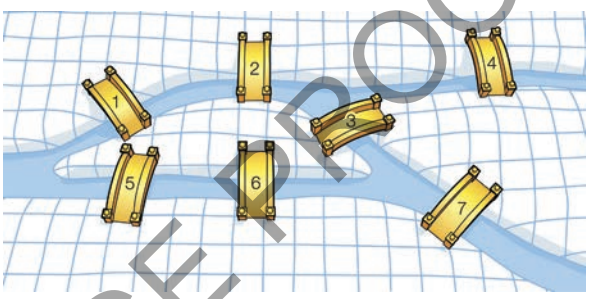
TOPIC 6
Functions and relations

6.1 Overview

6.1.1 Introduction



The concept of a function is fundamental in mathematical analysis. The reference to, and use of, the notation $y = f(x)$ is commonplace. The 18th-century mathematician Leonhard Euler is credited with first using the function notation $f(x)$, and that of inverse function notation $f^{-1}(x)$.

Euler's work contributed to most branches of mathematics as well as to the sciences, education, engineering, cartography, ship-building and music among other areas of study; his interests were vast and his published works prolific.

He devised much of the notation that we use today, including the symbol π for the ratio of the circumference of a circle to its diameter, the symbol Σ for summation, $\sin x$ and $\cos x$, $\log x$ and e^x, i as the symbol for the square root of -1, the convention of naming the angles of a triangle as A, B and C and the sides opposite these angles as a, b and c respectively, and much more.

Although Euler was born and educated in Basel, Switzerland, he spent much of his professional life in St Petersburg in Russia except for a 25-year period in Berlin under the patronage of Frederick the Great. In 1741 he was invited to return to St Petersburg by Catherine the Great and he remained there for the rest of his life. He suffered from poor eyesight from early adulthood, and shortly after returning to St Petersburg he lost his sight completely. This did not deter him; of the work he produced, almost half was completed when he was totally blind. He had a remarkable capacity to carry his calculations in his mind, and his sons helped to produce the books containing his discoveries.

One famous problem that Euler was able to solve was the Seven Bridges of Königsberg problem. Although only two of the original seven bridges remain, due to heavy bombardment in the Second World War, the legacy of the city (now Kaliningrad) is that Euler's solution led to a new branch of mathematics known as Graph Theory.

Euler's works and name live on in mathematics today.

LEARNING SEQUENCE

6.1 Overview
6.2 Functions and relations
6.3 The circle
6.4 The rectangular hyperbola and the truncus
6.5 The relation $y^2 = x$
6.6 Other functions and relations
6.7 Transformations of functions
6.8 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

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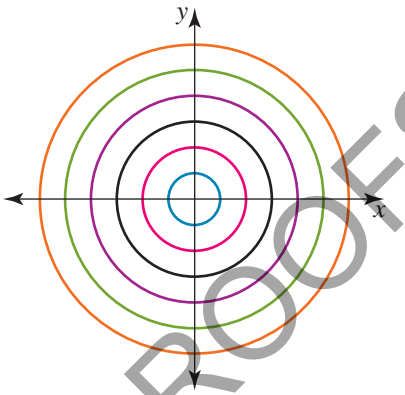
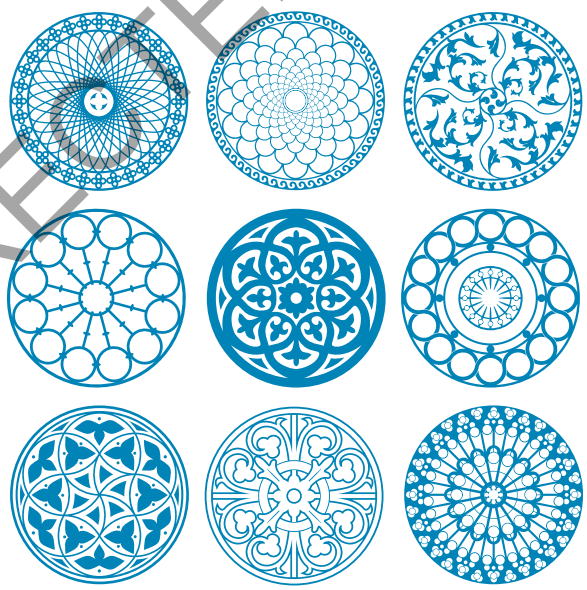
6.1.2 Kick off with CAS

Circles



1. Sketch the following circles using CAS technology.
   a. $x^2 + y^2 = 1$
   b. $x^2 + y^2 = 2^2$
   c. $x^2 + y^2 = 3^2$
   d. $x^2 + y^2 = 4^2$
   e. $x^2 + y^2 = 5^2$
   f. $x^2 + y^2 = 6^2$
2. Using CAS technology, enter $x^2 + y^2 = r^2$ into the equation entry line and use a slider to change the values of r.
3. Complete the following sentences.
   a. When sketching a circle, what is the effect of changing the value of r?
   b. When sketching a circle, can the value of r be negative? Explain your answer.
4. Sketch the following circles using CAS technology.
   a. $(x - 1)^2 + y^2 = 2^2$
   b. $(x - 2)^2 + y^2 = 2^2$
   c. $(x - 3)^2 + y^2 = 2^2$
   d. $(x + 1)^2 + y^2 = 2^2$
   e. $(x + 2)^2 + y^2 = 2^2$
   f. $(x + 3)^2 + y^2 = 2^2$
5. Using CAS technology, enter $(x - h)^2 + y^2 = 2^2$ into the equation entry line and use a slider to change the values of h.
   What effect does changing the value of h have on the graph of the circle?
6. Sketch the following circles using CAS technology.
   a. $x^2 + (y - 1)^2 = 2^2$
   b. $x^2 + (y - 2)^2 = 2^2$
   c. $x^2 + (y - 3)^2 = 2^2$
   d. $x^2 + (y + 1)^2 = 2^2$
   e. $x^2 + (y + 2)^2 = 2^2$
   f. $x^2 + (y + 3)^2 = 2^2$
7. Using CAS technology, enter $x^2 + (y - k)^2 = 2^2$ into the equation entry line and use a slider to change the values of k.
   What effect does changing the value of k have on the graph of the circle?
8. Using CAS technology and your answers to questions 1–7, create a pattern using circles.



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on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Functions and relations

The mathematical language, notation and functionality concepts that will be introduced in this topic will continue to be used throughout all subsequent topics. Within this topic, functionality concepts will be illustrated using the now-familiar polynomial relationships. They will also be applied to some new relationships.

6.2.1 Relations

A mathematical **relation** is any set of ordered pairs.

The ordered pairs may be listed or described by a rule or presented as a graph. Examples of relations could include $A = \{(-2, 4), (1, 5), (3, 4)\}$ where the ordered pairs have been listed, $B = \{(x, y) : y = 2x\}$ where the ordered pairs are described by a linear equation, and $C = \{(x, y) : y \leq 2x\}$ where the ordered pairs are described by a linear inequation. These relations could be presented visually by being graphed on coordinate axes. The graph of A would consist of three points, the graph of B would be a straight line and the graph of C would be a closed half-plane.

6.2.2 Domain and range

For a set of ordered pairs (x, y), the **domain** is the set of all the x-values of the ordered pairs and the **range** is the set of all the y-values of the ordered pairs.

For $A = \{(-2, 4), (1, 5), (3, 4)\}$ the domain is $\{-2, 1, 3\}$ and the range is $\{4, 5\}$. For both $B = \{(x, y) : y = 2x\}$ and $C = \{(x, y) : y \leq 2x\}$, the domain is R and the range is R.

The graph of any polynomial relation normally has a domain of R. For some practical situations, restrictions have been placed on the values of the variables in some polynomial models. In these cases the polynomial relation has been defined on a **restricted domain**. A restricted domain usually affects the range.

Set notation or interval notation should be used for domains and ranges.

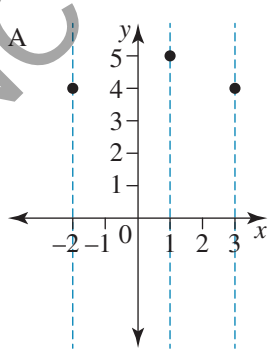
6.2.3 Functions

A **function** is a set of ordered pairs in which every x-value is paired to a unique y-value.

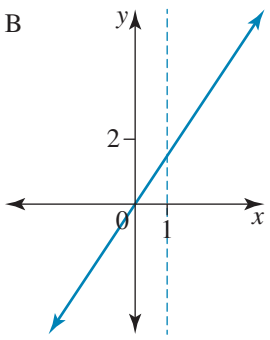
The relations $A = \{(-2, 4), (1, 5), (3, 4)\}$ and $B = \{(x, y) : y = 2x\}$ are functions, but not every relation is a function. The relation $C = \{(x, y) : y \leq 2x\}$ is not a function as it contains many points such as $(3, 1)$ and $(3, 2)$ which are in the region defined by $y \leq 2x$ but which have the same x-coordinate. No two ordered pairs of a function can have the same x-coordinate.

Vertical line test

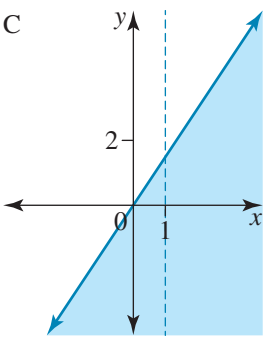
Functions can be recognised from their graphs by the **vertical line test**. Any vertical line which cuts the graph of a function will do so exactly once. If the vertical line cuts the graph at more than one place, the graph is not that of a function.



a vertical line cuts once



a vertical line cuts once



a vertical line cuts many times

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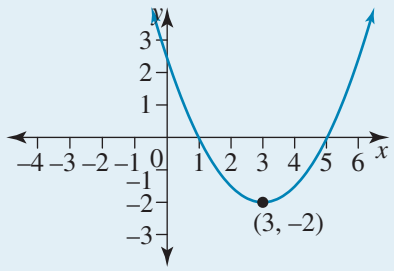
In the diagram, a vertical line only cuts the graphs of A and B in one place; it cuts the graph of C in many places. A and B are therefore functions; C is not a function. B is a linear polynomial function. The graph of any polynomial can be cut by a vertical line exactly once: every polynomial relation is a function.

WORKED EXAMPLE 1

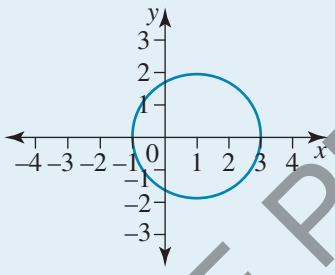
For each of the following, state the domain and range, and whether the relation is a function or not.

a. $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$

b.



c.



d. $\{(x, y) : y = 4 - x^3\}$

THINK

a. 1. State the domain.

2. State the range.

3. Are there any ordered pairs which have the same x-coordinate?

b. 1. State the domain.

2. State the range.

3. Use the vertical line test.

c. 1. State the domain and range.

2. Use the vertical line test.

WRITE

a. $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$

The domain is the set of x-values: $\{1, 2, 5\}$.

The range is the set of y-values: $\{-1, 0, 3, 4\}$.

The relation is not a function since there are two different points with the same x-coordinate: $(2, 0)$ and $(2, 3)$.

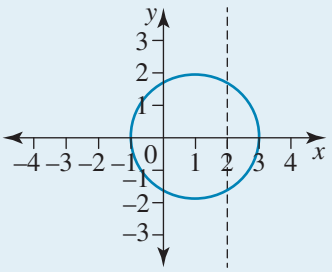
b. Reading from left to right horizontally in the direction of the x-axis, the graph uses every possible x-value.The domain is $(-\infty, \infty)$ or R.

Reading from bottom to top vertically in the direction of the y-axis, the graph's y-values start at -2 and increase from there. The range is $[-2, \infty)$ or $\{y : y \geq -2\}$.

This is a function since any vertical line cuts the graph exactly once.

c. The domain is $[-1, 3]$; the range is $[-2, 2]$.

This is not a function as a vertical line can cut the graph more than once.



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d. 1. State the domain.

2. State the range.

3. Is the relation a function?

d. $y = 4 - x^3$

This is the equation of a polynomial so its domain is R.

It is the equation of a cubic polynomial with a negative coefficient of its leading term, so as $x \to \pm\infty$, $y \to \mp\infty$. The range is R.

This is a function, because all polynomial relations are functions.

on Resources

Interactivity: Vertical and horizontal line tests (int-2570)

6.2.4 Type of correspondence

Functions and relations can be classified according to the **correspondence** between the coordinates of their ordered pairs. There are four possible types:

- **one-to-one correspondence** where each x-value is paired to a unique y-value, such as $\{(1, 2), (3, 4)\}$
- **many-to-one correspondence** where more than one x-value may be paired to the same y-value, such as $\{(1, 2), (3, 2)\}$
- **one-to-many correspondence** where each x-value may be paired to more than one y-value, such as $\{(1, 2), (1, 4)\}$
- **many-to-many correspondence** where more than one x-value may be paired to more than one y-value, such as $\{(1, 2), (1, 4), (3, 4)\}$.

Here 'many' means more than one.

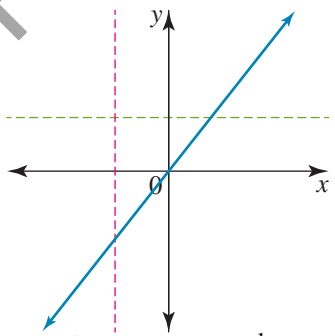
The graph of a function is recognised by the vertical line test and its type of correspondence is determined by the **horizontal line test**.

A function has a:

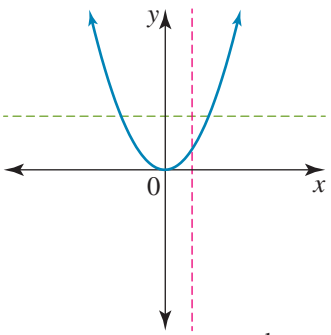
- one-to-one correspondence if a horizontal line cuts its graph exactly once
- many-to-one correspondence if a horizontal line cuts its graph more than once.

A relation which is not a function has a:

- one-to-many correspondence if a horizontal line cuts its graph exactly once
- many-to-many correspondence if a horizontal line cuts its graph more than once.



one-to-one correspondence function



many-to-one correspondence function

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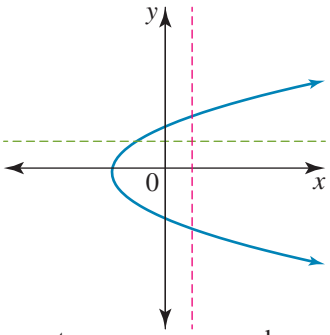
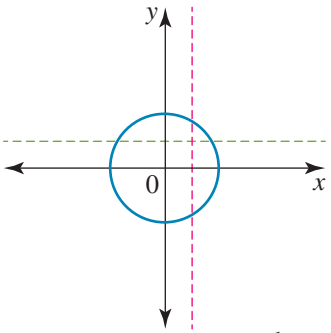


one-to-many correspondence
not a function



many-to-many correspondence
not a function

Graphically, the type of correspondence is determined by the number of intersections of a horizontal line (one or many) to the number of intersections of a vertical line (one or many) with the graph. Functions have either a one-to-one or many-to-one correspondence, since their graphs must pass the vertical line test.

WORKED EXAMPLE 2

Identify the type of correspondence and state whether each relation is a function or not.

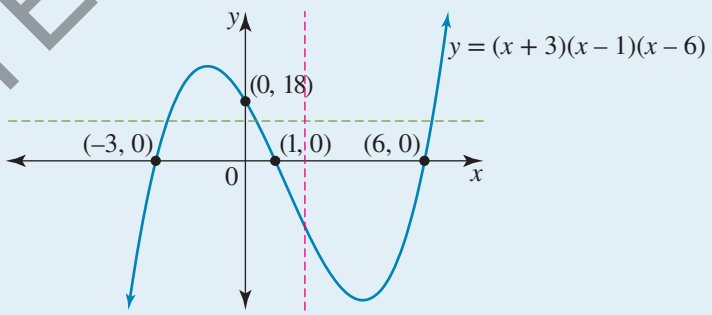
a. $(x, y) : y = (x + 3)(x - 1)(x - 6)$ **b.** $\{(1, 3), (2, 4), (1, 5)\}$

THINK

a. 1. Draw the graph.

WRITE

a. $y = (x + 3)(x - 1)(x - 6)$
x-intercepts: $(-3, 0), (1, 0), (6, 0)$
y-intercept: $(0, 18)$
The graph is a positive cubic.



2. Use the horizontal line test and the vertical line test to determine the type of correspondence.

A horizontal line cuts the graph in more than one place.
A vertical line cuts the graph exactly once.
This is a many-to-one correspondence.

3. State whether the relation is a function.

$y = (x + 3)(x - 1)(x - 6)$ is the equation of a polynomial function with a many-to-one correspondence.

b. 1. Look to see if there are points with the same x- or y-coordinates.

b. $\{(1, 3), (2, 4), (1, 5)\}$
$x = 1$ is paired to both $y = 3$ and $y = 5$. The relation has a one-to-many correspondence. It is not a function.

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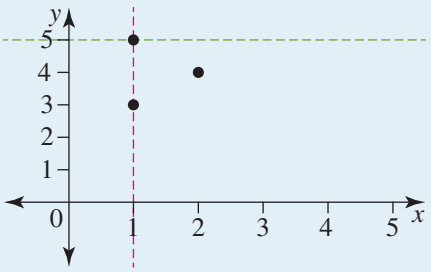
2. Alternatively, or as a check, plot the points and use the horizontal and vertical line tests.



one-to-many

A horizontal line cuts the graph exactly once.
A vertical line can cut the graph in more than one place.
This is a one-to-many correspondence.

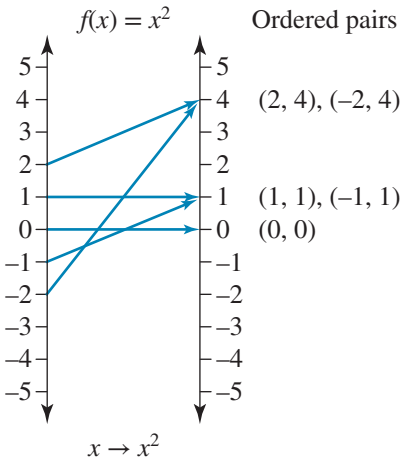
6.2.5 Function notation

The rule for a function such as $y = x^2$ will often be written as $f(x) = x^2$. This is read as 'f of x equals x^2'. We shall also refer to a function as $y = f(x)$, particularly when graphing a function as the set of ordered pairs (x, y) with x as the independent or explanatory variable and y as the dependent or response variable.

$f(x)$ is called the **image** of x under the function **mapping**, which means that if, for example, $x = 2$ then $f(2)$ is the y-value that $x = 2$ is paired with (mapped to), according to the function rule. We have used $P(x)$ notation for polynomial functions.

For $f(x) = x^2$, $f(2) = 2^2 = 4$. The image of 2 under the mapping f is 4; the ordered pair $(2, 4)$ lies on the graph of $y = f(x)$; 2 is mapped to 4 under f: these are all variations of the mathematical language that could be used for this function.

The ordered pairs that form the function with rule $f(x) = x^2$ could be illustrated on a mapping diagram. The mapping diagram shown uses two number lines, one for the x-values and one for the y-values, but there are varied ways to show mapping diagrams.



Under the mapping, every x-value in the domain is mapped to its square, $x \to x^2$. The range is the set of the images, or corresponding y-values, of each x-value in the domain. For this example, the polynomial function has a domain of R and a range of $[0, \infty)$, since squared numbers are not negative. Not all of the real numbers on the y number line are elements of the range in this example. The set of all the available y-values, whether used in the mapping or not, is called the **codomain**. Only the set of those y-values which are used for the mapping form the range. For this example, the codomain is R and the range is a subset of the codomain since $[0, \infty) \subset R$.

The mapping diagram also illustrates the many-to-one correspondence of the function defined by $y = x^2$.

Formal mapping notation

The mapping $x \to x^2$ is written formally as:

$$f: \underset{\text{name of function}}{\downarrow} \quad R \underset{\text{domain of } f}{\downarrow} \to \quad R, \underset{\text{codomain}}{\downarrow} \quad f(x) = x^2 \underset{\text{rule for, or equation of, } f}{\downarrow}$$

The domain of the function must always be specified when writing functions formally.

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We will always use R as the codomain. Mappings will be written as $f: D \rightarrow R$, where D is the domain. Usually a graph of the function is required in order to determine its range.

Note that f is a symbol for the name of the function or mapping, whereas $f(x)$ is an element of the range of the function: $f(x)$ gives the image of x under the mapping f. While f is the commonly used symbol for a function, other symbols may be used.

WORKED EXAMPLE 3

Consider $f: R \rightarrow R, f(x) = a + bx$, where $f(1) = 4$ and $f(-1) = 6$.

a. Calculate the values of a and b and state the function rule.

b. Evaluate $f(0)$.

c. Calculate the value of x for which $f(x) = 0$.

d. Find the image of -5.

e. Write the mapping for a function g which has the same rule as f but a domain restricted to R^+.

THINK	WRITE
a. 1. Use the given information to set up a system of simultaneous equations.	**a.** $f(x) = a + bx$ $f(1) = 4 \Rightarrow 4 = a + b \times 1$ $\therefore a + b = 4 \quad [1]$ $f[-1] = 6 \Rightarrow 6 = a + b \times -1$ $\therefore a - b = 6 \quad [2]$
2. Solve the system of simultaneous equations to obtain the values of a and b.	Equation [1] + equation [2] $2a = 10$ $a = 5$ Substitute $a = 5$ into equation [1] $\therefore b = -1$
3. State the answer.	$a = 5, b = -1$ $f(x) = 5 - x$
b. Substitute the given value of x.	**b.** $f(x) = 5 - x$ $f(0) = 5 - 0$ $= 5$
c. Substitute the rule for $f(x)$ and solve the equation for x.	**c.** $f(x) = 0$ $5 - x = 0$ $\therefore x = 5$
d. Write the expression for the image and then evaluate it.	**d.** The image of -5 is $f(-5)$. $f(x) = 5 - x$ $f(-5) = 5 - (-5)$ $= 10$ The image is 10.
e. Change the name of the function and change the domain.	**e.** $g: R^+ \rightarrow R, g(x) = 5 - x$

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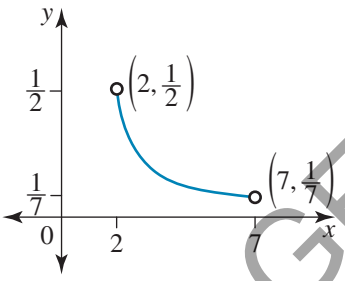
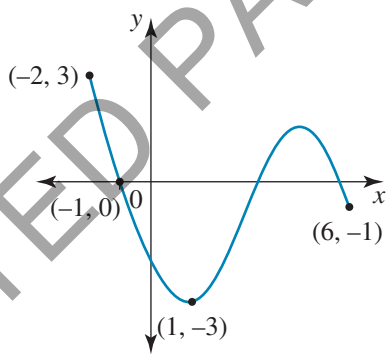
Exercise 6.2 Functions and relations

Technology free

1. **a.** State the domain and range of $\{\{-4, 7), (0, 10), (3, 5)\}$.
   b. Express $\{x : -10 \leq x < 4\}$ in interval notation.
   c. Use interval notation to describe the domain and range of the graph shown.

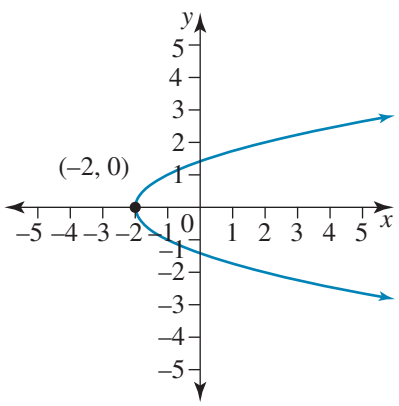


   d. State the domain and range of the given graph.



2. Calculate the coordinates of any endpoints of the following and state the domain and range in interval notation.
   a. $y = 2x, -1 < x \leq 3$
   b. $y = \frac{2}{3}x, x > 3$
   c. $y = -3x, -5 \leq x \leq 5$
   d. $y = x^2, x \in R$
   e. $y = x^2, x \in R^+$
   f. $y = x^2, x \in (-2, 2)$

3. WE1 For each of the following, state the domain and range and whether the relation is a function or not.
   a. $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$
   b.



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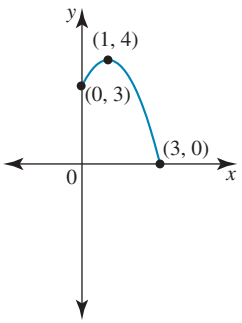
c.



d. $\{(x, y) : y = 4 - x^2\}$

4. Sketch the graph of $y = -4x, x \in [-1, 3)$ and state its domain and range.

5. Consider the relation $\{(x.y) : y = 2x - 1, -2 < x \leq 6\}$.

a. Sketch the graph of the relation.

b. State the domain and range.

c. Explain, with a reason, whether or not this relation is a function.

