2: degree = 3
FACE 3: degree = 3
FACE 4: degree = 4

3 FACE 1: degree = 3
FACE 2: degree = 3
FACE 3: degree = 3
FACE 4: degree = 4
FACE 5: degree = 4

4 FACE 1: degree = 4
FACE 2: degree = 3
FACE 3: degree = 3
FACE 4: degree = 6
FACE 5: degree = 6

5

6

7 $V = 9, E = 16, F = 9$, so $9 = 16 - 9 + 2$

8 $V = 4, E = 6, F = 4$, so $4 = 6 - 4 + 2$. 

9 Answers may vary. Below are suggested solutions.

a

b

Already planar

b

c

5

Already planar

b

5

c

Face I: degree = 3
Face II: degree = 3
Face III: degree = 4
Face IV: degree = 4
Face V: degree = 4

10 B

11 C

12 a

b

5

13

b

6

14 D

15 $V = 8, E = 13, F = 7$, so $8 = 13 - 7 + 2$

16 Since the degree of a single node is determined by the number of edges ($E$) 'leaving' it, and each such edge must be the 'entering' edge of another node, each edge is counted twice in the sum of degrees ($S$). Thus the sum must be an even number. And since each edge is counted twice $S = 2E$. 
17 B

18 D

19 F

The edges should represent the doorways between rooms. (Note: The hall space between rooms E and D belongs to room F. Similarly, the hall space between rooms B and C belongs to room A.)

20 a 28

\[
\begin{array}{cccccccc}
A & B & C & D & E & F & G & H \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

EXERCISE 9.4
1 A–B–C–D–E–F–G
2 E–F–G–D–C–B–A
3 1–2–5–3–4–5–1
4 B
5 1–3–4–5–6–7–8–5–3–2–1
6 D
7 2–1–6–5–4–3–7
8 C
9 2–1–6–5–4–7–3–2
10 E
13 D
15 7–6–5–4–3–2–1
16 A
17 a 2–1–3–4–5–2–3
   b Cannot be done (odd vertices)
   c 1–2–3–4–5
d 1–3–4–5–2–1

18 a Join 4 to 7 and 3 to 8.
   b An Euler trail
   b An Euler circuit
21 a R–Y–S–L–N
   b A Hamiltonian path
22 a C–D–A–E–F–B
   b C–A–E–F–B–D–C
   c Hamiltonian path, Hamiltonian cycle

EXERCISE 9.5
1 A–C–B; A–D–C–B; A–E–D–C–B
2 C
3 17 km
4 16 minutes
5 a 28 km
   b 17 km
6 28 km
7
8 B
9 B
10 a 32
   b 86
   (1 of 2 possible answers)
11 30 km
12 41 km
13 b, d
14 A–C–B; A–D–B; A–C–D–B; A–D–C–B; A–E–D–B; A–E–D–C–B
15 E
   b 22
17 C
18 D
19 44 km
20 a 5
   b 18
   c 7
   d 19
21 a $12 + 13 + 18 + 24 = 67$
b $6 + 8 + 9 + 16 + 15 = 54$
c $5 + 10 + 14 + 26 = 55$
d $4 + 4 + 6 + 7 = 21$

22 a 49

23 a

24 a

b 1845 passengers