6. Consider the relation $\{(x.y) : y = (x - 2)^3 + 4, -2 \leq x \leq 4\}$.

a. Sketch the graph of the relation.

b. State the domain and range.

c. Explain, with a reason, whether or not this relation is a function.

7. Consider the relation $\{(x.y) : y \geq -1\}$.

a. Sketch the graph of the relation.

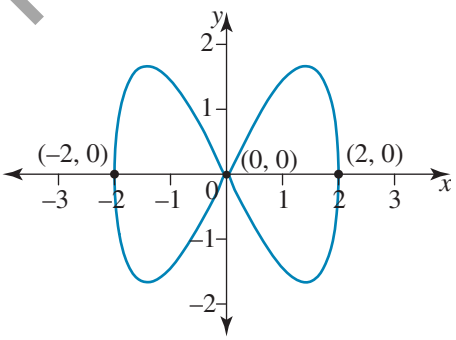
b. State the domain and range.

c. Explain, with a reason, whether or not this relation is a function.

8. WE2 Identify the type of correspondence and state whether each relation is a function or not:

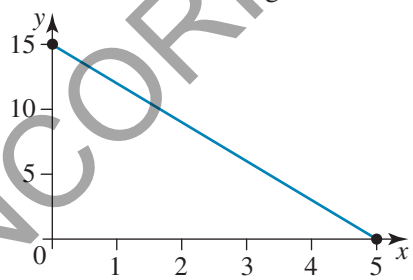
a. $\{(x, y) : y = 8(x + 1)^3 - 1\}$

b.

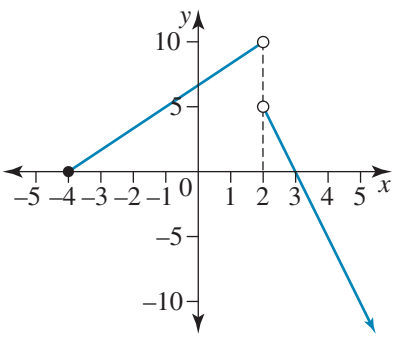


9. State the domain and range for each of the following relations.

a.



b.



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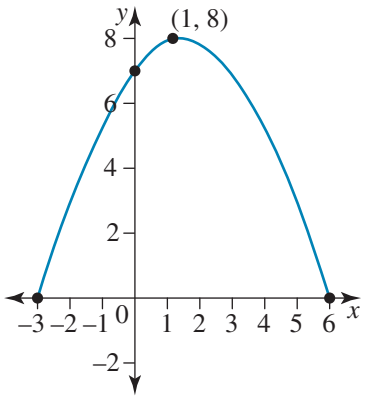
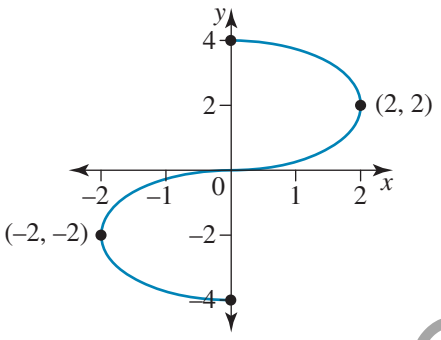
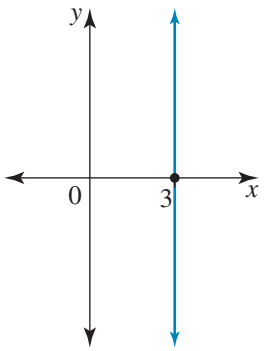
c.



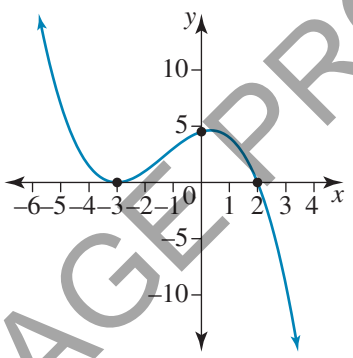
d.



e.



f.



10. Consider each of the graphs in question **9**.

a. State the type of correspondence of each relation.

b. Identify any of the relations which are not functions.

11. For each of the following relations, state the domain, range, type of correspondence and whether the relation is a function.

a. $\{(-11, 2), (-3, 8), (-1, 0), (5, 2)\}$

b. $\{(20, 6), (20, 20), (50, 10), (60, 10)\}$

c. $\{(-14, -7), (0, 0), (0, 2), (14, 7)\}$

d. $\{(x, y) : y = 2(x - 16)^3 + 13\}$

e. $\{(x, y) : y = 4 - (x - 6)^2\}$

f. $\{(x, y) : y = 3x^2(x - 5)^2\}$

12. Sketch each of the functions defined by the following rules and state the domain and range of each.

a. $f(x) = 4x + 2, x \in (-1, 1]$

b. $g(x) = 4x(x + 2), x \geq -2$

c. $h(x) = 4 - x^3, x \in R^+$

d. $y = 4, x \in R \setminus [-2, 2]$

13. **a.** Sketch the graph of $y = (x - 2)^2$, stating its domain, range and type of correspondence.

b. Restrict the domain of the function defined by $y = (x - 2)^2$ so that it will be a one-to-one and increasing function.

14. WE3 Consider $f : R \to R, f(x) = ax + b$, where $f(2) = 7$ and $f(3) = 9$.

a. Calculate the values of a and b and state the function rule.

b. Evaluate $f(0)$.

c. Calculate the value of x for which $f(x) = 0$.

d. Find the image of -3.

e. Write the mapping for a function g which has the same rule as f but a domain restricted to $(-\infty, 0]$.

15. Consider $f(x) = 1 - 4x$.

a. Calculate

i. $f(0)$ **ii.** $f(-5)$ **iii.** $f\left(\frac{1}{8}\right)$.

b. Find the value of x for which

i. $f(x) = 0$ **ii.** $f(x) = -1$ **iii.** $f(x) = x$.

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c. Identify the image of

i. -11 **ii.** 1.5 **iii.** $1-\sqrt{2}$.

d. Determine any value(s) of x for which $f(x) = g(x)$, given $g(x) = x^2 + 5$.

16. If $f(x) = x^2 + 2x - 3$, calculate the following.

a. **i.** $f(-2)$ **ii.** $f(9)$

b. **i.** $f(2a)$ **ii.** $f(1-a)$

c. $f(x+h) - f(x)$

d. $\{x : f(x) > 0\}$

e. The values of x for which $f(x) = 12$

f. The values of x for which $f(x) = 1 - x$

17. Consider $f : R \to R, f(x) = x^3 - x^2$.

a. Find the image of 2.

b. Sketch the graph of $y = f(x)$.

c. State the domain and range of the function f.

d. What is the type of correspondence?

e. Give a restricted domain so that f is one-to-one and increasing.

f. Calculate $\{x : f(x) = 4\}$.

18. Express $y = x^2 - 6x + 10, 0 \leq x < 7$ in mapping notation and state its domain and range.

19. Given $A = \{(1, 2), (2, 3), (3, 2), (k, 3)\}$, state the possible values for k and the type of correspondence, if the following apply.

a. A is not a function.

b. A is a function.

c. Draw a mapping diagram for the relation in part **a** using the chosen k values.

d. Draw a mapping diagram for the function in part **b** using the chosen k values.

20. Select the functions from the following list, express them in function mapping notation and state their ranges.

a. $y = x^2, x \in Z^+$ **b.** $2x + 3y = 6$

c. $y = \pm x$ **d.** $\{(x, 5), x \in R\}$

e. $\{(-1, y), y \in R\}$ **f.** $y = -x^5, x \in R^+$

Technology active

21. Consider the functions f and g where $f(x) = a + bx + cx^2$ and $g(x) = f(x - 1)$.

a. Given $f(-2) = 0, f(5) = 0$ and $f(2) = 3$, determine the rule for the function f.

b. Express the rule for g as a polynomial in x.

c. Calculate any values of x for which $f(x) = g(x)$.

d. On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ and describe the relationship between the two graphs.

22. a. A toy car moves along a horizontal straight line so that at time t its position x from a child at a fixed origin is given by the function $x(t) = 4 + 5t$. For the interval $t \in [0, 5]$, state the domain and range of this function and calculate how far the toy car travels in this interval.

b. A hat is thrown vertically into the air and at time t seconds its height above the ground is given by the function $h(t) = 10t - 5t^2$. Calculate how long it takes the hat to return to the ground and hence state the domain and range of this function.

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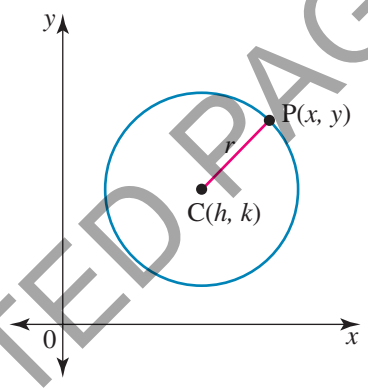
c. For part of its growth over a two-week period, the length of a leaf at time t weeks is given by the function $l(t) = 0.5 + 0.2t^3, 0 \leq t \leq 2$.
   i. State the domain and determine the range of this function.
   ii. Calculate how long it takes for the leaf to reach half the length that it reaches by the end of the time period.

23. Sketch $\{(x, y) : y^2 = x^2 + 1\}$ using CAS technology and hence state the domain and range, and determine if this relation is a function or not.
24. Define $f(x) = x^3 + lx^2 + mx + n$. Given $f(3) = -25, f(5) = 49, f(7) = 243$, answer the following questions.
   a. Calculate the constants l, m and n and hence state the rule for $f(x)$.
   b. What is the image of 1.2?
   c. Calculate, correct to 3 decimal places, the value of x such that $f(x) = 20$.
   d. Given $x \geq 0$, express the function as a mapping and state its range.

6.3 The circle

The **circle** is an example of a relation with a many-to-many correspondence. A circle is not a function.

6.3.1 Equation of a circle



To obtain the equation of a circle, consider a circle of radius r and centre at the point C(h, k). Let P(x, y) be any point on the circumference. CP, of length r, is the radius of the circle. Using the formula for the distance between two points:

$$\sqrt{(x-h)^2 + (y-k)^2} = \text{CP} = r$$

$$(x-h)^2 + (y-k)^2 = r^2$$

The equation of a circle with centre (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2$$

The endpoints of the horizontal diameter have coordinates $(h - r, k)$ and $(h + r, k)$; the endpoints of the vertical diameter are $(h, k - r)$ and $(h, k + r)$. These points, together with the centre point, are usually used to sketch the circle. The intercepts with the coordinate axes are not always calculated.

The domain and range are obtained from the endpoints of the horizontal and vertical diameters.

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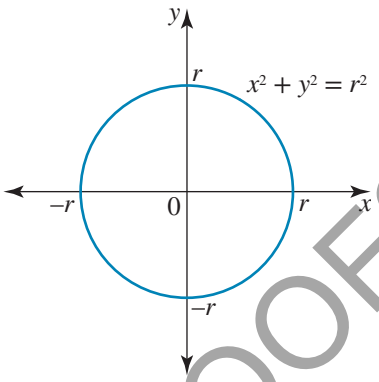
The circle with the centre (h, k) and radius r has domain $[h - r, h + r]$ and range $[k - r, k + r]$.

If the centre is at (0, 0), then the circle has equation $x^2 + y^2 = r^2$, with domain $[-r, r]$ and range $[-r, r]$.



General form of the equation of a circle

The general form of the equation of a circle is the expanded form of $(x - h)^2 + (y - k)^2 = r^2$.

Expanding gives $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$. This is equivalent to $x^2 + y^2 + ax + by + c = 0$ where $a = -2h$, $b = -2k$, $c = h^2 + k^2 - r^2$, and shows that three pieces of information are needed to calculate a, b and c in order to determine the equation.

The general form is converted into the standard centre–radius form by completing the square both on the x terms and on the y terms.

WORKED EXAMPLE 4

a. State the domain and range of the circle with equation $(x + 3)^2 + (y - 2)^2 = 16$ and sketch the graph.

b. Find the centre, radius, domain and range of the circle with equation $2x^2 + 2y^2 + 12x - 4y + 3 = 0$.

THINK

a. 1. Identify the centre and radius from the equation of the circle.

2. Use the x-coordinates of the endpoints of the horizontal diameter to state the domain.

3. Use the y-coordinates of the endpoints of the vertical diameter to state the range.

4. Sketch the circle using the endpoints of the domain and range, and the centre.

WRITE

a. $(x + 3)^2 + (y - 2)^2 = 16$

Centre $(-3, 2)$; radius $\sqrt{16} = 4$

Domain:

$x \in [h - r, h + r]$

$x \in [-3 - 4, -3 + 4]$

Therefore, the domain is $[-7, 1]$.

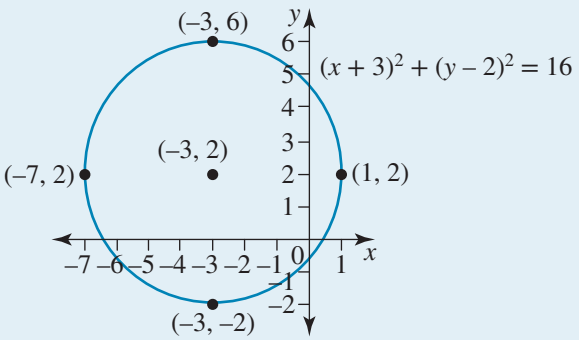
Range:

$y \in [k - r, k + r]$

$y \in [2 - 4, 2 + 4]$

Therefore, the range is $[-2, 6]$.

Circle has centre $(-3, 2)$ and contains the points $(-7, 2)$, $(1, 2)$, $(-3, -2)$, $(-3, 6)$.



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b. 1. Express the equation in the form where the coefficients of x^2 and y^2 are both 1.

b. $2x^2 + 2y^2 + 12x - 4y + 3 = 0$

Divide both sides by 2.

$\therefore x^2 + y^2 + 6x - 2y + \frac{3}{2} = 0$

2. Group the terms in x together and the terms in y together, and complete the squares.

$$x^2 + 6x + y^2 - 2y = -\frac{3}{2}$$

$$(x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - 1 = -\frac{3}{2}$$

$$(x + 3)^2 + (y - 1)^2 = -\frac{3}{2} + 9 + 1$$

$$(x + 3)^2 + (y - 1)^2 = \frac{17}{2}$$

3. State the centre and radius.

Centre $(-3, 1)$; radius $\sqrt{\frac{17}{2}} = \frac{\sqrt{34}}{2}$

4. State the domain and range.

Domain $\left[-3 - \frac{\sqrt{34}}{2}, -3 + \frac{\sqrt{34}}{2}\right]$

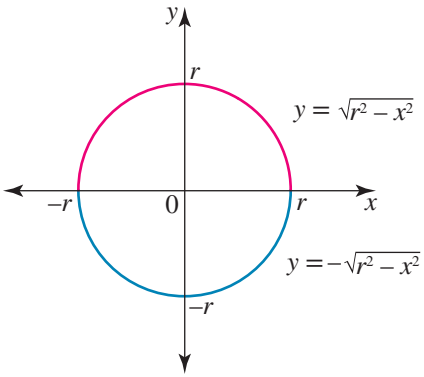
Range $\left[1 - \frac{\sqrt{34}}{2}, 1 + \frac{\sqrt{34}}{2}\right]$

6.3.2 Semicircles

The equation of the circle $x^2 + y^2 = r^2$ can be rearranged to make y the subject.

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$



The equation of the circle can be expressed as $y = \pm\sqrt{r^2 - x^2}$. This form of the equation indicates two semicircle functions which together make up the whole circle.

For $y = +\sqrt{r^2 - x^2}$, the y-coordinates must be positive (or zero) so this is the equation of the **semicircle** which lies above the x-axis.

For $y = -\sqrt{r^2 - x^2}$, the y-coordinates must be negative (or zero) so this is the equation of the semicircle which lies below the x-axis.

on Resources

Interactivity: Circles, semicircles and regions (int-2571)

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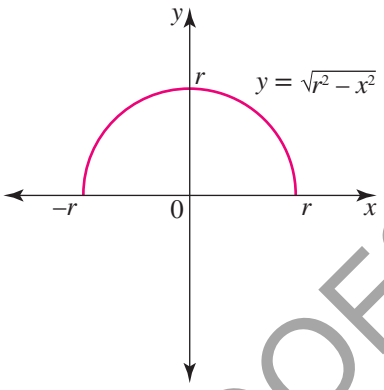
6.3.3 The semicircle $y = \sqrt{r^2 - x^2}$

The semicircle with equation $y = \sqrt{r^2 - x^2}$ is a function with a many-to-one correspondence. It is the top half of the circle, with centre (0, 0), radius r, domain $[-r, r]$ and range $[0, r]$.

The domain can be deduced algebraically since $\sqrt{r^2 - x^2}$ is only real if $r^2 - x^2 \geq 0$. From this the domain requirement $-r \leq x \leq r$ can be obtained.

For the circle with centre (h, k) and radius r, rearranging its equation $(x - h)^2 + (y - k)^2 = r^2$ gives the equation of the top, or upper, semicircle as $y = \sqrt{r^2 - (x - h)^2} + k$.

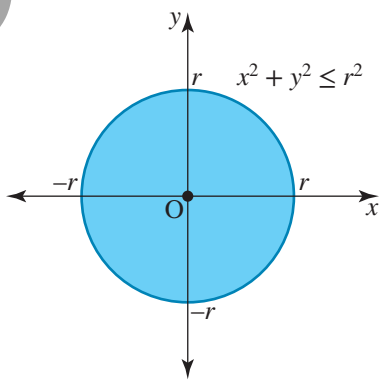


For the circle with centre (h, k), the equation of the semicircle

- **above the x-axis is $y = \sqrt{r^2 - (x - h)^2} + k$**
- **below the x-axis is $y = \sqrt{r^2 - (x - h)^2} + k$**

Regions

The region defined by $x^2 + y^2 \leq r^2$ could be determined by testing if $(0, 0)$ satisfies the inequation. Substitution gives the true statement that $0 \leq r^2$. The region is therefore the closed interior of the circle and includes its circumference. The region on or outside the circle is defined by the inequation $x^2 + y^2 \geq r^2$.



WORKED EXAMPLE 5

a. Sketch the graph of $y = \sqrt{5 - x^2}$ and state the domain and range.
b. Sketch $\{(x, y) : 4x^2 + 4y^2 < 1\}$.
c. For the circle with equation $4x^2 + 4y^2 = 1$, give the equation of its lower semicircle and state its domain and range.

THINK	WRITE
a. 1. State the centre and radius of the circle this semicircle is part of.	**a.** $y = \sqrt{5 - x^2}$ is the equation of a semicircle in the form of $y = \sqrt{r^2 - x^2}$. Centre: (0, 0) Radius: $r^2 = 5 \Rightarrow r = \sqrt{5}$ since r cannot be negative.

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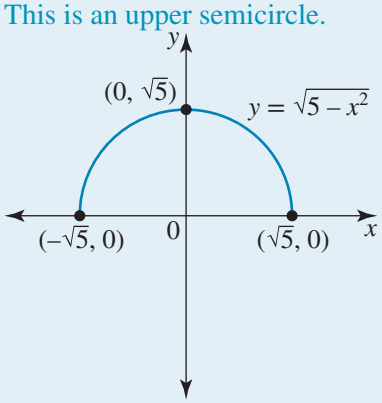
2. Sketch the graph.

This is an upper semicircle.



3. Read from the graph its domain and range.

Domain $[-\sqrt{5}, \sqrt{5}]$; range $[0, \sqrt{5}]$

b. 1. State the equation of the circle which is a boundary of this region.

b. $4x^2 + 4y^2 < 1$ represents the open region inside the circle $4x^2 + 4y^2 = 1$.

2. Express the equation in the form $x^2 + y^2 = r^2$.

$4x^2 + 4y^2 = 1$

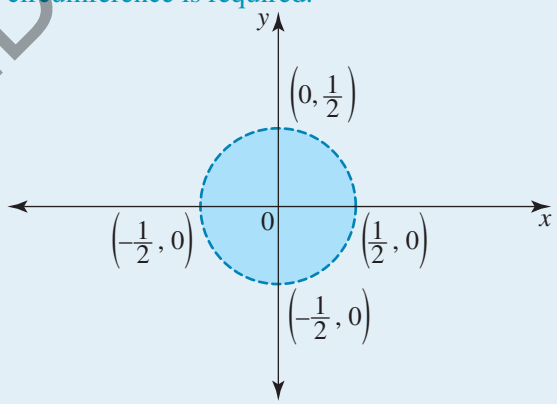
$$x^2 + y^2 = \frac{1}{4}$$

3. State the centre and radius of the circle.

Centre (0, 0); radius $r = \sqrt{\frac{1}{4}} = \frac{1}{2}$

4. Sketch the circle with an open boundary and shade the required region.

An open region inside the circle but not including the circumference is required.



c. 1. Rearrange the equation of the circle to make y the subject and state the equation of the lower semicircle.

c. $4x^2 + 4y^2 = 1$

Rearrange:

$4y^2 = 1 - 4x^2$

$$y^2 = \frac{1 - 4x^2}{4}$$

$$y = \pm\sqrt{\frac{1}{4} - x^2}$$

Therefore the lower semicircle has the equation

$y = -\sqrt{\frac{1}{4} - x^2}$.

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2. Alternatively, or as a check, use the centre and radius already identified in part **b**.

Check: using part **b**, the centre of the circle has coordinates (0, 0) and $r = \frac{1}{2}$. The lower semicircle has equation $y = -\sqrt{r^2 - x^2}$ with $r = \frac{1}{2}$.

$\therefore y = -\sqrt{\frac{1}{4} - x^2}$

or:

$y = -\sqrt{\frac{1 - 4x^2}{4}}$

$= -\frac{\sqrt{1 - 4x^2}}{2}$

$= -\frac{1}{2}\sqrt{1 - 4x^2}$

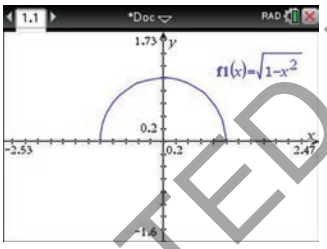
3. State the domain and range.

The domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This is the lower semicircle, so the range is $\left[-\frac{1}{2}, 0\right]$.

TI | THINK

a. 1. On a Graphs page, complete the entry line as:
$f1(x) = \sqrt{5 - x^2}$
Then press ENTER.
Press MENU and select:
4. Window/Zoom
3. Zoom in …
Then press ENTER.
Note: For a semi-circle, the graphing window must be square.

WRITE



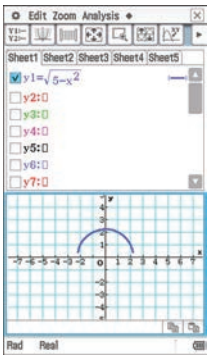
2. State the domain and range.

Domain is $[\sqrt{-5}, \sqrt{5}]$
Range is $[0, \sqrt{5}]$

CASIO | THINK

a. 1. On the Graphs & Table screen, complete the entry line as:
$y1(x) = \sqrt{5 - x^2}$
Then press EXE.
Note: In reality the graph touches the x-axis.

WRITE



2. State the domain and range.

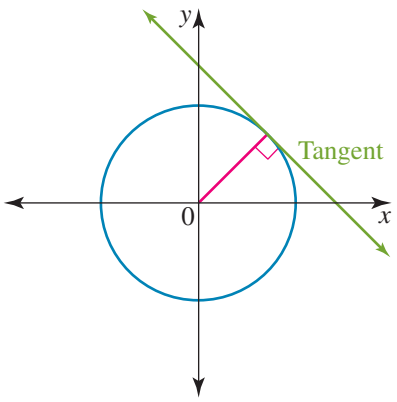
Domain is $[\sqrt{-5}, \sqrt{5}]$
Range is $[0, \sqrt{5}]$

6.3.4 Tangents to circles

A line and a circle can intersect in 2, 1 or 0 places. The coordinates of any points of intersection are found using simultaneous equations. If there are 2 points of intersection, the line is called a **secant**; a line segment joining these points is called a **chord**.

If there is exactly 1 point of intersection, the line is a **tangent** touching the circle at that point of contact. This tangent line is perpendicular to the radius drawn to the point of contact.

The gradient of the tangent and the gradient of the radius drawn to the point of contact must satisfy the relationship $m_{\text{tangent}} \times m_{\text{radius}} = -1$. Other coordinate geometry formulae may be required to determine the equation of a tangent or to calculate the length of a segment of the tangent.



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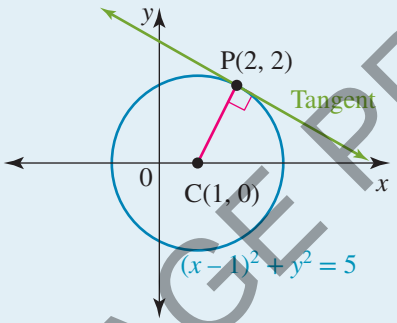
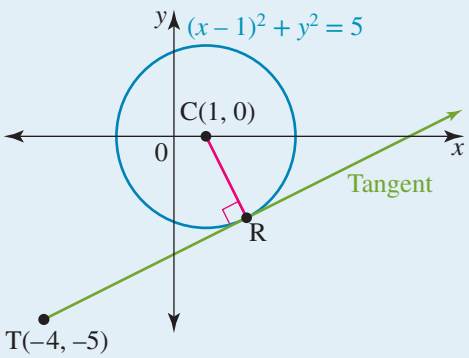
WORKED EXAMPLE 6

For the circle with equation $(x-1)^2+y^2=5$, determine:
a. the equation of the tangent at the point (2, 2) on the circumference of the circle
b. the length of the tangent drawn from the point (−4, −5) to its point of contact with the circle, R.
c. the number of intersections the line $y+3x+4=0$ makes with the circle.

THINK	WRITE
a. 1. State the centre and radius of the circle.	**a.** $(x-1)^2+y^2=5$ Centre $(1, 0)$ Radius: $r^2=5 \Rightarrow r=\sqrt{5}$
2. Draw a sketch of the circle and the tangent at the given point.	 Let C be the centre $(1, 0)$ and P the point $(2, 2)$.
3. Calculate the gradient of the tangent.	The tangent is perpendicular to the radius CP. $m_{CP}=\frac{2-0}{2-1}$ $=2$ As $m_{\text{tangent}} \times m_{CP}=-1, m_{\text{tangent}}=-\frac{1}{2}$.
4. Form the equation of the tangent line.	Equation of the tangent line: $y-y_1=m(x-x_1), m=-\frac{1}{2}, (x_1, y_1)=(2, 2)$ $y-2=-\frac{1}{2}(x-2)$ $y=-\frac{1}{2}x+3$
b. 1. Sketch the circle and a tangent drawn to the circle from the given point. *Note:* Two tangents can be drawn to the circle from a given point but the required length will be the same in either case.	**b.**  Let T be the point $(-4, -5)$ and R the point of contact with the circle.

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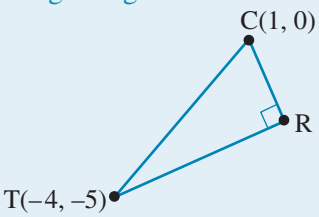
The centre C is the point (1, 0). The tangent is perpendicular to the radius CR, so the triangle CRT is right-angled.



2. Calculate the distance between the centre and the given external point.

Using the formula for the distance between two points:

$$TC = \sqrt{(-4-1)^2 + (-5-0)^2}$$
$$= \sqrt{25+25}$$
$$= \sqrt{50}$$

3. Calculate the required length of the tangent.

$r = \sqrt{5} \Rightarrow CR = \sqrt{5}$

Let TR = t, $t > 0$

Using Pythagoras' theorem: $(\sqrt{50})^2 = (\sqrt{5})^2 + t^2$

$$\therefore 50 = 5 + t^2$$
$$\therefore t^2 = 45$$
$$\therefore t = 3\sqrt{5}$$

4. State the answer.

The length of the tangent from the external point to its point of contact with the circle is $3\sqrt{5}$ units.

c. 1. Set up a system of simultaneous equations.

c. $y + 3x + 4 = 0$ [1]

$(x-1)^2 + y^2 = 5$ [2]

2. Use the substitution method to form the quadratic equation which determines the number of solutions.

From equation [1], $y = -3x - 4$.

Substitute this into equation [2].

$$(x-1)^2 + (-3x-4)^2 = 5$$
$$\therefore x^2 - 2x + 1 + 9x^2 + 24x + 16 = 5$$
$$\therefore 10x^2 + 22x + 12 = 0$$
$$\therefore 5x^2 + 11x + 6 = 0$$

3. Calculate the discriminant.

$\Delta = b^2 - 4ac \quad a = 5,\ b = 11,\ c = 6$

$$= (11)^2 - 4(5)(6)$$
$$= 121 - 120$$
$$= 1$$

4. State the answer.

Since $\Delta > 0$, there are two points of intersection.

UNCORRECTED PAGE PROOFS

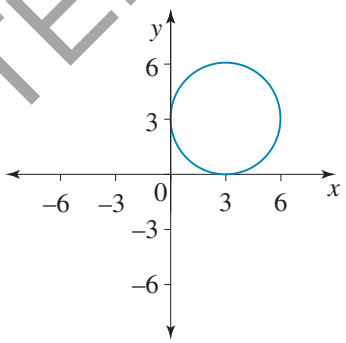
Resources

Interactivity: Tangents to a circle (int-2572)

Exercise 6.3 The circle

Technology free

1. A circle has the equation $(x-5)^2+(y-1)^2=16$.
   a. State the coordinates of its centre.
   b. What is its radius?
   c. Determine its domain and range.
   d. Sketch its graph.
2. The equation of a circle is $(x+4)^2+(y+3)^2=9$.
   a. State the coordinates of its centre.
   b. What is its radius?
   c. Determine its domain and range.
   d. Sketch its graph.
3. The equation of a circle is $4x^2+4y^2=25$.
   a. State the coordinates of its centre
   b. What is its radius?
   c. Determine its domain and range.
   d. Sketch its graph.
4. The equation of a circle is $(x+11)^2+y^2=49$.
   a. State its centre and radius.
   b. Determine its domain and range.
   c. Find the coordinates of the two points for which the x-coordinate is -8.
   d. Hence, explain why the circle is not a function.
5. a. Write down the equation of the circle with centre $(9,-2)$ and radius 9.
   b. Form the equation of the circle with centre $(0,10)$ and radius $\frac{2}{3}$.
   c. Form the equation of the circle shown in the diagram.



6. WE4 a. State the domain and range of the circle with equation $(x-1)^2+(y+3)^2=9$ and sketch the graph.
   b. Find the centre, radius, domain and range of the circle with equation $x^2+y^2+2x+8y=0$.
7. Sketch the following circles and state the centre, radius, domain and range of each.
   a. $x^2+(y-1)^2=1$
   b. $(x+2)^2+(y+4)^2=9$
   c. $16x^2+16y^2=81$
   d. $x^2+y^2-6x+2y+6=0$
   e. $16x^2+16y^2-16x-16y+7=0$
   f. $(2x+6)^2+(6-2y)^2=4$

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8. Form the equations of the following circles from the given information.
   a. Centre $(-8, 9)$; radius 6
   b. Centre $(7, 0)$; radius $2\sqrt{2}$
   c. Centre $(1, 6)$ and containing the point $(-5, -4)$
   d. Endpoints of a diameter are $\left(-\frac{4}{3}, 2\right)$ and $\left(\frac{4}{3}, 2\right)$.
9. A circle with centre $(-5, 0)$ passes through the point $(2, 3)$. Determine its equation and express it in general form.
10. a. Show that the equation of the circle with centre $(3, 5)$ and radius $\sqrt{5}$ can be expressed in expanded form as $x^2 + y^2 - 6x - 10y + 29 = 0$.
    b. Show that the equation of the circle with centre $\left(-4, -\frac{1}{2}\right)$ and radius 1 can be expressed in expanded form as $4x^2 + 4y^2 + 32x + 4y + 61 = 0$.
    c. By completing the square on both x and y, obtain the centre and radius of the circle $x^2 + y^2 + 2x - 4y - 2 = 0$.
    d. By completing the square on x, obtain the centre and radius of the circle $x^2 + y^2 - 8x - 20 = 0$.
11. WE5 a. Sketch the graph of $y = -\sqrt{2 - x^2}$, stating the domain and range.
    b. Sketch $\{(x, y) : 4x^2 + 4y^2 > 25\}$.
    c. For the circle with equation $4x^2 + 4y^2 = 25$, give the equation of its upper semicircle and state its domain and range.
12. Sketch the following relations, state the domain and range, and express the functions, if any, as mappings.
    a. $\{(x, y) : y = \sqrt{36 - x^2}\}$
    b. $\{(x, y) : y = -\sqrt{0.25 - x^2}\}$
    c. $\{(x, y) : y = \sqrt{1 - x^2} + 3\}$
    d. $\{(x, y) : x^2 + y^2 < 5\}$
    e. $\{(x, y) : (x - 2)^2 + (y - 4)^2 \geq 4\}$
    f. $\{(x, y) : y \leq \sqrt{4 - x^2}\}$
13. a. Does $(3, -3)$ lie on, inside or outside the circle $x^2 + y^2 - 3x + 3y + 3 = 0$?
    b. Find the two values of a so that the point $(a, 2)$ lies on the circle $x^2 + y^2 + 8x - 3y + 2 = 0$, and identify the equation of the semicircle (of the given circle) on which these points lie.
14. A semicircle has the equation $y = 2 + \sqrt{8 - 4x - x^2}$.
    a. Identify the equation of the circle of which it is part.
    b. State the domain and range of the semicircle.
15. a. Calculate the coordinates of the points of intersection of the line $y = 2x$ and the circle $(x - 2)^2 + (y - 2)^2 = 1$.
    b. Calculate the coordinates of the points of intersection of $y = 7 - x$ with the circle $x^2 + y^2 = 49$. On a diagram, sketch the region $\{(x, y) : y \geq 7 - x\} \cap \{(x, y) : x^2 + y^2 \leq 49\}$.
16. Determine the value(s) of m so that $y = mx - 3$ is a tangent to the circle $x^2 + y^2 = 4$.
17. a. Show that the line $y + 2x = 5$ is a tangent to the circle $x^2 + y^2 = 5$ and calculate the coordinates of the point of contact.
    b. Determine the value of k so the line $y = kx + 2$ and the circle $x^2 + y^2 + 5x - 4y + 10 = 0$ have:
       i. one intersection
       ii. two intersections
       iii. no intersections.

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18. WE6 For the circle with equation $(x+2)^2+(y-1)^2=10$, determine:

- **a.** the equation of the tangent at the point $(-3, -2)$ on the circumference of the circle
- **b.** the length of the tangent drawn from the point $(6, 0)$ to its point of contact with the circle
- **c.** the number of intersections the line $y+2x-5=0$ makes with the circle.

Technology active

19. Consider the circle defined by $x^2+y^2-6x+4y-12=0$.

- **a.** Specify its centre and radius.
- **b.** **i.** Show that the point $(-1, 1)$ lies on the circle.
  - **ii.** Find the equation of the tangent drawn to the circle at the point $(-1, 1)$.
- **c.** **i.** Show that the point $(3, 3)$ lies on the circle.
  - **ii.** What is the equation of the tangent at the point $(3, 3)$?
- **d.** The tangent drawn from the point T$(4, 10)$ meets the circle at a point R. Calculate the length of TR.
- **e.** Deduce the length of the tangent drawn from the point $(8, -7)$ to where it meets the circle.
- **f.** Obtain the coordinates of the point of intersection of the tangent in part **b** with the tangent in part **e**.

20. Consider the circle with equation $x^2+y^2-2x-4y-20=0$.

- **a.** Calculate the exact length of the intercept, or chord, cut off on the x-axis by the circle.
- **b.** Using clearly explained mathematical analysis, calculate the exact distance of the centre of the circle from the chord joining the points $(5, -1)$ and $(4, 6)$.

21. A circle passes through the three points $(1, 0)$, $(0, 2)$ and $(0, 8)$. The general equation of the circle is $x^2+y^2+ax+by+c=0$.

- **a.** Calculate the values of a, b and c.
- **b.** Determine the coordinates of the centre and the length of the radius.
- **c.** Sketch the circle labelling all intercepts with the coordinate axes with their coordinates.
- **d.** Calculate the shortest distance from the origin to the circle, giving your answer correct to 2 decimal places.
- **e.** What is the greatest distance from the origin to the circle? Express the answer correct to 2 decimal places.

22. **a.** Sketch the circle $x^2+y^2=16$ and two other circles that also pass through both of the points $(0, 4)$ and $(4, 0)$. Why is it possible for several circles to pass through these two points?

- **b.** Find the equation of the circle for which the points $(0, 4)$ and $(4, 0)$ are the endpoints of a diameter and show that this circle passes through the origin.
- **c.** Relative to a fixed origin O on the edge of a circular lake, the circumference of the circular shoreline passes through O and the points 4 km due north of O and 4 km due east of O, so that the equation of its circumference is that of the circle obtained in part **b**.

  Two friends, Sam and Rufus, have a competition to see who can first reach kiosk K which lies at the intersection of the line $y=x$ with the circle.

  - **i.** Find the coordinates of K.
  - **ii.** Sam decides to swim from O directly to K along a straight line while Rufus claims it would be faster to walk around the lake from O to K. Given that Sam swims at 2.5 km/h while Rufus walks at 4 km/h, who reaches the kiosk first and by how many minutes?

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23. Use the conic screen on a CAS calculator, or other technology, to sketch the circle $x^2 + y^2 + 4x - 7y + 2 = 0$, locating the intercepts with the coordinate axes to 3 decimal places and stating the centre and radius.
24. Sketch $(x - 6)^2 + (y + 4)^2 = 66$ and determine the endpoints of the domain and range to 3 decimal places. Why does CAS say the gradient of the tangent is undefined at the endpoints of the domain?

6.4 The rectangular hyperbola and the truncus

The family of functions with rules $y = x^n, n \in N$ are the now familiar polynomial functions. If n is a negative number, however, these functions cannot be polynomials. Here we consider the functions with rule $y = x^n$ where $n = -1$ and $n = -2$.

6.4.1 The graph of $y = \frac{1}{x}$

With $n = -1$, the rule $y = x^{-1}$ can also be written as $y = \frac{1}{x}$. This is the rule for a **rational function** called a **hyperbola**. Two things can be immediately observed from the rule:

- $x = 0$ must be excluded from the domain, since division by zero is not defined.
- $y = 0$ must be excluded from the range, since there is no number whose reciprocal is zero.

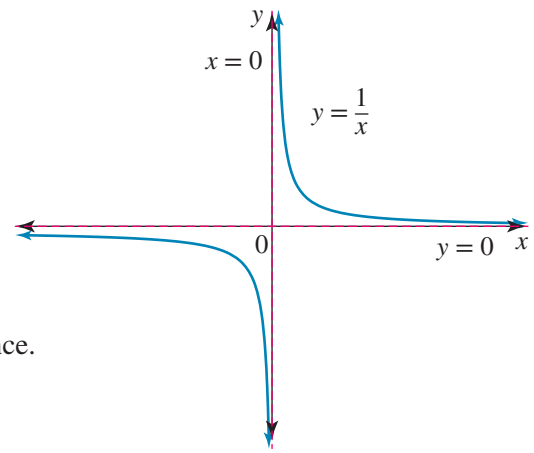
The lines $x = 0$ and $y = 0$ are **asymptotes**. An asymptote is a line the graph will approach but never reach. As these two asymptotes $x = 0$ and $y = 0$ are a pair of perpendicular lines, the hyperbola is known as a **rectangular hyperbola**. The asymptotes are a key feature of the graph of a hyperbola.

Completing a table of values can give us a 'feel' for this graph.

x	-10	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	10
y	$-\frac{1}{10}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-10	No value possible	10	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$

The values in the table illustrate that as $x \to \infty, y \to 0$ and as $x \to -\infty, y \to 0$.
The table also illustrates that as $x \to 0$, either $y \to -\infty$ or $y \to \infty$.
These observations describe the asymptotic behaviour of the graph.

The graph of the basic hyperbola $y = \frac{1}{x}$ is shown.



Key features:

- Vertical asymptote has equation $x = 0$ (the y-axis).
- Horizontal asymptote has equation $y = 0$ (the x-axis).
- Domain is $R \setminus \{0\}$.
- Range is $R \setminus \{0\}$.
- As $x \to \infty, y \to 0$ from above and as $x \to -\infty, y \to 0$ from below. This can be written as: $x \to \infty, y \to 0^+$ and as $x \to -\infty, y \to 0^-$.
- As $x \to 0^-, y \to -\infty$ and as $x \to 0^+, y \to \infty$.
- The graph is of a function with a one-to-one correspondence.
- The graph has two branches separated by the asymptotes.
- As the two branches do not join at $x = 0$, the function is said to be **discontinuous** at $x = 0$.
- The graph lies in **quadrants** 1 and 3 as defined by the asymptotes.

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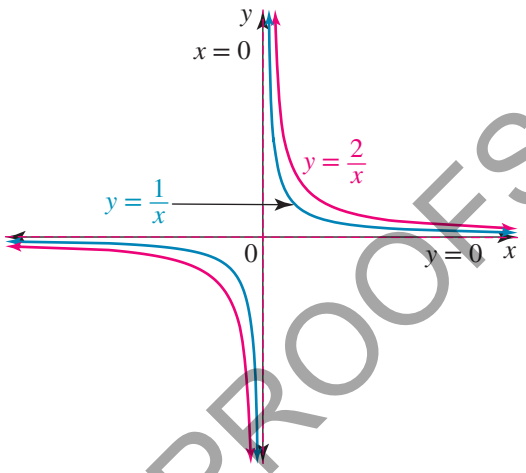
The asymptotes divide the Cartesian plane into four areas or quadrants. The quadrants formed by the asymptotes are numbered 1 to 4 anticlockwise.

With the basic shape of the hyperbola established, transformations of the graph of $y = \frac{1}{x}$ can be studied.



Dilation from the *x*-axis

The effect of a dilation factor on the graph can be illustrated by comparing $y = \frac{1}{x}$ and $y = \frac{2}{x}$.

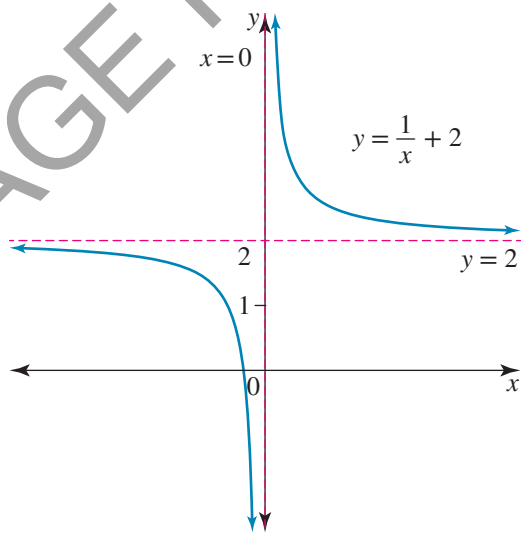
For $x = 1$ the point $(1, 1)$ lies on $y = \frac{1}{x}$ whereas the point $(1, 2)$ lies on $y = \frac{2}{x}$. The dilation effect on $y = \frac{2}{x}$ is to move the graph further out from the x-axis. The graph has a dilation factor of 2 from the x-axis.

Vertical translation

The graph of $y = \frac{1}{x} + 2$ illustrates the effect of a vertical translation of 2 units upwards.

Key features:

- The horizontal asymptote has equation $y = 2$.
  This means that as $x \to \pm\infty, y \to 2$.
- The vertical asymptote is unaffected and remains $x = 0$.
- Domain is $R \setminus \{0\}$.
- Range is $R \setminus \{2\}$.



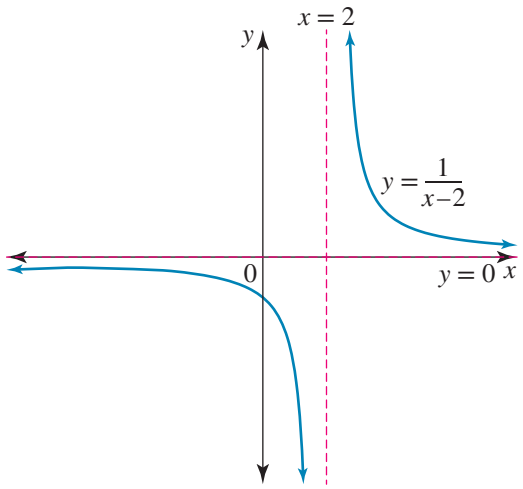
Horizontal translation

For $y = \frac{1}{x-2}$ as the denominator cannot be zero, $x - 2 \neq 0 \Rightarrow x \neq 2$. The domain must exclude $x = 2$, so the line $x = 2$ is the vertical asymptote.

Key features:

- Vertical asymptote has equation $x = 2$.
- Horizontal asymptote is unaffected by the horizontal translation and still has the equation $y = 0$.
- Domain is $R \setminus \{2\}$.
- Range is $R \setminus \{0\}$.

The graph of $y = \frac{1}{x-2}$ demonstrates the same effect that we have seen with other graphs that are translated 2 units to the right.



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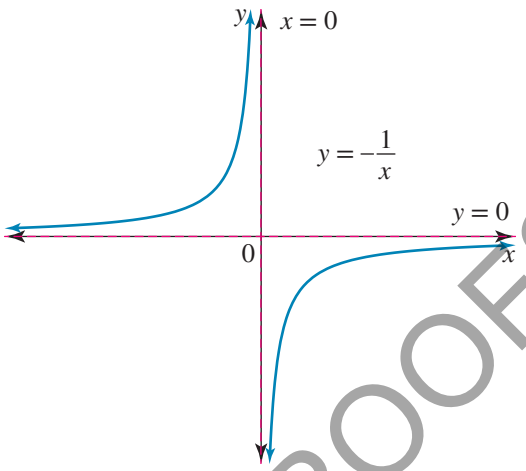
Reflection in the *x*-axis

The graph of $y = -\frac{1}{x}$ illustrates the effect of inverting the graph by reflecting $y = \frac{1}{x}$ in the *x*-axis.

The graph of $y = -\frac{1}{x}$ lies in quadrants 2 and 4 as defined by the asymptotes.



Resources

Interactivity: The rectangular hyperbola (int-2573)

6.4.2 General equation of a hyperbola

The general equation of the hyperbola:

$$y = \frac{a}{x-h} + k$$

Key features:

- Vertical asymptote has the equation $x = h$
- Horizontal asymptote has the equation $y = k$
- Domain is $R \setminus \{h\}$
- Range is $R \setminus \{k\}$
- Asymptotic behaviour: as $x \to \pm\infty, y \to k$ and as $x \to h, y \to \pm\infty$
- There are two branches to the graph and the graph is discontinuous at $x = h$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 3.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4.
- $|| a ||$ gives the dilation factor from the *x*-axis.

If the equation of the hyperbola is in the form $y = \frac{a}{bx+c} + k$, then the vertical asymptote can be identified by finding the *x*-value for which the denominator term $bx + c = 0$. The horizontal asymptote is $y = k$ because as $x \to \pm\infty$, $\frac{a}{bx+c} \to 0$ and therefore $y \to k$.

WORKED EXAMPLE 7

Sketch the graphs of the following functions, stating the domain and range.

a. $y = \frac{-1}{x-2} + 1$

b. $y = \frac{4}{x} - 2$

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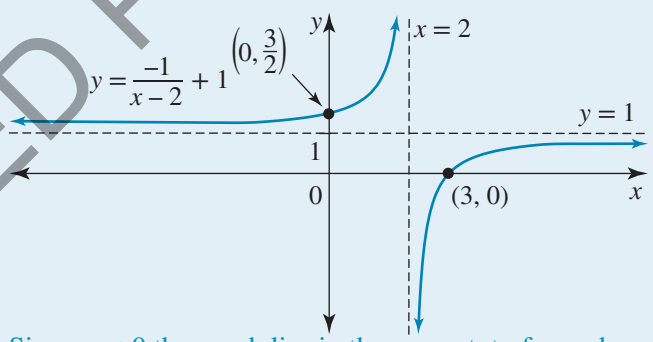
THINK	WRITE
a. 1. State the equations of the asymptotes.	**a.** $y = \dfrac{-1}{x-2} + 1$ Vertical asymptote occurs when $x - 2 = 0$. Vertical asymptote: $x = 2$ Horizontal asymptote occurs as $x \to \pm\infty$. Horizontal asymptote: $y = 1$
2. Calculate the coordinates of any axis intercepts.	y-intercept: let $x = 0$ $y = \dfrac{-1}{-2} + 1$ $= \dfrac{3}{2}$ $\left(0, \dfrac{3}{2}\right)$ is the y-intercept. x-intercept: let $y = 0$ $\dfrac{-1}{x-2} + 1 = 0$ $\dfrac{1}{x-2} = 1$ $x - 2 = 1$ $x = 3$ $(3, 0)$ is the x-intercept.
3. Sketch the graph ensuring it approaches but does not touch the asymptotes. *Note:* Check the graph has the shape anticipated.	 Since $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4, as expected.
4. State the domain and range.	The domain is $R\backslash\{2\}$; the range is $R\backslash\{1\}$.
b. 1. State the equations of the asymptotes.	**b.** $y = \dfrac{4}{x} - 2$ Vertical asymptote: $x = 0$ Horizontal asymptote: $y = -2$
2. Calculate the coordinates of any axes intercepts.	No y-intercept since the y-axis is the vertical asymptote x-intercept: let $y = 0$ $0 = \dfrac{4}{x} - 2$ $\dfrac{4}{x} = 2$ $4 = 2x$ $x = 2$ $(2, 0)$ is the x-intercept.

UNCORRECTED PAGE PROOFS

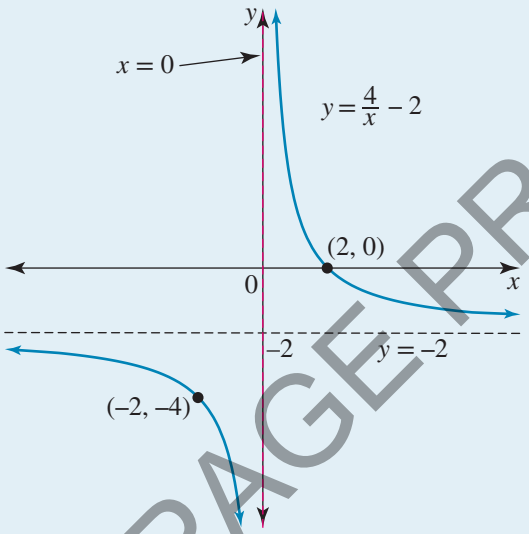
3. Anticipating the position of the graph, calculate the coordinates of a point so each branch of the graph will have a known point.

Graph lies in asymptote-formed quadrants 1 and 3.
A point to the left of the vertical asymptote is required.
Point: let $x = -2$

$$y = \frac{4}{-2} - 2$$
$$= -4$$

$(-2, -4)$ is a point.

4. Sketch the graph.



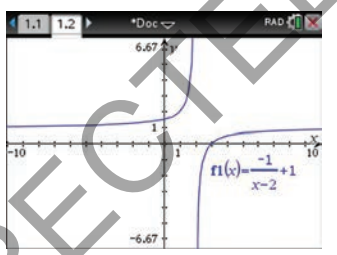
5. State the domain and range.

Domain is $R\backslash\{0\}$; the range is $R\backslash\{-2\}$.

TI | THINK

a. 1. On a Graphs page, complete the entry line as:
$f1(x) = \frac{-1}{x-2} + 1$
Then press ENTER.

WRITE



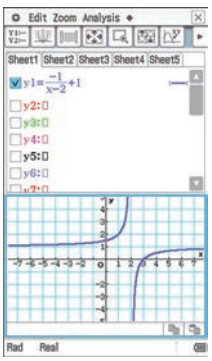
2. State the domain and range.

Domain is $R\backslash\{2\}$
Range is $R\backslash\{1\}$

CASIO | THINK

a. 1. On the Graphs & Table screen, complete the entry line as:
$y1 = \frac{-1}{x-2} + 1$
Then press EXE.

WRITE



2. State the domain and range.

Domain is $R\backslash\{2\}$
Range is $R\backslash\{1\}$

Proper rational functions

The equation of the hyperbola $y = \frac{a}{x-h} + k$ is expressed in **proper rational function** form. This means the rational term, $\frac{a}{x-h}$, has a denominator of a higher degree than that of the numerator. For example, $y = \frac{x-1}{x-2}$ is not expressed as a proper rational function: the numerator has the same degree as the denominator. Using division, it can be converted to the proper form $y = \frac{1}{x-2} + 1$ which is recognisable as a hyperbola and from which the asymptotes can be obtained.

UNCORRECTED PAGE PROOFS

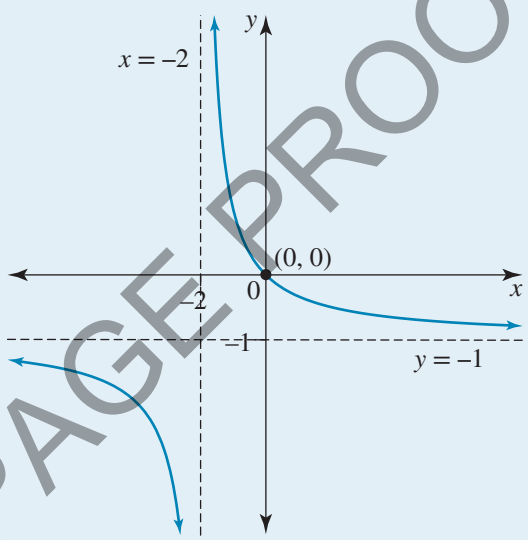
6.4.3 Forming the equation

From the equation $y = \frac{a}{x-h} + k$ it can be seen that three pieces of information will be needed to form the equation of a hyperbola. These are usually the equations of the asymptotes and the coordinates of a point on the graph.

WORKED EXAMPLE 8

a. Identify the asymptotes of the hyperbola with equation $y = \frac{2x-3}{5-2x}$.

b. Form the equation of the hyperbola shown.



THINK

a. 1. The equation is in improper form so reduce it to proper form using division.
Note: The long-division algorithm could also have been used to reduce the function to proper form.

2. State the equations of the asymptotes.

b. 1. Recall the general equation of the hyberbola, where the vertical asymptote is $x = h$, and the horizontal asymptote is $y = k$.

2. Substitute the equations of the asymptotes shown on the graph into the general equation of a hyperbola.

WRITE

a. $y = \frac{2x-3}{5-2x}$

$= \frac{-1(5-2x)+2}{5-2x}$

$= -1 + \frac{2}{5-2x}$

$y = \frac{2}{5-2x} - 1$ is the proper rational function form.

Vertical asymptote when $5 - 2x = 0$

$\therefore x = \frac{5}{2}$

Horizontal asymptote is $y = -1$.

b. The general equation of the graph is $y = \frac{a}{x-h} + k$.

From the graph, asymptotes have equations $x = -2, y = -1$, substituting into the general form gives

$\therefore y = \frac{a}{x+2} - 1$

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3. Use a known point on the graph to determine the remaining unknown constant.

Point (0, 0) lies on the graph.

$$0 = \frac{a}{2} - 1$$

$$\therefore 1 = \frac{a}{2}$$

$$\therefore a = 2$$

4. State the equation of the hyperbola.

The equation is $y = \frac{2}{x+2} - 1$.

6.4.4 Inverse proportion

The hyperbola is also known as the **inverse proportion** graph. To illustrate this, consider the time taken to travel a fixed distance of 60 km.

The time to travel a fixed distance depends on the speed of travel. For a distance of 60 km, the times taken for some different speeds are shown in the table.

Speed, v (km/h)	10	15	20	30
Time, t (hours)	6	4	3	2

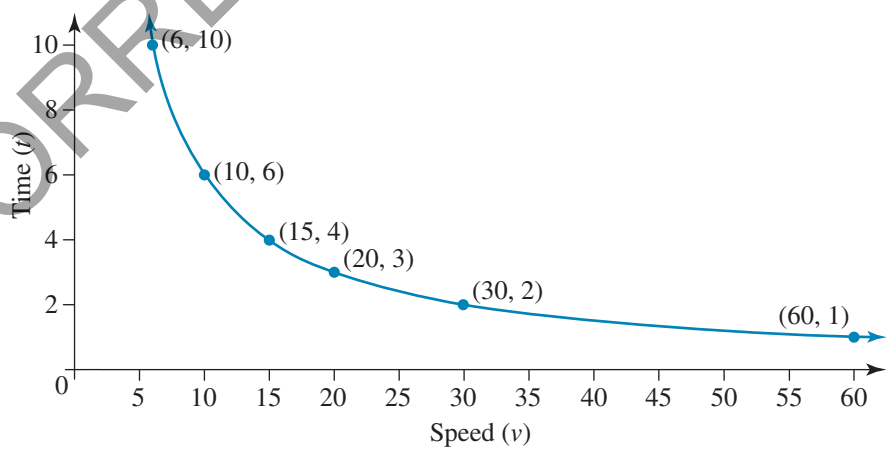
As the speed increases, the time will decrease; as the speed decreases, the time will increase. The time is inversely proportional to the speed, or the time varies inversely as the speed.

From the table:

$$t \times v = 60$$

$$\therefore t = \frac{60}{v}$$

This is the equation of a hyperbola where v is the independent variable and t the dependent variable.



The graph of time versus speed exhibits the asymptotic behaviour typical of a hyperbola. As the speed increases, becoming faster and faster, the time decreases, becoming smaller and smaller. In other words, as $v \to \infty, t \to 0$, but t can never reach zero nor can speed reach infinity. Further, as $v \to 0, t \to \infty$. Only one branch of the hyperbola is given since neither time nor speed can be negative.

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In general, the following rules apply.

- **'y is inversely proportional to x' is written as $y \propto \frac{1}{x}$.**
- **If y is inversely proportional to x, then $y = \frac{k}{x}$ where k is the constant of proportionality.**
- **This relationship can also be expressed as $xy = k$ so if the product of two variables is constant, the variables are in inverse proportion.**

If $y = \frac{k}{x}$, then it could also be said that y is directly proportional to $\frac{1}{x}$; the graph of y against $\frac{1}{x}$ is linear. The graph of y against x is a hyperbola.

Functions of variables may be in inverse proportion. For example, the strength of a radio signal I varies inversely as the square of the distance d from the transmitter, so $I = \frac{k}{d^2}$. The graph of I against d is not a hyperbola; it is called a **truncus** and is from the family $y = x^n, n = -2$.

WORKED EXAMPLE 9

Boyle's law says that if the temperature of a given mass of gas remains constant, its volume V is inversely proportional to the pressure P.
A container of volume 50 cm^3 is filled with a gas under a pressure of 75 cm of mercury.
a. Find the relationship between the volume and pressure.
b. The container is connected by a hose to an empty container of volume 100 cm^3. Find the pressure in the two containers.

THINK

a. 1. Write the rule for the inverse proportion relation.

2. Use the given data to find k and hence the rule.

b. 1. State the total volume.

WRITE

a. $V \propto \frac{1}{P}$

$\therefore V = \frac{k}{P}$

Substitute $V = 50$, $P = 75$.

$50 = \frac{k}{75}$

$k = 50 \times 75$

$= 3750$

Hence $V = \frac{3750}{P}$.

b. The two containers are connected and can be thought of as one. Therefore, the combined volume is $100 + 50 = 150$ cm^3.

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2. Calculate the pressure for this volume.

$$V = \frac{3750}{P}$$

When $V = 150$,

$$150 = \frac{3750}{P}$$

$$P = \frac{3750}{150}$$

$$= 25$$

3. State the answer.

The gas in the containers is under a pressure of 25 cm of mercury.

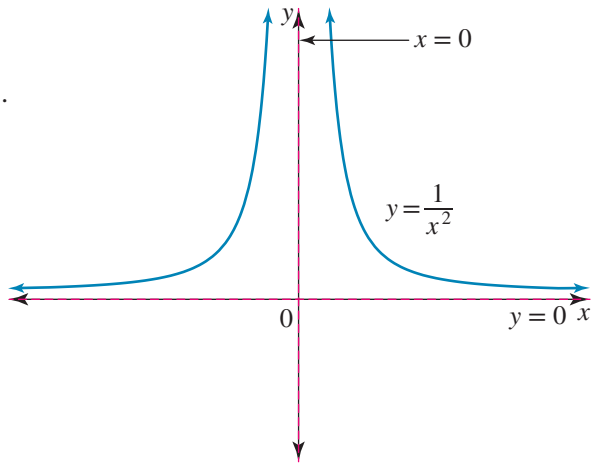
6.4.5 The graph of the truncus $y = \frac{1}{x^2}$

With $n = -2$, the rule $y = x^{-2}$ is also written as $y = \frac{1}{x^2}$. The rational function with this rule is called a truncus. Its graph shares similarities with the graph of the hyperbola $y = \frac{1}{x}$ as the rule shows that $x = 0$ and $y = 0$ are vertical and horizontal asymptotes, respectively.

A major difference between the two curves is that for a truncus, whether $x < 0$ or $x > 0$, the y-values must be positive. This means the graph of the truncus $y = \frac{1}{x^2}$ will have two branches, one in quadrant 1 and the other in quadrant 2. The quadrants are formed by the asymptote positions.

Key features:

- Vertical asymptote has equation $x = 0$ (the y-axis).
- Horizontal asymptote has equation $y = 0$ (the x-axis).
- Domain is $R \setminus \{0\}$.
- Range is R^+.
- As $x \to \pm\infty, y \to 0^+$.
- As $x \to 0, y \to \infty$.
- The graph is of a function with a many-to-one correspondence.
- The graph has two symmetric branches separated by the vertical asymptote.
- The graph lies in quadrants 1 and 2 as defined by the asymptotes.
- The function is discontinuous at $x = 0$.



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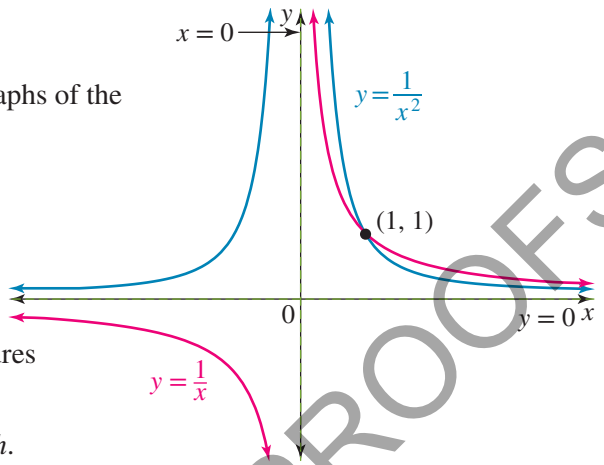
The branches of the truncus $y = \frac{1}{x^2}$ approach the horizontal asymptote more rapidly than those of the hyperbola $y = \frac{1}{x}$ because $\frac{1}{x^2} < \frac{1}{x}$, for $x > 1$. This can be seen when the graphs of the hyperbola and truncus are drawn on the same diagram.



Under a dilation of factor a from the x axis and translations of h units horizontally and k units vertically, $y = \frac{1}{x^2}$ becomes $y = \frac{a}{(x-h)^2} + k$.

From this general form, $y = \frac{a}{(x-h)^2} + k$, the key features can be identified:

- There is a vertical asymptote with the equation $x = h$.
- There is a horizontal asymptote with the equation $y = k$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 2.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 3 and 4.
- Domain is $R \setminus \{h\}$.
- If $a > 0$, range is (k, ∞); if $a < 0$, range is $(-\infty, k)$.

WORKED EXAMPLE 10

Sketch the graphs of the following functions, stating the domain and range.

a. $y = \frac{1}{(x+1)^2} + 2$ **b.** $y = 1 - \frac{4}{(x-2)^2}$

THINK

a. 1. State the equations of the asymptotes.

2. Calculate the coordinates of any axis intercepts.

3. Sketch the graph ensuring that each branch has at least one known point.

WRITE

a. $y = \frac{1}{(x+1)^2} + 2$

Vertical asymptote occurs when $(x+1)^2 = 0$.

$x + 1 = 0$

$\therefore x = -1$

Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 2$

y-intercept: let $x = 0$

$y = \frac{1}{(1)^2} + 2$

$= 3$

$(0, 3)$ is the y-intercept.

Since $a > 0$ the graph lies above the horizontal asymptote, $y = 2$.

Hence, there is no x-intercept.

Let $x = -2$.

$y = \frac{1}{(-1)^2} + 2$

$= 3$

$(-2, 3)$ lies on the graph.

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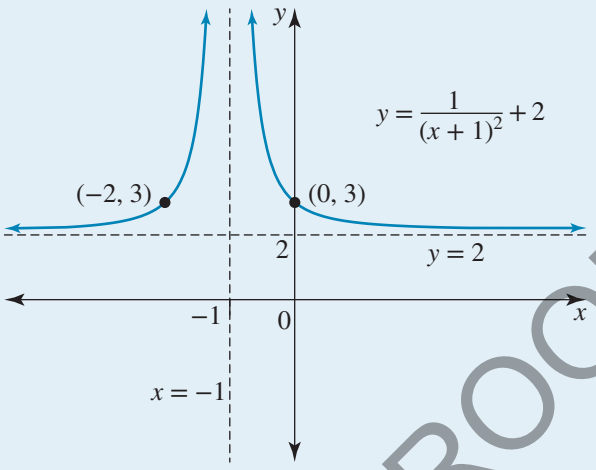


4. State the domain and range.

Domain is $R \setminus \{-1\}$; range is $(2, \infty)$.

b. 1. State the equations of the asymptotes.

b. $y = 1 - \frac{4}{(x-2)^2}$

$= \frac{-4}{(x-2)^2} + 1$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 1$

2. Calculate the coordinates of any axis intercepts.
Note: The symmetry of the graph about its vertical asymptote may enable key points to be identified.

y-intercept: let $x = 0$

$y = \frac{-4}{(-2)^2} + 1$

$= 0$

The graph passes through the origin $(0, 0)$.

x-intercepts: let $y = 0$

$\frac{-4}{(x-2)^2} + 1 = 0$

$\frac{4}{(x-2)^2} = 1$

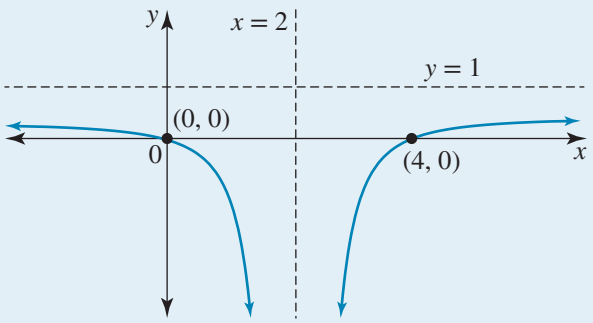
$(x-2)^2 = 4$

$x - 2 = \pm 2$

$\therefore x = 0, x = 4$

$(4, 0)$ is also an x-intercept.

3. Sketch the graph ensuring that each branch has at least one known point.



4. State the domain and range.

Domain $R \setminus \{2\}$; range $(-\infty, 1)$

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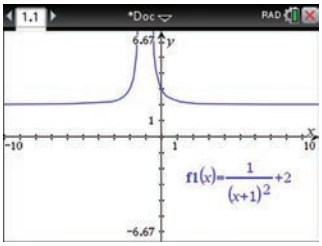
TI | THINK

a. 1. On a Graphs page, complete the entry line as:

$f1(x) = \dfrac{1}{(x+1)^2} + 2$

Then press ENTER.

WRITE



2. State the domain and range.

Domain is $R \setminus \{-1\}$
Range is $(2, \infty)$

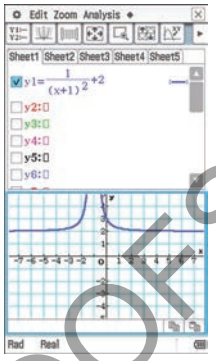
CASIO | THINK

a. 1. On the Graphs & Table screen, complete the entry line as:

$f1(x) = \dfrac{1}{(x+1)^2} + 2$

Then press EXE.

WRITE



2. State the domain and range.

Domain is $R \setminus \{-1\}$
Range is $(2, \infty)$

6.4.6 Modelling with the hyperbola and truncus

As for inverse proportionality, practical applications involving hyperbola or truncus models may need domain restrictions. Unlike many polynomial models, neither the hyperbola nor the truncus has maximum and minimum turning points, so the asymptotes are often where the interest will lie. The horizontal asymptote is often of particular interest as it represents the limiting value of the model.

WORKED EXAMPLE 11

A relocation plan to reduce the number of bats in a public garden is formed and t months after the plan is introduced the number of bats N in the garden is thought to be modelled by $N = 250 + \dfrac{30}{t+1}$.

a. How many bats were removed from the garden in the first 9 months of the relocation plan?

b. Sketch the graph of the bat population over time using the given model and state its domain and range.

c. What is the maximum number of bats that will be relocated according to this model?

THINK

a. Find the number of bats at the start of the plan and the number after 9 months and calculate the difference.

WRITE

a. $N = 250 + \dfrac{30}{t+1}$

When $t = 0$, $N = 250 + \dfrac{30}{1}$.

Therefore there were 280 bats when the plan was introduced.

When $t = 9$, $N = 250 + \dfrac{30}{10}$.

Therefore 9 months later there were 253 bats.

Hence, over the first 9 months, 27 bats were removed.

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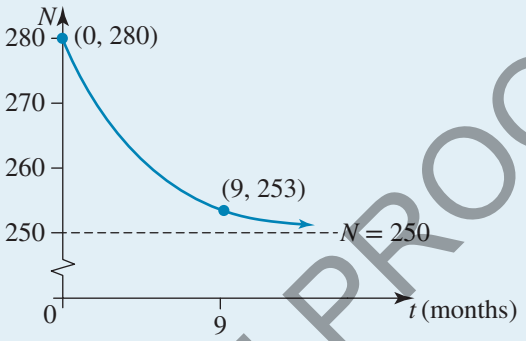
b. 1. Identify the asymptotes and other key features which are appropriate for the restriction $t \geq 0$.

b. $N = 250 + \dfrac{30}{t+1}, t \geq 0$
Vertical asymptote $t = -1$ (not applicable)
Horizontal asymptote $N = 250$
Initial point is (0, 280).

2. Sketch the part of the graph of the hyperbola that is applicable and label axes appropriately.
Note: The vertical scale is broken in order to better display the graph.



3. State the domain and range for this model.

Domain $\{t : t \geq 0\}$
Range (250, 280]

c. 1. Interpret the meaning of the horizontal asymptote.

c. The horizontal asymptote shows that as $t \to \infty, N \to 250$.
This means $N = 250$ gives the limiting population of the bats.

2. State the answer.

Since the population of bats cannot fall below 250 and there were 280 initially, the maximum number of bats that can be relocated approaches 30.

study on

Units 1 & 2 > AOS 1 > Topic 5 > Concept 3

The rectangular hyperbola and the truncus Summary screen and practice questions

Exercise 6.4 The rectangular hyperbola and the truncus

Technology free

1. State the equations of the asymptotes of the following hyperbolas.

a. $y = \dfrac{1}{x+5} + 2$

b. $y = \dfrac{8}{x} - 3$

c. $y = \dfrac{-3}{4x}$

d. $y = \dfrac{-3}{14+x} - \dfrac{3}{4}$

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2. For each of the following, state the:

i. equations of the asymptotes

ii. domain

iii. range.

a. $y = \dfrac{1}{x-6} + 1$ **b.** $y = \dfrac{1}{3+x} - 7$

c. $y = 4 + \dfrac{8}{2-x}$ **d.** $y = \dfrac{-5}{4(x-1)} - 3.$

3. **WE7** Sketch the graphs of the following functions, stating the domain and range.

a. $y = \dfrac{2}{x+2} - 1$ **b.** $y = \dfrac{-1}{x-2}$

4. Consider the hyperbola with equation $y = \dfrac{1}{x-1} + 5$.

a. State the equations of its asymptotes.

b. Calculate the coordinates of its y-intercept.

c. Calculate the coordinates of any x-intercept.

d. Show that the point $(2, 6)$ lies on the graph of the hyperbola and sketch the graph.

5. Consider the hyperbola with equation $y = 5 - \dfrac{6}{x+3}$.

a. State the equations of its asymptotes.

b. Calculate the coordinates of its y-intercept.

c. Calculate the coordinates of any x-intercept.

d. Sketch the graph.

6. Sketch the graph of the following functions, stating the domain and range.

a. $y = \dfrac{1}{x+1} - 3$

b. $y = 4 - \dfrac{3}{x-3}$

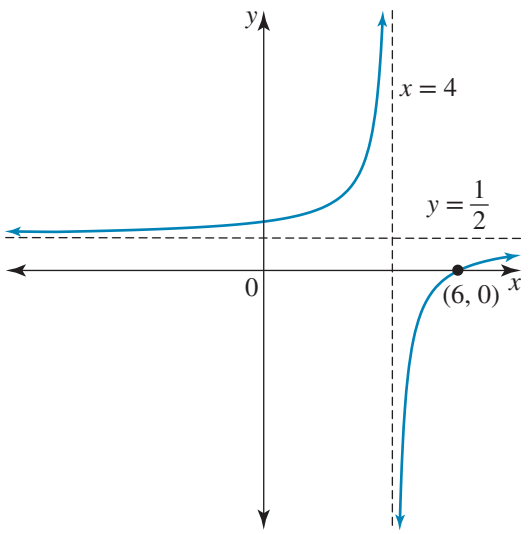
c. $y = -\dfrac{5}{3+x}$

d. $y = -\left(1 + \dfrac{5}{2-x}\right)$

7. For $f : R \setminus \left\{\dfrac{1}{2}\right\} \to R, f(x) = 4 - \dfrac{3}{1-2x}$, state the equations of the asymptotes and the asymptote-formed quadrants in which the graph would lie.

8. **WE8** **a.** Identify the asymptotes of the hyperbola with equation $y = \dfrac{6x}{3x+2}$.

b. Form the equation of the hyperbola shown.



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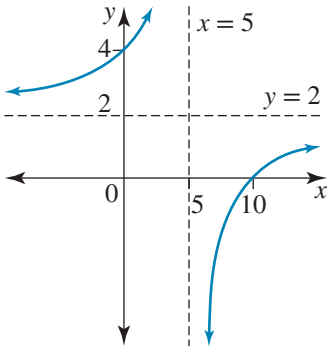
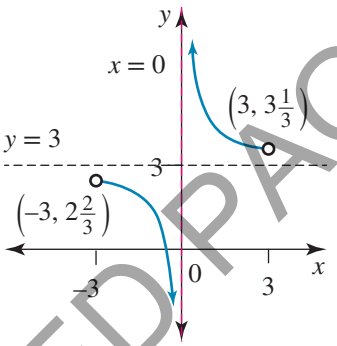
9. a. The equations of the asymptotes of a hyperbola are $x = -4$ and $y = 2$. The graph passes through the point $(0, 8)$. Express its equation in the form $y = \dfrac{a}{x-h} + k$.

b. The point $(2, -1)$ lies on the graph of a hyperbola. Given the equations of the asymptotes are $x = 0$ and $y = 5$, determine the equation of the hyperbola.

c. Form the equation of the hyperbola shown in the diagram.



d. Part of a hyperbola is shown in the diagram.



i. State the domain and range of the graph.

ii. Form the equation of the graph.

10. a. If $\dfrac{11-3x}{4-x} = a - \dfrac{b}{4-x}$ calculate the values of a and b.

b. Hence, sketch the graph of $y = \dfrac{11-3x}{4-x}$.

c. For what values of x is $\dfrac{11-3x}{4-x} > 0$?

11. Express in the form $y = \dfrac{a}{bx+c} + d$ and state the equations of the asymptotes for each of the following.

a. $y = \dfrac{x}{4x+1}$

b. $(x-4)(y+2) = 4$

c. $y = \dfrac{1+2x}{x}$

d. $2xy + 3y + 2 = 0$

12. Find the domain and range of the hyperbola with equation $xy - 4y + 1 = 0$.

13. MC The equations of the asymptotes of a truncus are $x = 7$ and $y = 0$. Which of the following could be its equation?

A. $y = \dfrac{1}{x^2 - 7}$ **B.** $y = \dfrac{2}{x-7}$ **C.** $y = \dfrac{-3}{(7-x)^2}$

D. $y = \dfrac{4}{(x+7)^2}$ **E.** $y = \dfrac{5}{49+x^2}$

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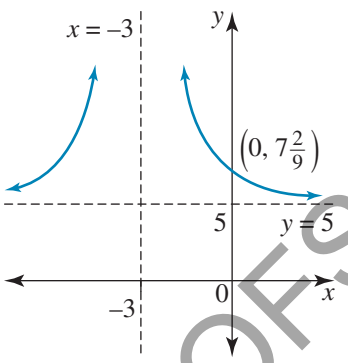
14. The diagram shown at right shows the graph of a truncus.

a. State the domain and range of the graph.

b. Form the equation of the graph.



15. **WE9** From Ohm's Law, the electrical resistance, R ohms, of a metal conductor in which the voltage is constant is inversely proportional to the current, I amperes.
When the current is 0.6 amperes, the resistance is 400 ohms.

a. Find the relationship between the resistance and current.

b. If the current is increased by 20%, what is the resistance?
You may choose to use technology complete your answer.

16. **WE10** Sketch the graphs of the following functions, stating the domain and range.

a. $y = \dfrac{1}{(x-3)^2} - 1$ **b.** $y = \dfrac{-8}{(x+2)^2} - 4$

17. For each of the following:

i. state the equations of the asymptotes

ii. state the domain and range

iii. calculate any intercepts with the coordinate axes

iv. sketch the graph.

a. $y = \dfrac{4}{3x^2} - 1$

b. $y = \dfrac{-2}{(x-1)^2} + 4.$

18. Identify the equations of the asymptotes and sketch the graphs of the following, stating the domain and range.

a. $y = \dfrac{12}{(x-2)^2} + 5$ **b.** $y = \dfrac{-24}{(x+2)^2} + 6$

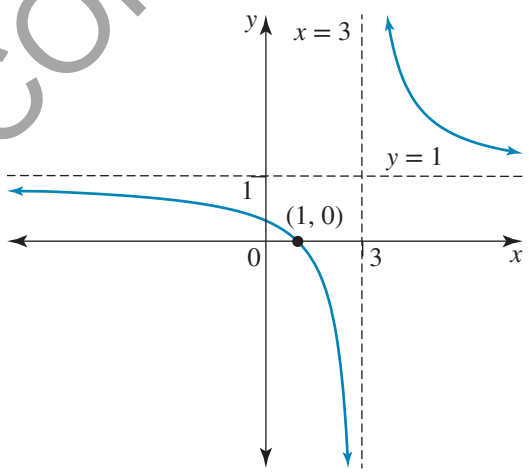
c. $y = 7 - \dfrac{1}{7x^2}$ **d.** $y = \dfrac{4}{(2x-1)^2}$

e. $y = -2 - \dfrac{1}{(2-x)^2}$ **f.** $y = \dfrac{x^2+2}{x^2}$

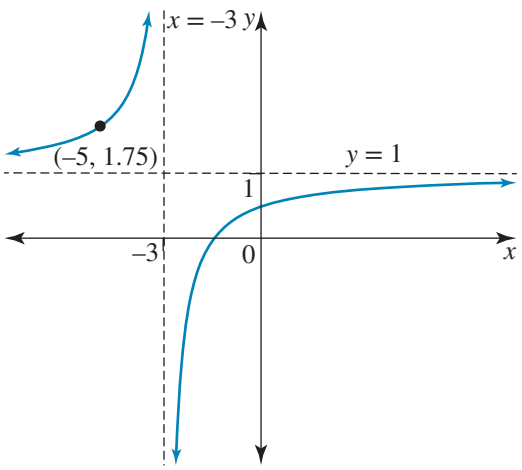
19. Determine the equations of the asymptotes and state the domain and range of the truncus with equation $y = \dfrac{3}{2(1-5x)^2}$.

20. The diagrams in parts **a** to **d** are either of a hyperbola or a truncus. Form the equations of each graph.

a.



b.



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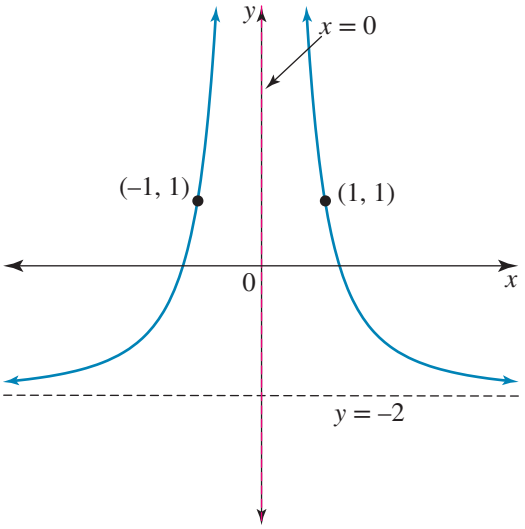
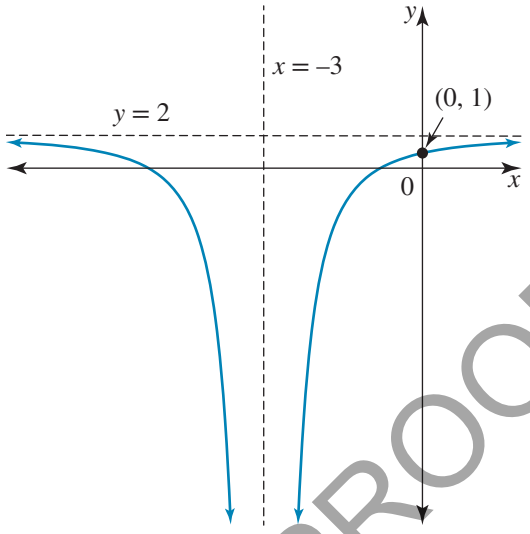
c.



d.



e. The graph of a truncus has the same shape as $y = \frac{4}{x^2}$ but its vertical asymptote has the equation $x = -2$. Give its rule and write the function in mapping notation.

f. A hyperbola is undefined when $x = \frac{1}{4}$. As $x \to -\infty$ its graph approaches the line $y = -\frac{1}{2}$ from below. The graph cuts the x-axis where $x = 1$.

i. Determine the equation of the hyperbola and express it in the form $y = \frac{ax + b}{cx + d}$ where $a, b, c, d \in Z$.

ii. Write the function in mapping notation.

Technology active

21. **WE11** The number P of cattle owned by a farmer at a time t years after purchase is modelled by $P = 30 + \frac{100}{2 + t}$.

a. By how many cattle is the herd reduced after the first 2 years?

b. Sketch the graph of the number of cattle over time using the given model and state its domain and range.

c. What is the minimum number the herd of cattle is expected to reach according to this model?

22. Select the table of data which shows $y \propto \frac{1}{x}$, complete the table and state the rule for y in terms of x.

a.

x	0.5	1	2	4		8
y	20.5	11	7	6.5	6.4	

b.

x	0.5	1	2	4		8
y	14.4	7.2	3.6	1.8	6.4	

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23. The strength of a radio signal I is given by $I = \frac{k}{d^2}$, where d is the distance from the transmitter.

a. Draw a sketch of the shape of the graph of I against d.

b. Describe the effect on the strength of the signal when the distance from the transmitter is doubled.

24. The time t taken to travel a fixed distance of 180 km is given by $t = \frac{k}{v}$ where v is the speed of travel.

a. What is the constant of proportionality, k?

b. Sketch a graph to show the nature of the relationship between the time and the speed.

c. What speed needs to be maintained if the entire journey is to be completed in $2\frac{1}{4}$ hours?

25. The height h metres of a hot air balloon above ground level t minutes after take-off is given by $h = 25 - \frac{100}{(t+2)^2}, t \geq 0$.

a. How long, to the nearest second, does it take the balloon to reach an altitude of 12.5 metres above ground level?

b. What is its limiting altitude?

26. a. Calculate the coordinates of the point(s) of intersection of $xy = 2$ and $y = \frac{x^2}{4}$.

b. On the same axes sketch the graphs of $xy = 2$ and $y = \frac{x^2}{4}$.

c. Calculate the coordinates of the point(s) of intersection of $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$ and sketch the graphs of $y = \frac{4}{x^2}$ and $y = \frac{x^2}{4}$ on the same set of axes.

d. Express the coordinates of the point(s) of intersection of $y = \frac{a}{x^2}$ and $y = \frac{x^2}{a}$ in terms of a, for $a > 0$.

27. In an effort to protect a rare species of stick insect, 20 of the species were captured and relocated to a small island where there were few predators. After 2 years the population size grew to 240 stick insects.

A model for the size N of the stick insect population after t years on the island is thought to be defined by the function $N: R^+ \cup \{0\} \to R, N(t) = \frac{at+b}{t+2}$.

a. Calculate the values of a and b.

b. After what length of time, to the nearest month, did the stick insect population reach 400?

c. Show that $N(t+1) - N(t) = \frac{880}{(t+2)(t+3)}$.

d. Hence, or otherwise, find the increase in the stick insect population during the 12th year and compare this with the increase during the 14th year. What is happening to the growth in population?

e. When would the model predict the number of stick insects reaches 500?

f. How large can the stick insect population grow?

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28. Use CAS technology to sketch $y = \dfrac{x+1}{x+2}$ together with its asymptotes and use the graphing screen to obtain:

a. the number of intersections of $y = x$ with $y = \dfrac{x+1}{x+2}$

b. the values of k for which $y = x + k$ intersects $y = \dfrac{x+1}{x+2}$ once, twice or not at all.

29. a. Sketch the graphs of $xy = 1$ and $x^2 - y^2 = 2$ using the conic screen or other technology and give the equations of their asymptotes.

b. These hyperbolas are the same but they are sketched on different orientations of the axes. Suggest a way to transform one graph into the other.

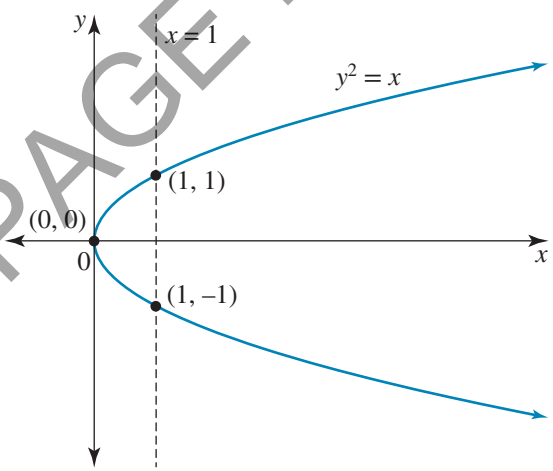
6.5 The relation $y^2 = x$

Here we shall consider a relation with a one-to-many correspondence. This relation is not a function.

6.5.1 The relation $y^2 = x$

The relation $y^2 = x$ cannot be a function since, for example, $x = 1$ is paired with both $y = 1$ and $y = -1$; the graph of $y^2 = x$ therefore fails the vertical line test for a function.

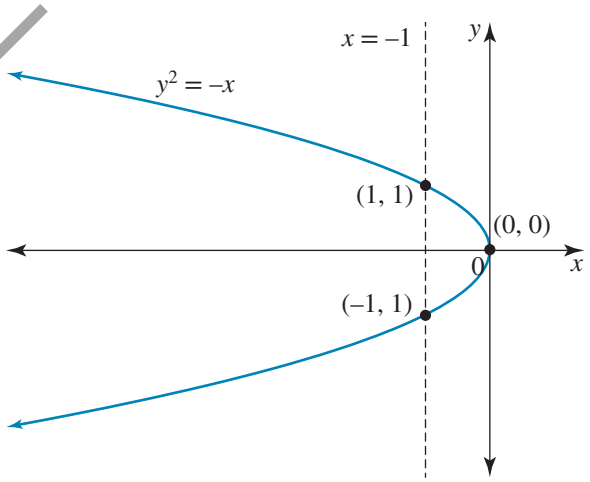
The shape of the graph of $y^2 = x$ could be described as a **sideways parabola** opening to the right, like the reflector in a car's headlight.



Key features of the graph of $y^2 = x$ are:

- domain $R^+ \cup \{0\}$ with the graph opening to the right
- range R
- turning point, usually called a vertex, at $(0, 0)$
- axis of symmetry is horizontal with equation $y = 0$ (the x-axis)
- one-to-many correspondence.

The graph of $y^2 = -x$ will open to the left, with domain $R^- \cup \{0\}$.



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on Resources

Interactivity: The relation $y^2 = x$ (int-2574)

6.5.2 Transformations of the graph of $y^2 = x$

Horizontal and vertical translations are identifiable from the coordinates of the vertex, just as they are for translations of the parabolic function $y = x^2$; the analysis of the curve is very similar to that applied to the parabolic function.

The general form of the relation of $y^2 = x$ is $(y - k)^2 = a(x - h)$

Key features:

- The vertex has coordinates (h, k), due to the horizontal and vertical translations h and k respectively.
- The axis of symmetry has equation $y = k$.
- If $a > 0$, the graph opens to the right; if $a < 0$, it opens to the left.
- There is always one x-intercept obtained by substituting $y = 0$.
- There may be two, one or no y-intercepts, determined by substituting $x = 0$ and solving the resulting quadratic equation for y.

By considering the sign of a and the position of the vertex, it is possible to deduce whether or not there will be a y-intercept. If there is no y-intercept, this consideration can avoid wasted effort in attempting to solve a quadratic equation for which there are no real solutions.

If the equation of the graph is not given in the vertex form $(y - k)^2 = a(x - h)$, completing the square on the y-terms may be necessary to transform the equation into this form.

WORKED EXAMPLE 12

For each of the following relations, state the coordinates of the vertex and sketch the graph stating its domain and range.

a. $(y - 1)^2 = 8(x + 2)$ **b.** $y^2 = 6 - 3x$

THINK

a. 1. State the coordinates of the vertex.

2. Calculate any intercepts with the axes.

WRITE

a. $(y - k)^2 = a(x - h)$ has vertex (h, k)
$(y - 1)^2 = 8(x + 2)$ has vertex $(-2, 1)$
x-intercept: let $y = 0$
$(-1)^2 = 8(x + 2)$

$$8x = -15$$
$$\therefore\ x = -\frac{15}{8}$$

x-intercept $\left(-\frac{15}{8}, 0\right)$
y-intercepts: let $x = 0$
$(y - 1)^2 = 16$

$$y - 1 = \pm 4$$
$$\therefore\ y = -3 \text{ or } y = 5$$

y-intercepts $(0, -3)$, $(0, 5)$

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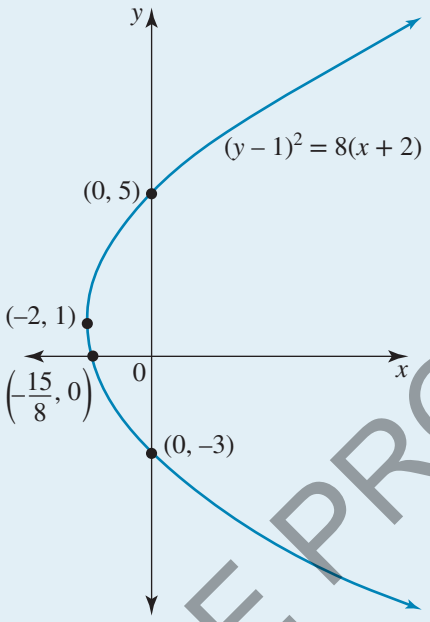
3. Sketch the graph showing the key features and state the domain and range.



Domain $[-2, \infty)$; range R.

b. 1. Express the equation in the form $(y-k)^2 = a(x-h)$ and state the vertex.

b. $y^2 = 6 - 3x$

$= -3(x-2)$

Vertex is (2, 0).

2. Calculate any intercepts with the axes.

x-intercept is the vertex (2, 0).

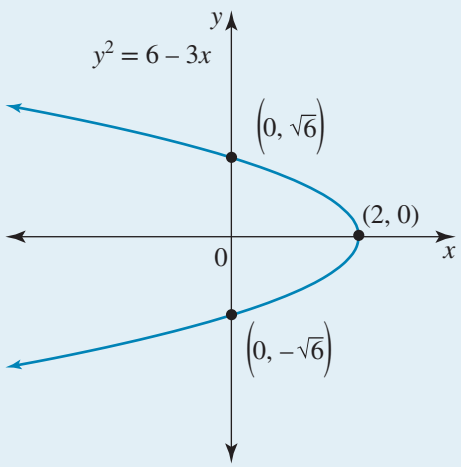
y-intercepts: in $y^2 = 6 - 3x$, let $x = 0$

$y^2 = 6$

$\therefore\ y = \pm\sqrt{6}$

y-intercepts $(0, \pm\sqrt{6})$

3. Sketch the graph and state the domain and range.



Domain $(-\infty, 2]$; range R.

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6.5.3 Determining the rule for the sideways parabola

Since the most common form given for the equation of the sideways parabola is the vertex form $(y-k)^2 = a(x-h)$, once the coordinates of the vertex are known, a second point can be used to obtain the value of a. Other sets of three pieces of information and analysis could also determine the equation, including that the axis of symmetry lies midway between the y-intercepts.

WORKED EXAMPLE 13

a. Determine the equation of the relation with rule $(y-k)^2 = a(x-h)$ and vertex (3, 5) which passes through the point (5, 3).

b. Determine the equation of the sideways parabola which contains the three points (0, 0), (0, −4), (3, 2).

THINK	WRITE
a. 1. Substitute the coordinates of the vertex into the general form of the equation.	**a.** $(y-k)^2 = a(x-h)$ Vertex (3, 5) $\Rightarrow (y-5)^2 = a(x-3)$
2. Use the given point on the graph to determine the remaining unknown constant.	Point (5, 3) is on the curve. $\Rightarrow (3-5)^2 = a(5-3)$ $4 = 2a$ $\therefore a = 2$
3. State the equation.	The equation is $(y-5)^2 = 2(x-3)$.
b. 1. Calculate the equation of the axis of symmetry. *Note:* An alternative approach would be to set up a system of 3 simultaneous equations using the coordinates of the 3 given points.	**b.** Two of the given points, (0, 0) and (0, −4), lie on the y-axis, so the axis of symmetry lies midway between these two points. Equation of axis of symmetry is: $y = \frac{y_1 + y_2}{2}$ $= \frac{0 + (-4)}{2}$ $= -2$ $y = -2$ is the equation of the axis of symmetry.
2. Substitute the equation of the axis of symmetry into the general equation of a sideways parabola.	Let the equation be $(y-k)^2 = a(x-h)$. Axis of symmetry $y = -2$ $\therefore (y+2)^2 = a(x-h)$

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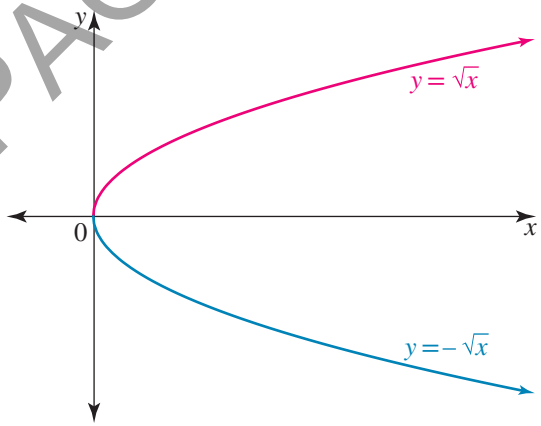
3. Use the third point and one of the y-intercepts to form a system of two simultaneous equations.	Substitute the point (0, 0). $(2)^2 = a(-h)$ $4 = -ah$ Substitute the point (3, 2). $(2 + 2)^2 = a(3 - h)$ $16 = a(3 - h)$ $= 3a - ah$
4. Solve the simultaneous equations to obtain a and h.	$4 = -ah$ [1] $16 = 3a - ah$ [2] Equation [2] – equation [1] $12 = 3a$ $a = 4$ Equation [1] $\Rightarrow h = -1$
5. State the answer.	The equation of the sideways parabola is $(y + 2)^2 = 4(x + 1)$.

6.5.4 The square root function

The **square root function** is formed as part of the relation $y^2 = x$ in much the same way as the semicircle forms part of a circle.

Although $y^2 = x$ is not a function, it is made up of two branches, the upper branch and the lower branch, each of which is a function. Since $y^2 = x \Rightarrow y = \pm\sqrt{x}$, the upper branch has the equation $y = \sqrt{x}$ and the lower branch has the equation $y = -\sqrt{x}$.

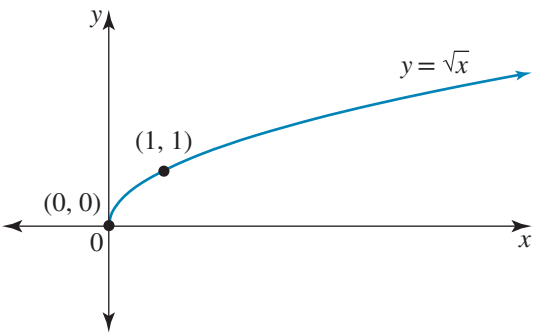


6.5.5 The graph of $y = \sqrt{x}$

The square root function has the rule $y = x^n$, $n = \frac{1}{2}$. It is not a polynomial function. Since $x^{\frac{1}{2}} = \sqrt{x}$, the square root function is defined by the equation $y = \sqrt{x}$ or $y = x^{\frac{1}{2}}$. The y-values of this function must be such that $y \geq 0$. No term under a square root symbol can be negative so this function also requires that $x \geq 0$. The graph of the square root function is shown in the diagram.

Key features of the graph of $y = \sqrt{x}$ are:

- Endpoint (0, 0)
- As $x \to \infty, y \to \infty$
- Defined only for $x \geq 0$ so the domain is $R^+ \cup \{0\}$
- y-values cannot be negative so the range is $R^+ \cup \{0\}$
- One-to-one correspondence
- Upper half of the sideways parabola $y^2 = x$



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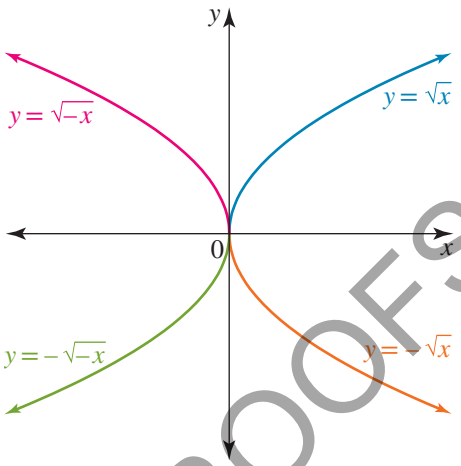
Variations of the basic graph

The term under the square root cannot be negative so $y = \sqrt{-x}$ has a domain requiring $-x \geq 0 \Rightarrow x \in (-\infty, 0]$. This graph can be obtained by reflecting the graph of $y = \sqrt{x}$ in the y-axis. The four variations in position of the basic graph are shown in the diagram.

The diagram could also be interpreted as displaying the graphs of the two relations $y^2 = x$ (on the right of the y-axis) and $y^2 = -x$ (on the left of the y-axis). The endpoints of the square root functions are the vertices of these sideways parabolas.



6.5.6 Transformations of the square root function

The general form of the square root function is $y = a\sqrt{x - h} + k$

Key features:

- Endpoint (h, k)
- If $a > 0$, the endpoint is the minimum point
- If $a < 0$, the endpoint is the maximum point
- Either one or no x-intercepts
- Either one or no y-intercepts
- Domain $[h, \infty)$ since $x - h \geq 0 \Rightarrow x \geq h$
- Range is $[k, \infty)$ if $a > 0$ or $(-\infty, k]$ if $a < 0$

The graph of $y = a\sqrt{-(x - h)} + k$ also has its endpoint at (h, k) but its domain is $(-\infty, h]$.

WORKED EXAMPLE 14

a. Sketch $y = 2\sqrt{x + 1} - 4$, stating its domain and range.

b. For the function $f(x) = \sqrt{2 - x}$:

i. state its domain

ii. sketch the graph of $y = f(x)$

iii. form the equation of the sideways parabola of which it is part.

THINK	WRITE
a. 1. State the coordinates of the endpoint.	**a.** $y = 2\sqrt{x + 1} - 4$ Endpoint $(-1, -4)$

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2. Calculate any intercepts with the coordinate axes.

y-intercept: let $x = 0$

$y = 2\sqrt{1} - 4$

$= -2$

y-intercept $(0, -2)$

x-intercept: let $y = 0$

$2\sqrt{x+1} - 4 = 0$

$\therefore \sqrt{x+1} = 2$

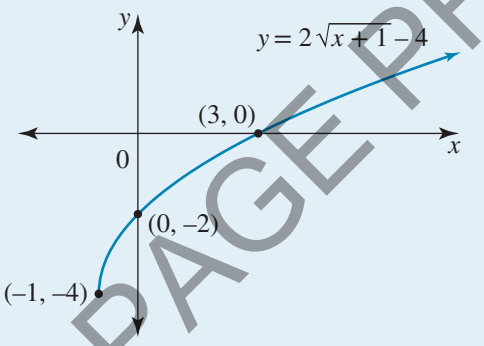
Square both sides:

$x + 1 = 4$

$\therefore\ x = 3$

x-intercept $(3, 0)$

3. Sketch the graph and state its domain and range.



Domain $[-1, \infty)$ range $[-4, \infty)$

b. **i.** State the requirement on the expression under the square root and then state the domain.

b. i. $f(x) = \sqrt{2 - x}$

Domain: require $2 - x \geq 0$

$-x \geq -2$

$\therefore x \leq 2$

Therefore the domain is $(-\infty, 2]$.

ii. Identify the key points of the graph and sketch.

ii. $f(x) = \sqrt{2 - x}$

$y = \sqrt{2 - x}$

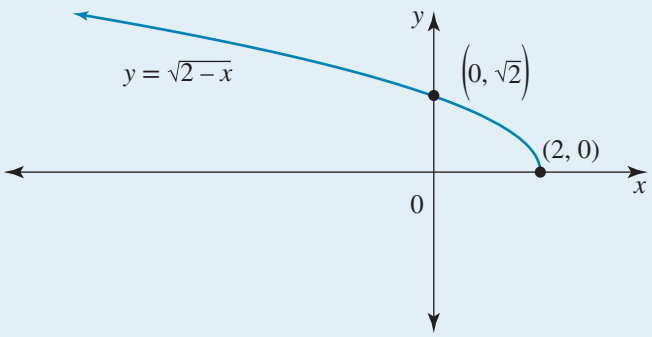
$= \sqrt{-(x - 2)}$

Endpoint $(2, 0)$ is also the x-intercept.

y-intercept: let $x = 0$ in $y = \sqrt{2 - x}$

$y = \sqrt{2}$

y-intercept is $(0, \sqrt{2})$.



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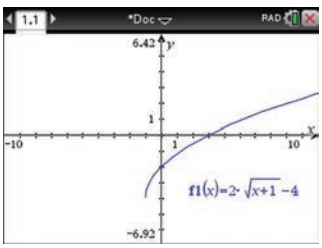
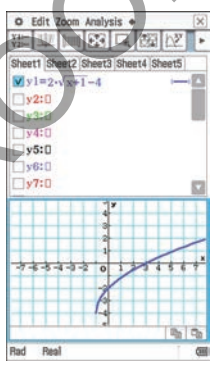
iii. Form the equation of the sideways parabola.
Note: The required equation cannot be written using $f(x)$ notation since the sideways parabola is not a function.

iii. Square root function: $y = \sqrt{2 - x}$
Square both sides:
$y^2 = 2 - x$
The equation of the sideways parabola is
$y^2 = 2 - x$ or $y^2 = -(x - 2)$ or $y = \pm\sqrt{2 - x}$.

TI | THINK	WRITE
a. 1. On a Graphs page, complete the entry line as: $f1(x) = 2\sqrt{x+1} - 4$ Then press ENTER.	
2. State the domain and range.	Domain is $[-1, \infty)$ Range is $[-4, \infty)$

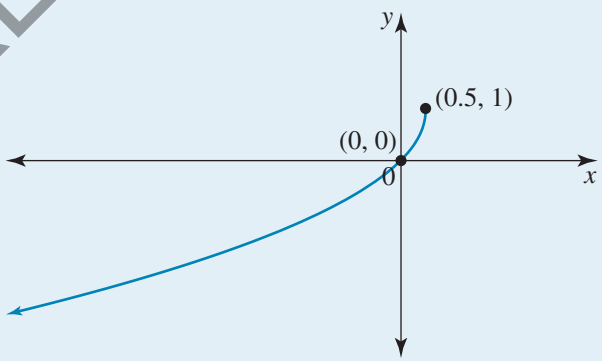
CASIO | THINK	WRITE
a. 1. On the Graphs & Table screen, complete the entry line as: $y1 = 2\sqrt{x+1} - 4$ Then press EXE.	
2. State the domain and range.	Domain is $[-1, \infty)$ Range is $[-4, \infty)$

6.5.7 Determining the equation of a square root function

If a diagram is given, the direction of the graph will indicate whether to use the equation in the form of $y = a\sqrt{x - h} + k$ or of $y = a\sqrt{-(x - h)} + k$. If a diagram is not given, the domain or a rough sketch of the given information may clarify which form of the equation to use.

WORKED EXAMPLE 15

Form a possible equation for the square root function shown.



THINK	WRITE
1. Note the direction of the graph to decide which form of the equation to use.	Graph opens to the left with $x \leq 0.5$. Let equation be $y = a\sqrt{-(x - h)} + k$.

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2. Substitute the coordinates of the endpoint into the equation.

Endpoint $(0.5, 1)$

$y = a\sqrt{-(x - 0.5)} + 1$

$= a\sqrt{-x + 0.5} + 1$

3. Use a second point on the graph to determine the value of a.

$(0, 0)$ lies on the graph.

$0 = a\sqrt{0.5} + 1$

$-1 = a \times \frac{1}{\sqrt{2}}$

$a = -\sqrt{2}$

4. State the equation of the graph.

The equation is $y = -\sqrt{2}\sqrt{-x + 0.5} + 1$.

5. Express the equation in a simplified form.

$y = -\sqrt{2}\sqrt{-x + \frac{1}{2}} + 1$

$= -\sqrt{2}\sqrt{\frac{-2x + 1}{2}} + 1$

$= -\sqrt{2}\frac{\sqrt{-2x + 1}}{\sqrt{2}} + 1$

$= -\sqrt{-2x + 1} + 1$

Therefore the equation is $y = 1 - \sqrt{1 - 2x}$.

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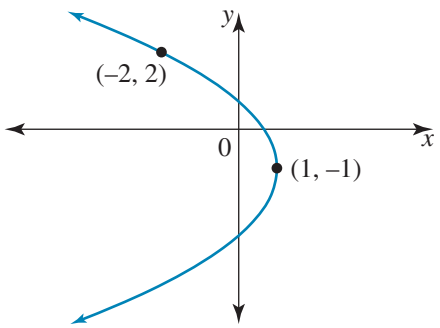
Units 1 & 2 > AOS 1 > Topic 5 > Concept 4 **The relation $y^2 = x$** Summary screen and practice questions

Exercise 6.5 The relation $y^2 = x$

Technology free

1. On the same diagram, sketch the graphs of $y^2 = x$, $y^2 = 4x$ and $y^2 = \frac{1}{4}x$ and comment on the effect of the change of the coefficient of the x-term.
2. a. Form the equation of the graph of the parabola relation shown.



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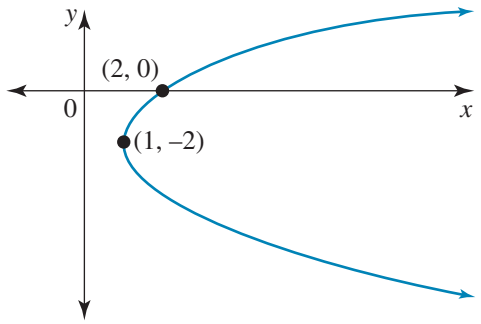
b. Give a possible equation for the graph shown.



3. **WE12** For each of the following relations state the coordinates of the vertex and sketch the graph, stating its domain and range.

a. $(y+3)^2 = 4(x-1)$ **b.** $(y-3)^2 = -9x$

4. Sketch the following, labelling the coordinates of the vertex and any axis intercepts.

a. $(y+1)^2 = 3x$ **b.** $9y^2 = x+1$

c. $(y+2)^2 = 8(x-3)$ **d.** $(y-4)^2 = 2x+1$

5. Sketch the following, stating the coordinates of the vertex and the exact coordinates of any intercept with the axes.

a. $y^2 = -2x$ **b.** $(y+1)^2 = -2(x-4)$

c. $(6-y)^2 = -8-2x$ **d.** $x = -(2y-6)^2$

6. **WE13** **a.** Determine the equation of the relation with rule $(y-k)^2 = a(x-h)$ which passes through the point $(-10, 0)$ and has a vertex at $(4, -7)$.

b. Determine the equation of the sideways parabola which contains the points $(0, 0), (0, 6)$ and $(9, -3)$.

7. A parabola touches the y-axis at $y = 3$ and cuts the x-axis at $x = 2$. Explain whether this parabola is a function or not and form its equation.

8. Express the relation given by $y^2 + 8y - 3x + 20 = 0$ in the form $(y-k)^2 = a(x-h)$ and hence state the coordinates of its vertex and the equation of its axis of symmetry.

9. Express the following equations in the form $(y-k)^2 = a(x-h)$ and hence state the coordinates of the vertex and the domain.

a. $y^2 + 16y - 5x + 74 = 0$ **b.** $y^2 - 3y + 13x - 1 = 0$

c. $(5+2y)^2 = 8-4x$ **d.** $(5-y)(1+y) + 5(x-1) = 0$

10. For each of the following square root functions, state:

i. its domain

ii. the coordinates of its endpoint

iii. its range.

a. $y = \sqrt{x-9}$ **b.** $y = \sqrt{4-x}$

c. $y = -\sqrt{x+3}$ **d.** $y = 3 + \sqrt{-x}$

e. $y = \sqrt{3x-6} - 7$ **f.** $y = 4 - \sqrt{1-2x}$

11. **WE14** **a.** Sketch $y = \sqrt{x-1} - 3$, stating its domain and range.

b. For the function $f(x) = -\sqrt{2x+4}$:

i. state its domain

ii. sketch the graph of $y = f(x)$

iii. form the equation of the sideways parabola of which it is part.

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12. Sketch the graph of each of the following square root functions, showing endpoints and any intersections with the coordinate axes.

a. $y = \sqrt{x+4} - 1$ **b.** $y = 3 - \sqrt{3x}$

c. $y = -\sqrt{6-x}$ **d.** $y = \sqrt{9-2x} + 1$

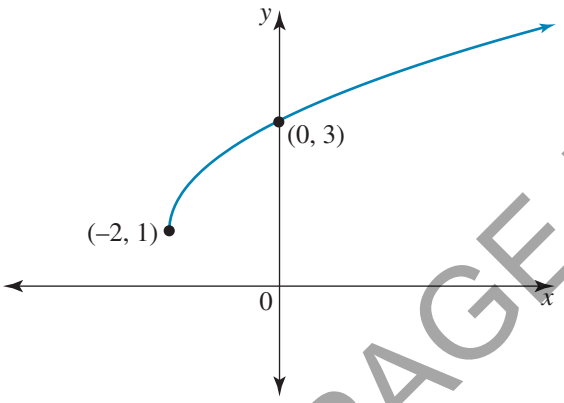
13. Sketch the following relations and state their domains and ranges.

a. $y = \sqrt{x+3} - 2$ **b.** $y = 5 - \sqrt{5x}$

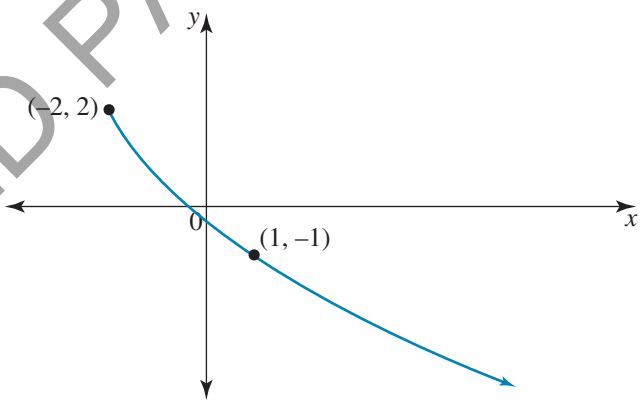
c. $y = 2\sqrt{9-x} + 4$ **d.** $y = \sqrt{49-7x}$

e. $y = 2 \pm \sqrt{x+4}$ **f.** $y + 1 + \sqrt{-2x+3} = 0$

14. **WE15** Form a possible equation for the square root function shown.



15. **a.** Determine the equation for the graph shown, given it represents a square root function with endpoint $(-2, 2)$.

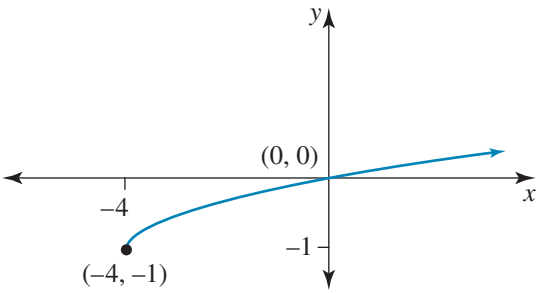


b. Form the equation of the square root function with endpoint $(4, -1)$ and containing the point $(0, 9)$. At what point does this function cut the x-axis?

c. Give the equation of the function which has the same shape as $y = \sqrt{-x}$ and an endpoint with coordinates $(4, -4)$.

16. **a.** A square root function graph has an equation in the form $y = a\sqrt{x-h} + k$. If the endpoint is $(3, 2)$ and the graph passes through the point $(4, 6)$, what is the equation of the graph?

b. Form the equation of the square root function shown in the diagram, expressing it in the form $y = a\sqrt{x-h} + k$.



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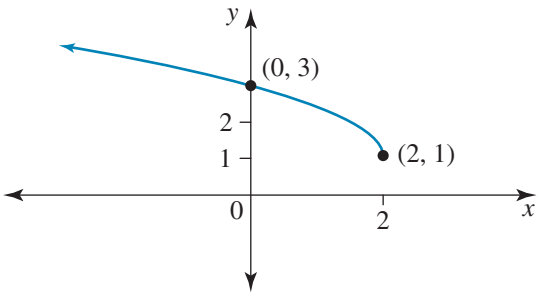
c. The equation of the graph shown has the form $y = \sqrt{a(x-b)} + c$. Determine the values of a, b and c, and state the equation.



17. Consider the relation $S = \{(x, y) : (y+2)^2 = 9(x-1)\}$.

a. Find the coordinates of its vertex and x-intercept and hence sketch its graph.

b. Form the equation of the square root function which forms the lower half of S.

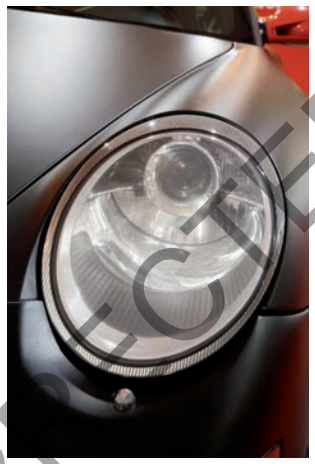
18. A small object falls to the ground from a vertical height of h metres. The time, t seconds, is proportional to the square root of the height. If it takes 2 seconds for the object to reach the ground from a height of 19.6 metres, obtain the rule for t in terms of h.

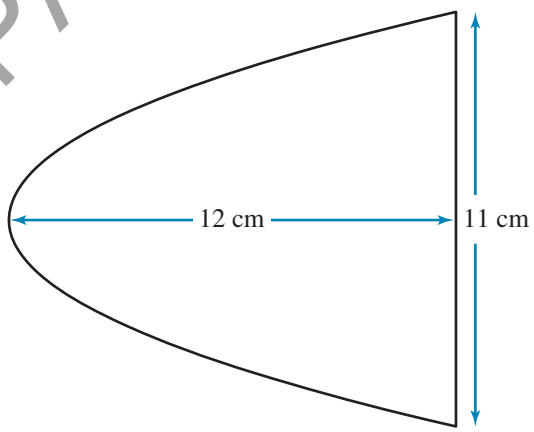
19. a. A curve in the shape of a sideways parabola touches the y-axis and passes through the points $(1, 12)$ and $(1, -4)$.

i. State the equation of its axis of symmetry.

ii. Determine the equation of the curve.

b. The reflector in a car's headlight has a parabolic shape.





Placing the coordinate axes with the origin at the vertex of the parabola, form the equation of the parabola relative to these axes.

20. Consider the relation $S = \{(x, y) : (y-2)^2 = 1 - x\}$.

a. Express the equation of the relation S with y as the subject.

b. Define, as mappings, the two functions f and g which together form the relation S.

c. Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = g(x)$.

d. Give the image of -8 for each function.

21. The relation $(y-a)^2 = b(x-c)$ has a vertex at $(2, 5)$ and cuts the x-axis at $x = -10.5$. Determine the values of a, b and c, and hence state the equation of the relation and its domain and range.

22. A function is defined as $f : [0, \infty) \to R$, $f(x) = \sqrt{mx} + n$, $f(1) = 1$ and $f(4) = 4$.

a. Determine the values of m and n.

b. Calculate the value of x for which $f(x) = 0$.

c. Sketch the graph of $y = f(x)$ together with the graph of $y = x$ and state $\{x : f(x) > x\}$.

d. State the equation of the sideways parabola of which the function f is one of its branches.

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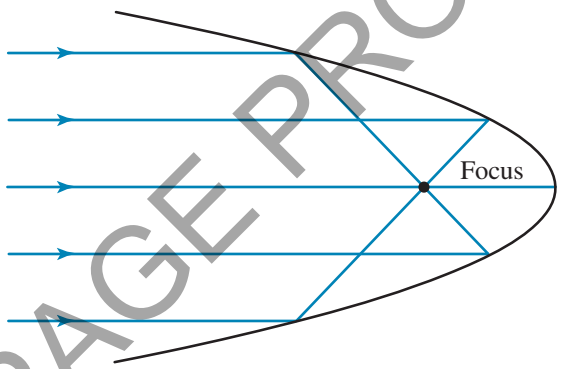
23. Consider the curve with equation $y^2 = -8x$.

a. State the domain of the curve and show that the point P$(-3, 2\sqrt{6})$ lies on the curve. Identify which branch of the curve it lies on.

b. Show that both the vertex V and the point P$(-3, 2\sqrt{6})$ are at positions which are equidistant from the point F$(-2, 0)$ and the vertical line D with equation $x = 2$.

c. Q is a point on the other branch of the curve to P, where $x = a, a < 0$. Express the coordinates of Q in terms of a and show that Q is also equidistant from the point F$(-2, 0)$ and the vertical line D with equation $x = 2$.

d. A property of a parabola is that rays travelling parallel to its axis of symmetry are all reflected through a point called the focus. A radio telescope is designed on this principle so that signals received from outer space will be concentrated at its focus.



Consider the equation $y^2 = -8x$ as a two-dimensional model of a telescope dish. Its focus is the point F$(-2, 0)$. A signal, travelling parallel to the axis of symmetry, strikes the dish at the point P$(-3, 2\sqrt{6})$ and is reflected through the focus F, striking the curve at point Q where $x = a, a < 0$. Calculate the value of a.

24. Consider the functions with rules $f(x) = 3\sqrt{x+1} + 2$ and $g(x) = \sqrt{4 - x^2} + 2$.

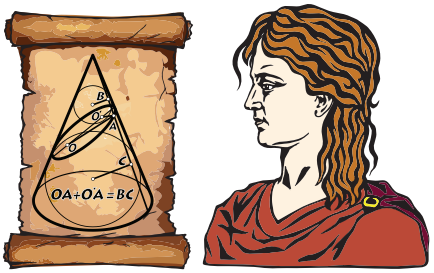
a. State their domains and what types of function they are.

b. Obtain the coordinates of any points of intersection of the graphs of $y = f(x)$ and $y = g(x)$, expressing the coordinates to 1 decimal place.

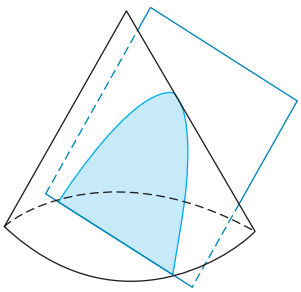
c. The function f is a branch of a relation A with the same domain as f; the function g is a branch of a relation B with the same domain as g. Form the rules for the two relations A and B.

d. Give the coordinates of the points of intersection of the two relations A and B obtained in part **c**, to 1 decimal place.

Hypatia, who was born in Alexandria around 350 BC, is the first recorded female mathematician of note. Amongst her mathematical interests were the curves, known as the conic sections, which arise from a plane cutting a cone. When a plane parallel to the slant height of a right cone cuts the cone, the cross-section obtained is a parabola. The hyperbola is another of these conics.



HYPATIA



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6.6 Other functions and relations

In this section we will use the functions we have learnt about so far to shed insight into and explore some connections between other functions and relations.

6.6.1 Maximal domains

When we write the function for which $f(x) = x^2$ we imply that this function has domain R. This is called the **maximal** or **implied** domain of the function. The knowledge acquired about the domains of polynomial, hyperbola, truncus, semicircle and square root functions can now be applied to identify the maximal domains of algebraic functions with which we are not familiar.

We know, for example, that the maximal domain of any polynomial function is R; the domain of the square root function $y = \sqrt{x}$ is $R^+ \cup \{0\}$ since the expression under the square root symbol cannot be negative; and the domain of the hyperbola $y = \frac{1}{x}$ is $R \setminus \{0\}$ since the denominator cannot be zero.

The maximal domain of any function must exclude:

- any value of x for which the denominator would become zero
- any value of x which would create a negative number or expression under a square root sign.

Hence, to obtain maximal domains, the following conditions would have to be satisfied:

$$y = \frac{g(x)}{f(x)} \Rightarrow f(x) \neq 0$$
$$y = \sqrt{f(x)} \Rightarrow f(x) \geq 0$$
$$y = \frac{g(x)}{\sqrt{f(x)}} \Rightarrow f(x) > 0$$

Numerators can be zero. For $y = \frac{g(x)}{f(x)}$, if $g(a) = 0$ and $f(a) = 2$, for example, the value of $\frac{g(a)}{f(a)} = \frac{0}{2} = 0$, which is defined.

At a value of x for which its denominator would be zero, a function will be discontinuous and its graph will have a vertical asymptote. As noted, this value of x is excluded from the function's domain.

Where a function is composed of the sum or difference of two functions, its domain must be the set over which all of its parts are defined.

The maximal domain of $y = f(x) \pm g(x)$ is $d_f \cap d_g$ where d_f and d_g are the domains of f and g respectively.

WORKED EXAMPLE 16

Identify the maximal domains of the functions with the following rules.

a. $y = \frac{1}{x^2 - 9}$ b. $y = \frac{x - 4}{x^2 + 9}$ c. $y = \frac{1}{\sqrt{x^3 - 8}}$

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THINK	WRITE
a. 1. Determine any values of x for which the denominator would become zero and exclude these values from the domain.	**a.** $y = \frac{1}{x^2 - 9}$ If the denominator $x^2 - 9 = 0$, then: $(x + 3)(x - 3) = 0$ $\therefore\ x = -3, x = 3$
2. State the maximal domain.	The values $x = -3, x = 3$ must be excluded from the domain. Therefore, the maximal domain is $R \setminus \{\pm 3\}$.
b. 1. Determine any values of x for which the denominator would become zero.	**b.** $y = \frac{x - 4}{x^2 + 9}$ If the denominator is zero: $x^2 + 9 = 0$ $x^2 = -9$ There is no real solution. As the denominator $x^2 + 9$ is the sum of two squares, it can never be zero.
2. State the maximal domain.	The denominator cannot be zero and both the numerator and denominator are polynomials, so they are always defined for any x-value. Therefore the maximal domain of the function is R.
c. 1. State the condition the expression contained under the square root sign must satisfy.	**c.** $y = \frac{1}{\sqrt{x^3 - 8}}$ The term $\sqrt{x^3 - 8}$ requires $x^3 - 8 \geq 0$. However, this term cannot be allowed to be zero since it is in the denominator. Hence, $x^3 - 8 > 0$.
2. Solve the inequation and state the maximal domain.	$x^3 > 8$ $\therefore\ x > 2$ The maximal domain is $(2, \infty)$.

6.6.2 Inverse relations and functions

The relation $A = \{(-1, 4), (0, 3), (1, 5)\}$ is formed by the mapping:

$$-1 \rightarrow 4$$
$$0 \rightarrow 3$$
$$1 \rightarrow 5$$

The **inverse** relation is formed by 'undoing' the mapping where:

$$4 \rightarrow -1$$
$$3 \rightarrow 0$$
$$5 \rightarrow 1$$

The inverse of A is the relation $\{(4, -1), (3, 0), (5, 1)\}$.

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The x- and y-coordinates of the points in relation A have been interchanged in its inverse. This causes the domains and ranges to be interchanged also.

Domain of $A = \{-1, 0, 1\}$ = range of its inverse

Range of $A = \{3, 4, 5\}$ = domain of its inverse

- For any relation, the inverse is obtained by interchanging the x- and y-coordinates of the ordered pairs.
- Domains and ranges are interchanged between a pair of inverse relations.

6.6.3 The equation of the inverse

The effect of the operation 'multiply by 2' is undone by the operation 'divide by 2'; the effect of the operation 'subtract 2' is undone by the operation 'add 2', and so on. These are examples of inverse operations and we use these 'undoing' operations to solve equations. The 'undoing' operations can also be used to deduce the equation of the inverse of a given relation.

The 'multiply by 2' rule is that of the relation $y = 2x$, so its inverse with the 'divide by 2' rule is $y = \frac{x}{2}$.

Algebraically, interchanging x and y in the equation $y = 2x$ gives $x = 2y$. Rearranging this to make y the subject gives $y = \frac{x}{2}$, the equation of its inverse.

To obtain the equation of the inverse:

- **interchange x and y in the equation of the original relation or function and rearrange to make y the subject.**

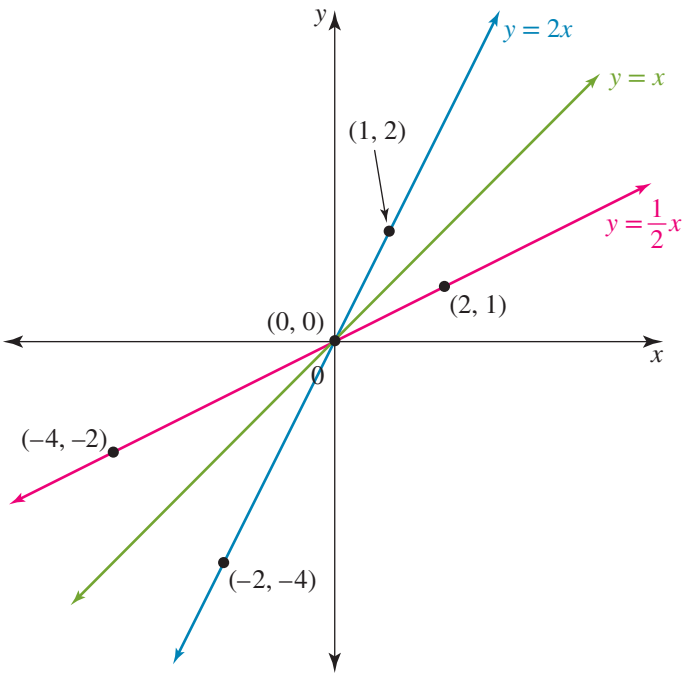
In some cases, the domain will need to be included when stating the equation of the inverse. For example, to find the equation of the inverse of the function $y = \sqrt{x}$, interchanging coordinates gives $x = \sqrt{y}$. Expressing $x = \sqrt{y}$ with y as the subject gives $y = x^2$. This rule is not unexpected since 'square root' and 'squaring' are inverse operations. However, as the range of the function $y = \sqrt{x}$ is $[0, \infty)$, this must be the domain of its inverse. Hence, the equation of the inverse of $y = \sqrt{x}$ is $y = x^2$, with the restriction that $x \geq 0$.

Graphs of inverse relations

The graphs of the pair of inverses, $y = 2x$ and $y = \frac{x}{2}$ show that their graphs are symmetric about the line $y = x$. The coordinates of symmetric points relative to $y = x$ are interchanged on each graph. Both graphs contain the point (0, 0) since interchanging coordinates of this point does not change the point.

Given the graph of a function or another relation, reflection of this graph in the line $y = x$ will create the graph of its inverse.

The coordinates of known points, such as the axes intercepts, are interchanged by this reflection. If the graphs intersect, they will do so on the line $y = x$ since interchanging the coordinates of any point on $y = x$ would not cause any alteration to the coordinates.



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6.6.4 Notation for inverse functions

Neither the original relation nor its inverse have to be functions. However, when both a relation and its inverse are functions, there is a special symbol for the name of the inverse.

If the inverse of a function f is itself a function, then the inverse function is denoted by f^{-1}.

The equation of the inverse of the square root function $f(x) = \sqrt{x}$ can be written as $f^{-1}(x) = x^2$, $x \geq 0$.

In mapping notation, if $f: [0, \infty) \to R,\ f(x) = \sqrt{x}$, then the inverse function is

$$f^{-1}: [0, \infty) \to R,\ f^{-1}(x) = x^2.$$

The domain of f^{-1} equals the range of f and the range of f^{-1} equals the domain of f; that is,

$$d_{f^{-1}} = r_f \text{ and } r_{f^{-1}} = d_f.$$

Note that f^{-1} is a function notation and cannot be used for relations which are not functions. Note also that the inverse function f^{-1} and the reciprocal function $\frac{1}{f}$ represent different functions: $f^{-1} \neq \frac{1}{f}$.

WORKED EXAMPLE 17

Consider the linear function $f: [0, 3) \to R, f(x) = 2x - 2$.

a. State the domain and determine the range of f.

b. State the domain and range of f^{-1}, the inverse of f.

c. Form the rule for the inverse and express the inverse function in mapping notation.

d. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

THINK	WRITE
a. State the domain and use its endpoints to determine the range.	**a.** $f: [0, 3) \to R, f(x) = 2x - 2$ Domain $[0, 3)$ Endpoints: when $x = 0, f(0) = -2$ (closed); when $x = 3, f(3) = 4$ (open) Therefore the range is $[-2, 4)$.
b. State the domain and range of the inverse.	**b.** $d_{f^{-1}} = r_f$ $= [-2, 4)$ $r_{f^{-1}} = d_f$ $= [0, 3)$ The inverse has domain $[-2, 4)$ and range $[0, 3)$.
c. 1. Interchange x- and y-coordinates to form the equation of the inverse function.	**c.** $f(x) = 2x - 2$ Let $y = 2x - 2$ Function: $y = 2x - 2$ Inverse: $x = 2y - 2$
2. Rearrange the equation to make y the subject.	$x + 2 = 2y$ $\therefore\ y = \frac{x + 2}{2}$ Therefore the rule for the inverse is $f^{-1}(x) = \frac{x + 2}{2}$.

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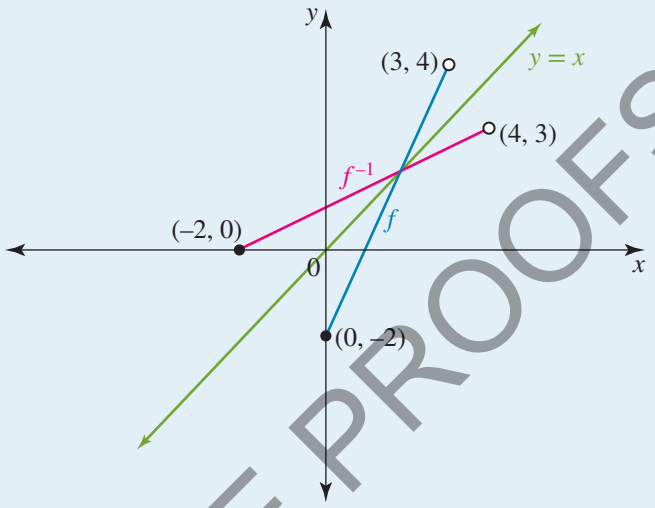
3. Express the inverse function as a mapping.

The inverse function is:
$f^{-1} : [-2, 4) \to R, f^{-1}(x) = \frac{x+2}{2}$

d. Sketch each line using its endpoints and include the line $y = x$ as part of the graph.
Note: The graph of f^{-1} could be deduced from the graph of f by interchanging coordinates of the endpoints.

d.



f and f^{-1} intersect on the line $y = x$.

Resources

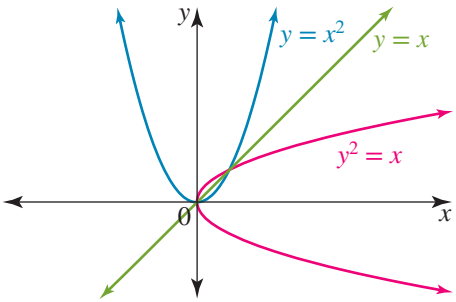
Interactivity: Inverse functions (int-2575)

6.6.5 Condition for the inverse function f^{-1} to exist

The inverse of the parabola with equation $y = x^2$ can be obtained algebraically by interchanging x- and y-coordinates to give $x = y^2$ as the equation of its inverse.

Graphically, the inverse of the parabola is obtained by reflecting the graph of the parabola in the line $y = x$. This not only illustrates that the sideways parabola, $y^2 = x$, is the inverse of the parabola $y = x^2$, but it also illustrates that although the parabola is a function, its inverse is not a function.

The reflection has interchanged the types of correspondence, as well as the domains and ranges. The many-to-one function $y = x^2$ has an inverse with a one-to-many correspondence and therefore its inverse is not a function. This has an important implication for functions.



For f^{-1} to exist, f must be a one-to-one function.

To ensure the inverse exists as a function, the domain of the original function may need to be restricted in order to ensure its correspondence is one-to-one.

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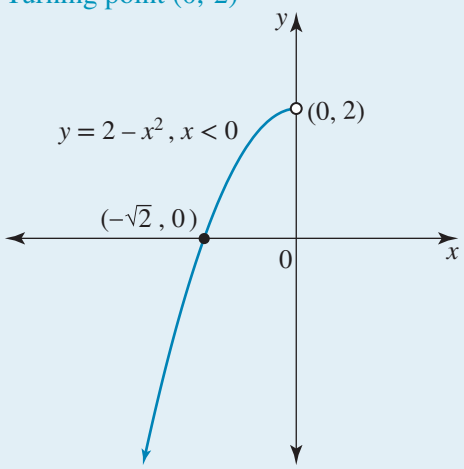
WORKED EXAMPLE 18

Consider the quadratic function defined by $y = 2 - x^2$.
a. Form the rule for its inverse and explain why the inverse is not a function.
b. Sketch the graph of $y = 2 - x^2$, $x \in R^-$ and use this to sketch its inverse on the same diagram.
c. Form the equation of the inverse of $y = 2 - x^2$, $x \in R^-$.
d. At what point do the two graphs intersect?

THINK	WRITE
a. 1. Interchange x- and y-coordinates to form the rule for its inverse.	**a.** $y = 2 - x^2$ Inverse: $x = 2 - y^2$ $\therefore y^2 = 2 - x$ is the rule for the inverse.
2. Explain why the inverse is not a function.	The quadratic function is many-to-one so its inverse has a one-to-many correspondence. Therefore the inverse is not a function.
b. 1. Sketch the graph of the function for the restricted domain.	**b.** $y = 2 - x^2, x \in R^-$ Domain: R^- y-intercept: (0, 2) is open since $x \in R^-$. x-intercept: let $y = 0$ $2 - x^2 = 0$ $x^2 = 2$ $\therefore x = \pm\sqrt{2}$ $\Rightarrow x = -\sqrt{2}$ since $x \in R^-$ x-intercept $(-\sqrt{2}, 0)$ Turning point (0, 2)
	
2. Deduce the key features of the inverse and sketch its graph and the line $y = x$ on the same diagram as the graph of the function.	For the inverse, (2, 0) is an open point on the x-axis and $(0, -\sqrt{2})$ is the y-intercept. Its graph is the reflection of the graph of $y = 2 - x^2$, $x \in R^-$ in the line $y = x$.

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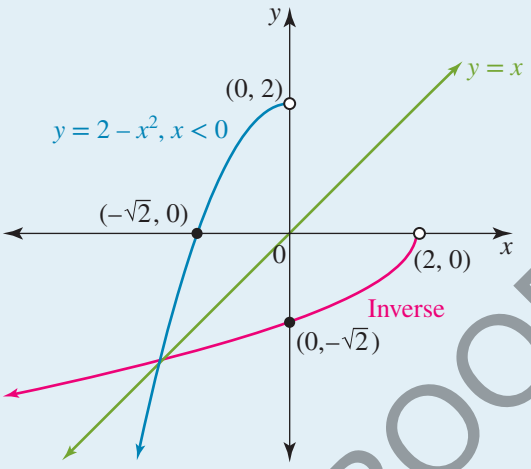

c. Use the range of the inverse to help deduce its equation.

c. From part **a**, the inverse of $y = 2 - x^2$ is:

$y^2 = 2 - x$

$\therefore\ y = \pm\sqrt{2 - x}$

The range of the inverse must be R^-, so the branch with the negative square root is required.

Therefore the equation of the inverse is

$y = -\sqrt{2 - x}$.

d. Choose two of the three equations that contain the required point and solve this system of simultaneous equations.

d. Point of intersection lies on $y = x$, $y = 2 - x^2$ and $y = -\sqrt{2 - x}$.

Using the first two equations, at intersection:

$x = 2 - x^2,\ x \in R^-$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, x = 1$

Reject $x = 1$ since $x \in R^-$.

Therefore $(-2, -2)$ is the point of intersection.

6.6.6 The inverse of $y = x^3$

The cubic function $y = x^3$ has a one-to-one correspondence, and both its domain and range are R. Its inverse will also be a function with both the domain and range of R.

The rule for the inverse is obtained by interchanging the x- and y-coordinates.

Function: $y = x^3$

Inverse function: $x = y^3$

$$\therefore y = \sqrt[3]{x}$$

The inverse of the cubic function $y = x^3$ is the cube root function $f : R \to R, f(x) = \sqrt[3]{x}$.

In index form $y = \sqrt[3]{x}$ is written as $y = x^{\frac{1}{3}}$, just as the square root function $y = \sqrt{x}$ can be expressed as $y = x^{\frac{1}{2}}$. Both $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{2}}$ belong to a category of functions known as power functions of the form $y = x^{\frac{p}{q}}$, $p, q \in N$.

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The graph of $y = x^n$, $n = \frac{1}{3}$

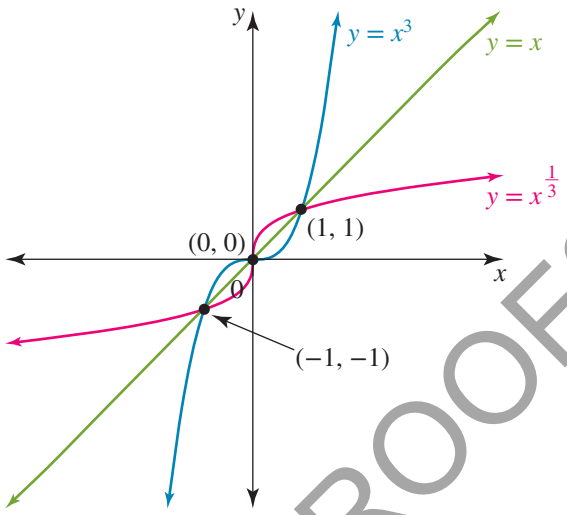
The graph of $y = x^{\frac{1}{3}}$ is obtained by reflecting the graph of $y = x^3$ in the line $y = x$.

Note that the points $(-1, -1), (0, 0), (1, 1)$ lie on $y = x^3$ so the same three points must lie on $y = x^{\frac{1}{3}}$.

Key features:

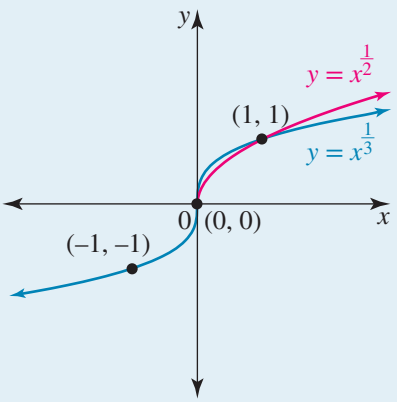
- Point of inflection at $(0, 0)$
- As $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$
- Domain R and range R

Unlike the square root function, the cube root function has domain R since cube roots of negative numbers can be calculated.



WORKED EXAMPLE 19

On the same set of axes sketch the graphs of $y = x^n$ for $n = \frac{1}{2}$ and $n = \frac{1}{3}$ and hence state $\{x : x^{\frac{1}{3}} > x^{\frac{1}{2}}\}$.

THINK	WRITE
1. State the features of the first graph.	The function $y = x^n$ for $n = \frac{1}{2}$ is the square root function, $y = x^{\frac{1}{2}}$ or $y = \sqrt{x}$. Endpoint: (0, 0) Point: the point (1, 1) lies on the graph. Domain requires $x \geq 0$.
2. State features of the second graph.	The function $y = x^n$ for $n = \frac{1}{3}$ is the cube root function, $y = x^{\frac{1}{3}}$ or $y = \sqrt[3]{x}$. Point of inflection: (0, 0) Point: the point (1, 1) lies on the graph. Domain allows $x \in R$.
3. Sketch the graphs on the same set of axes.	

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4. State the answer to the inequation.	For $x > 1$ the square root values are larger than the cube root values. For $0 < x < 1$ the cube root values are the larger. $\{x : x^{\frac{1}{3}} > x^{\frac{1}{2}}\}$ is $\{x : 0 < x < 1\}$

6.6.7 Hybrid functions

Note: Although hybrid functions are not part of the Units 1 & 2 course, they are included here in preparation for Units 3 & 4.

A **hybrid function** is one in which the rule may take a different form over different sections of its domain. An example of a simple hybrid function is one defined by the rule:

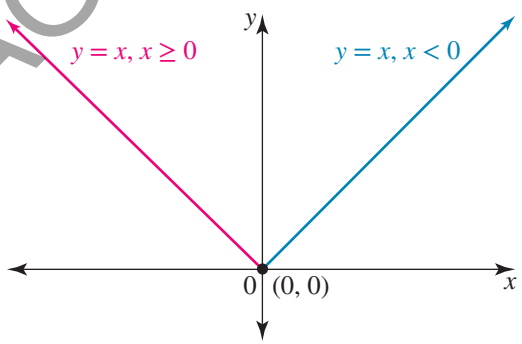
$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Graphing this function would give a line with positive gradient to the right of the y-axis and a line with negative gradient to the left of the y-axis.

This hybrid function is continuous at $x = 0$ since both of its branches join, but that may not be the case for all hybrid functions. If the branches do not join, the function is not continuous for that value of x: it is discontinuous at that point of its domain.

Sketching a hybrid function is like sketching a set of functions with restricted domains all on the same graph. Each branch of the hybrid rule is valid only for part of the domain and, if the branches do not join, it is important to indicate which endpoints are open and which are closed.

As for any function, each x-value can only be paired to exactly one y-value in a hybrid function. To calculate the corresponding y-value for a given value of x, the choice of which branch of the hybrid rule to use depends on which section of the domain the x-value belongs to.



WORKED EXAMPLE 20

Consider the function:

$$f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$$

a. Evaluate
i. $f(-2)$ **ii.** $f(1)$ **iii.** $f(2)$.

b. Sketch the graph of $y = f(x)$ and state the domain and range.

c. State any value of x for which the function is not continuous.

THINK	WRITE
a. Decide for each x-value which section of the domain it is in and calculate its image using the branch of the hybrid function's rule applicable to that section of the domain.	**a. i.** $f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$ $f(-2)$: Since $x = -2$ lies in the domain section $x < 1$, use the rule $f(x) = x^2$. $f(-2) = (-2)^2$ $\therefore f(-2) = 4$

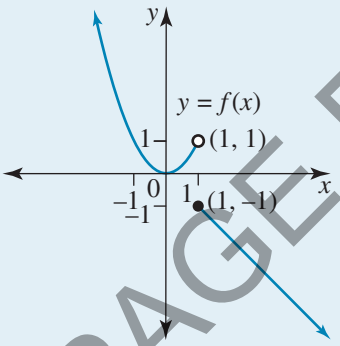
UNCORRECTED PAGE PROOFS

ii. $f(1)$: Since $x = 1$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.
$\therefore f(1) = -1$

iii. $f(2)$: Since $x = 2$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.
$\therefore f(2) = -2$

b. 1. Sketch each branch over its restricted domain to form the graph of the hybrid function.

b. Sketch $y = x^2, x < 1$
Parabola, turning point (0, 0) open endpoint (1, 1)
Sketch $y = -x, x \geq 1$
Line, closed endpoint (1, −1)
Point $x = 2 \Rightarrow (2, -2)$



2. State the domain and range.

The domain is R.
The range is $(-\infty, -1] \cup [0, \infty)$.

c. State any value of x where the branches of the graph do not join.

c. The function is not continuous at $x = 1$ because there is a break in the graph.

studyon

Units 1 & 2 > AOS 1 > Topic 5 > Concept 5

Other functions and relations Summary screen and practice questions

Exercise 6.6 Other functions and relations

Technology free

1. Identify the maximal domains of the functions with the following rules.

a. $y = \sqrt{3 - 2x}$ **b.** $y = \dfrac{1}{1 - 2x}$ **c.** $y = 2x^2 - 3x + 1$

2. WE16 Give the maximal domains of the functions with the following rules.

a. $y = \dfrac{1}{16 - x^2}$ **b.** $y = \dfrac{2 - x}{x^2 + 3}$ **c.** $y = \dfrac{1}{\sqrt{x^3 + 1}}$

d. $y = \sqrt{x^2 - 4}$ **e.** $y = \dfrac{2x}{x^2 - 4}$ **f.** $y = \dfrac{1}{\sqrt{4 - x}}$

g. $y = \sqrt{x} + \sqrt{2 - x}$ **h.** $y = 4x^6 + 6x^{-1} + 2x^{\frac{1}{2}}$ **i.** $y = \dfrac{\sqrt{x^2 + 2}}{x^2 + 8}$

UNCORRECTED PAGE PROOFS

3. Give the implied domains of the functions defined by the following rules.
   a. $f(x) = \dfrac{1}{x^2 + 5x + 4}$ b. $g(x) = \sqrt{x + 3}$ c. $h(x) = f(x) + g(x)$
4. Consider the set of points $S = \{(2, 9), (3, 10), (4, 11), (5, 12)\}$.
   a. State the domain and range of the set S.
   b. Write down the set of points which form the inverse of the set S and give its domain and range.
   c. Form a rule for the points in set S.
   d. What is the rule for the points in the inverse set?
5. **WE17** Consider the linear function $f: (-\infty, 2] \to R$, $f(x) = 6 - 3x$.
   a. State the domain and determine the range of f.
   b. State the domain and range of f^{-1}, the inverse of f.
   c. Form the rule for the inverse and express the inverse function in mapping notation.
   d. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.
6. For each of the following functions
   i. state the domain and range
   ii. form the rule for the inverse
   iii. state the domain and range of the inverse.
   a. $y = \dfrac{x + 3}{4}$ b. $y = 2 - \dfrac{5}{x}$
   c. $y = 10 - 2x$, $x \in [-1, 2]$ d. $y = \sqrt{x - 2}$
7. Consider the linear function $f: R \to R$, $f(x) = 3x - 2$.
   a. Sketch the graph of $y = f(x)$.
   b. Hence sketch the graph of its inverse.
   c. What are the coordinates of the point of intersection of the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$?
   d. Form the mapping for the inverse function f^{-1}.
8. Find the rule for the inverse of each of the following.
   a. $4x - 8y = 1$ b. $y = -\dfrac{2}{3}x - 4$ c. $y^2 = 4x$
   d. $y = 4x^2$ e. $x^2 + (y - 3)^2 = 1$ f. $y = \sqrt{2x + 1}$
9. Consider the hyperbola function with the rule $f(x) = \dfrac{1}{x - 2}$ defined on its maximal domain.
   a. State the maximal domain and the equations of the asymptotes of the function.
   b. Obtain the rule for its inverse.
   c. State the equations of the asymptotes of the inverse.
   d. For any hyperbola with equation $y = \dfrac{1}{x - a} + b$, what would the equations of the asymptotes of its inverse be? Write down the equation of the inverse.
10. Consider $f: [-2, 4) \to R$, $f(x) = 4 - 2x$.
    a. What are the domain and range of f^{-1}?
    b. Obtain the rule for f^{-1} and write f^{-1} as a mapping.
    c. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.
    d. For what value of x does $f(x) = f^{-1}(x)$?
11. Consider the function with rule $y = (x + 4)^2 + 2$.
    a. State the maximal domain and explain why the inverse of $y = (x + 4)^2 + 2$ is not a function.
    b. State the domain and range of the inverse of $f: [-4, 0] \to R$, $f(x) = (x + 4)^2 + 2$ and explain why the inverse is a function.
    c. Form the rule $f^{-1}(x)$.

UNCORRECTED PAGE PROOFS

12. **a.** Explain why the inverse of $g : R \to R, g(x) = (x - 1)^3 + 2$ will be a function.

b. Form the inverse function of $g : R \to R, g(x) = (x - 1)^3 + 2$ and state the domain and range of this inverse.

13. **a.** State the range and the implied domain of $y = 6x - x^2$.

b. Given $h : (-\infty, k] \to R, h(x) = 6x - x^2$, find the largest value of k so that the inverse of h will be a function.

14. **WE18** Consider the quadratic function $y = (x + 1)^2$ defined on its maximal domain.

a. Form the rule for its inverse and explain why the inverse is not a function.

b. Sketch the graph of $y = (x + 1)^2, x \in [-1, \infty)$ and use this to sketch its inverse on the same diagram.

c. Form the equation of the inverse of $y = (x + 1)^2, x \in [-1, \infty)$.

d. At what point do the two graphs intersect?

15. Consider the semicircle with equation $y = \sqrt{4 - x^2}$.

a. Sketch the semicircle and its inverse on the same set of axes; state the type of correspondence for each graph.

b. Is the inverse a function?

16. Given $f : [a, \infty) \to R, f(x) = x^2 - 4x + 9$, find the smallest value of a so that f^{-1} exists.

17. **a.** Sketch the graphs of the following by reflecting the given relation in the line $y = x$.

i. $y = 1$ and its inverse

ii. $y = \sqrt{-x}$ and its inverse

iii. $y = \dfrac{1}{x^2}$ and its inverse

iv. $y = x^3 - 1$ and its inverse

v. $y = 1 - \sqrt{1 - x^2}$ and its inverse

vi. $(y - 2)^2 = 1 - x$ and its inverse

b. For those relations in part **a** that are functions, state which of them have inverses that are also functions and give the inverses' rules.

18. **WE19** On the same set of axes sketch the graphs of $y = x^n$ for $n = 1$ and $n = \dfrac{1}{3}$; hence, state $\{x : x^{\frac{1}{3}} > x\}$.

19. **a.** Calculate the coordinates of the points of intersection of the two functions $y = x^n$, $n = -1$ and $n = \dfrac{1}{3}$.

b. Sketch the graphs of $y = x^n$, $n = -1$ and $n = \dfrac{1}{3}$ on the same set of axes.

20. **WE20** Consider the function for which $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \geq 1 \end{cases}$

a. Evaluate the following.

i. $f(-2)$ **ii.** $f(1)$ **iii.** $f(2)$

b. Sketch the graph of $y = f(x)$ and state the domain and range.

c. State any value of x for which the function is not continuous.

Technology active

21. Sketch the graphs of each of the following hybrid functions and state their domain and range.

a. $y = \begin{cases} 2, & x \leq 0 \\ 1 + x^2 & x > 0 \end{cases}$

b. $y = \begin{cases} x^3, & x < 1 \\ x, & x \geq 1 \end{cases}$

c. $y = \begin{cases} -2x, & x < -1 \\ 2, & -1 \leq x \leq 1 \\ 2x, & x > 1 \end{cases}$

d. $y = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ \dfrac{1}{2 - x}, & x > 2 \end{cases}$

e. $y = \begin{cases} x^{\frac{1}{3}}, & x \geq 0 \\ -x^{-2}, & x < 0 \end{cases}$

f. $y = \begin{cases} x^3, & x < -1 \\ \sqrt[3]{x}, & -1 \leq x \leq 1 \\ 3x, & x > 1 \end{cases}$

UNCORRECTED PAGE PROOFS

22. Consider $f: R \to R, f(x) = \begin{cases} -x-1, & x < -1 \\ \sqrt{1-x^2}, & -1 \le x \le 1 \\ x+1, & x > 1 \end{cases}$

a. Calculate the value of:

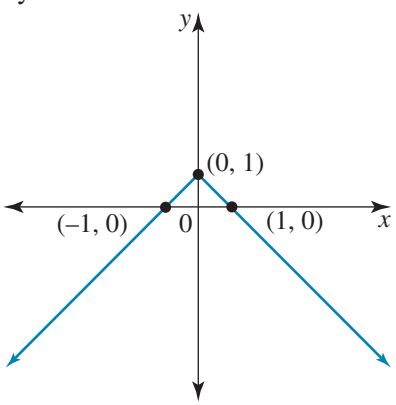
i. $f(0)$ **ii.** $f(3)$ **iii.** $f(-2)$ **iv.** $f(1)$

b. Show the function is not continuous at $x = 1$.

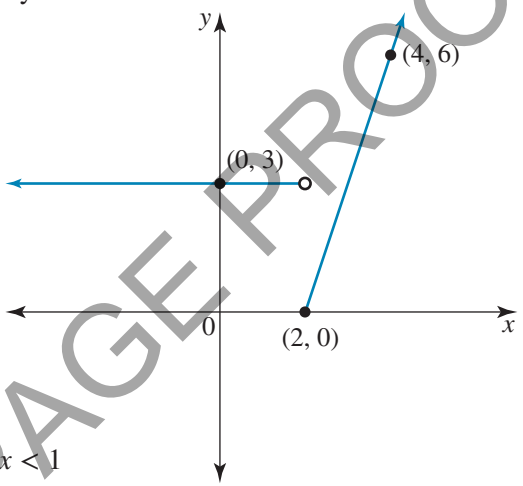
c. Sketch the graph of $y = f(x)$ and state the type of correspondence.

d. Determine the value of a such that $f(a) = a$.

23. a. Form a rule for the graph of the hybrid function.



b. Form the rule for the graph of the hybrid function.



24. a. Consider the function defined by $f(x) = \begin{cases} 4x + a, & x < 1 \\ \frac{2}{x}, & 1 \le x \le 4 \end{cases}$

i. Determine the value of a so the function will be continuous at $x = 1$.

ii. Explain whether the function is continuous at $x = 0$.

b. Determine the values of a and b so that the function with the rule

$$f(x) = \begin{cases} a, & x \in (-\infty, -3] \\ x+2, & x \in (-3, 3) \\ b, & x \in [3, \infty) \end{cases}$$

is continuous for $x \in R$; for these values, sketch the graph of $y = f(x)$.

25. In an effort to reduce the time her children spend in the shower, a mother introduced a penalty scheme with fines to be paid from the children's pocket money according to the following: If someone spends more than 5 minutes in the shower, the fine in dollars is equal to the shower time in minutes; if someone spends up to and including 5 minutes in the shower, there is no fine. If someone chooses not to shower at all, there is a fine of $2 because that child won't be nice to be near.

Defining appropriate symbols, express the penalty scheme as a mathematical rule in hybrid form and sketch the graph which represents it.

26. Consider the function $g: D \to R, g(x) = x^2 + 8x - 9$.

a. Give two possible domains D for which g^{-1} exists.

b. If the domain of g is chosen to be R^+, form g^{-1} and state its range.

c. **i.** For this domain R^+, sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes.

ii. Calculate the exact coordinates of any point(s) of intersection of the two graphs.

d. What number is its own image under the mapping g^{-1}?

UNCORRECTED PAGE PROOFS

27. A function is defined by the rule $f(x) = \begin{cases} \sqrt{x-4}+2, & x > 4 \\ 2, & 1 \leq x \leq 4 \\ \sqrt{1-x}+2, & x < 1 \end{cases}$

a. Calculate the values of $f(0)$ and $f(5)$.

b. Sketch $y = f(x)$ and state its domain and range.

c. For what values of x does $f(x) = 8$?

d. Will the inverse of this function also be a function? Draw a sketch of the graph of the inverse on a new set of axes.

28. a. Use CAS technology to find algebraically the rule for the inverse of $y = x^2 + 5x - 2$.

b. Use CAS technology to draw the graph of $y = x^2 + 5x - 2$ and its inverse and find, to 2 decimal places, the coordinates of the points of intersection of $y = x^2 + 5x - 2$ with its inverse.

29. Consider the hybrid function with the rule $y = \begin{cases} \sqrt{2+x^2}, & x \leq -2 \\ x\sqrt{2+x}, & -2 < x < 2 \\ \dfrac{1}{\sqrt{2+x}}, & x \geq 2 \end{cases}$

a. Identify any points of discontinuity.

b. Graph the function and give its exact range.

6.7 Transformations of functions

Once the basic shape of a function is known, its features can be identified after various **transformations** have been applied to it simply by interpreting the transformed equation of the image. For example, we recognise the equation $y = (x-1)^2 + 3$ to be a parabola formed by translating the basic parabola $y = x^2$ horizontally 1 unit to the right and vertically 3 units up. In exactly the same way, we recognise $y = \dfrac{1}{x-1} + 3$ to be a hyperbola formed by translating the basic hyperbola $y = \dfrac{1}{x}$ horizontally 1 unit to the right and vertically 3 units up. The only difference is the basic shape.

In this section we extend such interpretation of the transformed equation to consider the image of a general function $y = f(x)$ under a sequence of transformations.

6.7.1 Horizontal and vertical translations of $y = f(x)$

Translations parallel to the x- and y-axes move graphs horizontally to the left or right and vertically up or down, respectively.

Under a horizontal translation of h units to the right, the following effect is seen:

$$\begin{aligned} y = x^2 &\rightarrow y = (x-h)^2; \\ y = \frac{1}{x} &\rightarrow y = \frac{1}{x-h}; \\ y = \sqrt{x} &\rightarrow y = \sqrt{x-h}; \end{aligned}$$

and so, for any function, $y = f(x) \rightarrow y = f(x-h)$.

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Under a vertical translation of k units upwards:

$$\begin{aligned} y = x^2 &\rightarrow y = x^2 + k; \\ y = \frac{1}{x} &\rightarrow y = \frac{1}{x} + k; \\ y = \sqrt{x} &\rightarrow y = \sqrt{x} + k; \end{aligned}$$

and so, for any function, $y = f(x) \rightarrow y = f(x) + k$

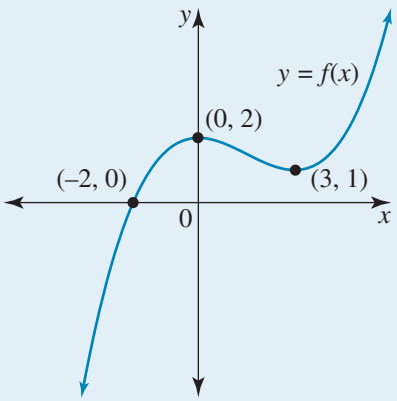
Thus, for any function:

- **$y = f(x - h)$ is the image of $y = f(x)$ under a horizontal translation of h units to the right**
- **$y = f(x) + k$ is the image of $y = f(x)$ under a vertical translation of k units upwards**
- **under the combined transformations of h units parallel to the x-axis and k units parallel to the y-axis, $y = f(x) \rightarrow y = f(x - h) + k$.**

WORKED EXAMPLE 21

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$ $(0, 2)$ $(3, 1)$.

Sketch the graph of $y = f(x+1)$ using the images of these three points.



THINK

1. Identify the transformation required.

2. Find the image of each key point.

WRITE

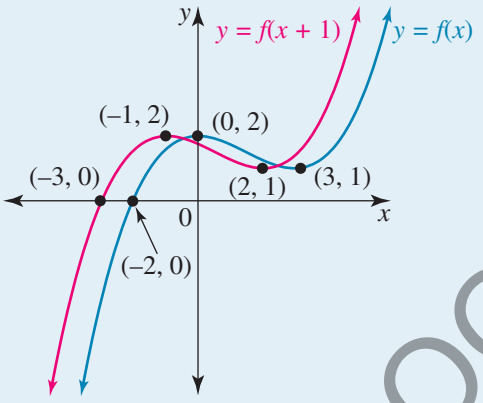
$y = f(x + 1)$
This is a horizontal translation 1 unit to the left of the graph of $y = f(x)$.

Under this transformation:

$(-2, 0) \rightarrow (-3, 0)$
$(0, 2) \rightarrow (-1, 2)$
$(3, 1) \rightarrow (2, 1)$

UNCORRECTED PAGE PROOFS

3. Sketch the image.

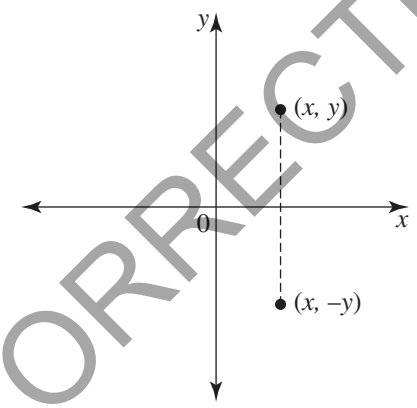


on Resources

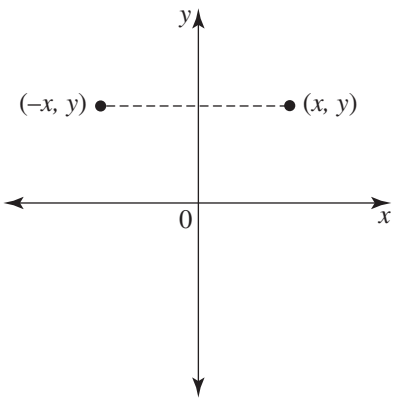
Interactivity: Transformations of functions (int-2576)

6.7.2 Reflections in the coordinate axes

The point (x, y) becomes $(x, -y)$ when reflected in the x-axis and $(-x, y)$ when reflected in the y-axis.



Reflection in the x-axis



Reflection in the y-axis

Reflecting the graph of $y = \sqrt{x}$ in the x-axis gives the graph of $y = -\sqrt{x}$, so under a reflection in the x-axis, $y = f(x) \rightarrow y = -f(x)$.

Reflecting the graph of $y = \sqrt{x}$ in the y-axis gives the graph of $y = \sqrt{-x}$, so under a reflection in the y-axis, $y = f(x) \rightarrow y = f(-x)$.

For any function:

- **$y = -f(x)$ is the image of $y = f(x)$ under a reflection in the x-axis**
- **$y = f(-x)$ is the image of $y = f(x)$ under a reflection in the y-axis.**

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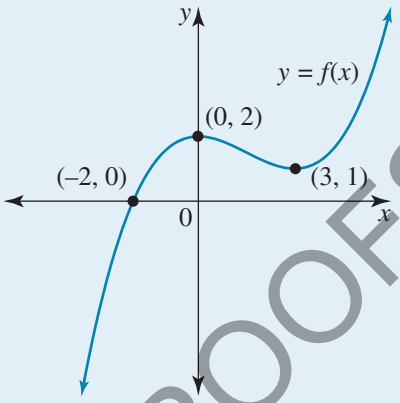
WORKED EXAMPLE 22

Consider again the graph in Worked example 21.
Sketch the graph of $y = f(-x)$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.



THINK

1. Identify the transformation required.

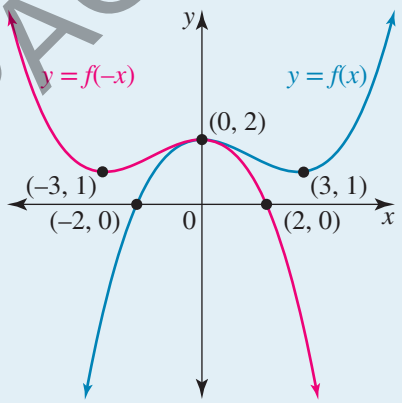
WRITE

$y = f(-x)$
This is a reflection in the y-axis of the graph of $y = f(x)$.

2. Find the image of each key point.

Under this transformation,
$(x, y) \rightarrow (-x, y)$
$(-2, 0) \rightarrow (2, 0)$
$(0, 2) \rightarrow (0, 2)$
$(3, 1) \rightarrow (-3, 1)$

3. Sketch the image.



6.7.3 Dilations from the coordinate axes

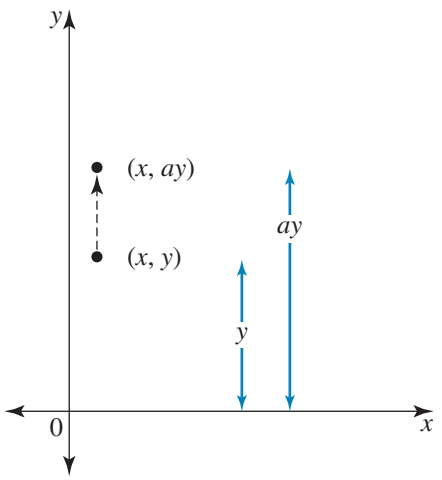
A dilation from an axis either stretches or compresses a graph from that axis, depending on whether the dilation factor is greater than 1 or between 0 and 1, respectively.

6.7.4 Dilation from the x-axis by factor a

A dilation from the x-axis acts parallel to the y-axis, or in the y-direction.

The point $(x, y) \rightarrow (x, ay)$ when dilated by a factor a from the x-axis.

A dilation of factor a from the x-axis transforms $y = x^2$ to $y = ax^2$ and, generalising, under a dilation of factor a from the x-axis, $y = f(x) \rightarrow y = af(x)$.



Dilation of factor a, $(a > 1)$, from the x-axis

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For any function:

- **$y = af(x)$ is the image of $y = f(x)$ under a dilation of factor a from the x-axis, parallel to the y-axis.**

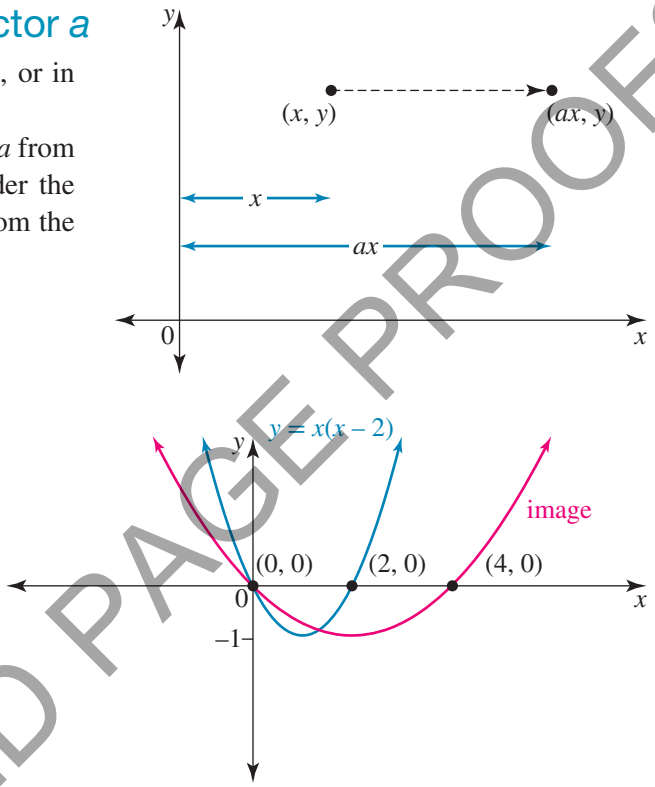
6.7.5 Dilation from the *y*-axis by factor *a*

A dilation from the y-axis acts parallel to the x-axis, or in the x-direction.

The point $(x, y) \rightarrow (ax, y)$ when dilated by a factor a from the y-axis. To see the effect of this dilation, consider the graph of $y = x(x - 2)$ under a dilation of factor 2 from the y-axis. Choosing the key points, under this dilation:

$$\begin{aligned} (0, 0) &\rightarrow (0, 0) \\ (1, -1) &\rightarrow (2, -1) \\ (2, 0) &\rightarrow (4, 0) \end{aligned}$$

and the transformed graph is as shown.



The equation of the image of $y = x(x-2)$ can be found by fitting the points to a quadratic equation. Its equation is

$$y = (0.5x)(0.5x - 2) \Rightarrow y = \left(\frac{x}{2}\right)\left(\left(\frac{x}{2}\right) - 2\right).$$

This illustrates that dilating $y = f(x)$ by a factor a from the y-axis gives the image $y = f\left(\frac{x}{a}\right)$.

For any function:

- $y = f(ax)$ is the image of $y = f(x)$ under a dilation of factor $\frac{1}{a}$ from the y-axis, parallel to the x-axis.
- $y = f\left(\frac{x}{a}\right)$ is the image of $y = f(x)$ under a dilation of factor a from the y-axis, parallel to the x-axis.

Since $y = af(x)$ could be written as $\frac{y}{a} = f(x)$, and $y = f(ax)$ as $y = f\left(\frac{x}{\frac{1}{a}}\right)$, dilations from the coordinate axes can be summarised as follows.

- **Under a dilation of factor a from the x-axis (in the y-direction), $y = f(x) \rightarrow \frac{y}{a} = f(x)$.**
- **Under a dilation of factor a from the y-axis (in the x-direction), $y = f(x) \rightarrow y = f\left(\frac{x}{a}\right)$.**

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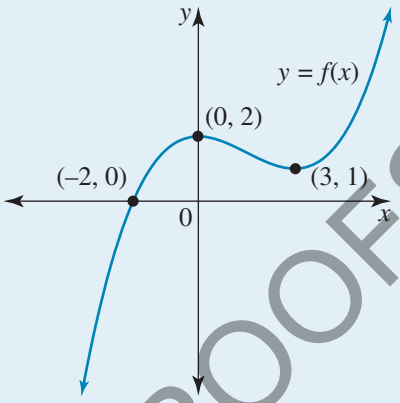
WORKED EXAMPLE 23

For the graph given in Worked examples 21 and 22, sketch the graph of $y = f(2x)$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.



THINK

1. Identify the transformation.

WRITE

$y = f(2x) \Rightarrow y = f\left(\frac{x}{\frac{1}{2}}\right)$

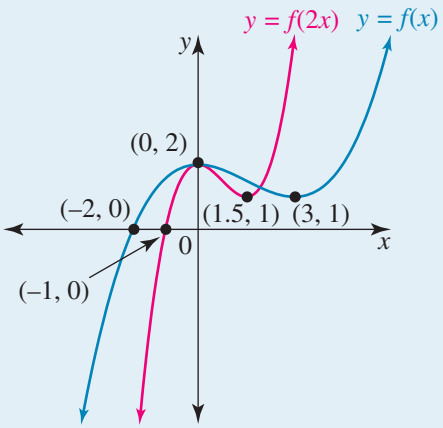
The transformation is a dilation from the y-axis of factor $\frac{1}{2}$.
This dilation acts in the x-direction.

2. Find the image of each key point.

Under this dilation $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$

$(-2, 0) \rightarrow (-1, 0)$
$(0, 2) \rightarrow (0, 2)$
$(3, 1) \rightarrow (1.5, 1)$

3. Sketch the image.



6.7.6 Combinations of transformations

The graph of $y = af(n(x - h)) + k$ is the graph of $y = f(x)$ under a set of transformations which are identified as follows.

- a gives the dilation factor $|a|$ from the x-axis, parallel to the y-axis.
- If $a < 0$, there is a reflection in the x-axis.
- n gives the dilation factor $\frac{1}{|n|}$ from the y-axis, parallel to the x-axis.
- If $n < 0$, there is a reflection in the y-axis.
- h gives the horizontal translation parallel to the x-axis.
- k gives the vertical translation parallel to the y-axis.

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When applying transformations to $y = f(x)$ to form the graph of $y = af(n(x-h))+k$, the order of operations can be important, so any dilation or reflection should be applied before any translation.

It is quite possible that more than one order or more than one set of transformations may achieve the same image. For example, $y = 4x^2$ could be considered a dilation of $y = x^2$ by factor 4 from the x-axis or, as $y = (2x)^2$, it's also a dilation of $y = x^2$ by a factor $\frac{1}{2}$ from the y-axis.

WORKED EXAMPLE 24

a. Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4 - 2f(3x + 2)$.

b. Describe the transformations applied to the graph of $y = \sqrt[3]{x}$ to obtain $y = \sqrt[3]{6 - 2x}$.

THINK	WRITE
a. 1. Express the image equation in the summary form.	**a.** $y = 4 - 2f(3x + 2)$ $= -2f\left(3\left(x + \frac{2}{3}\right)\right) + 4$
2. State the values of a, n, h, k from the summary form.	$y = af(n(x - h)) + k$ $a = -2, n = 3, h = -\frac{2}{3}, k = 4$
3. Interpret the transformations, leaving the translations to last.	Dilation of factor 2 from the x-axis, followed by a reflection in the x-axis; then, a dilation of factor $\frac{1}{3}$ from the y-axis; then, a horizontal translation $\frac{2}{3}$ units to the left; finally, a vertical translation upwards of 4 units
b. 1. Express the image equation in the summary form.	**b.** $y = \sqrt[3]{6 - 2x}$ $= \sqrt[3]{-2(x - 3)}$
2. Identify the transformations in the correct order.	Dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the y-axis; then, a horizontal translation 3 units to the right

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Units 1 & 2 > AOS 1 > Topic 5 > Concept 6

Transformations of functions Summary screen and practice questions

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Exercise 6.7 Transformations of functions

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1. Identify the transformations that would be applied to the graph of $y = x^2$ to obtain each of the following graphs.
   a. $y = 3x^2$ b. $y = -x^2$ c. $y = x^2 + 5$ d. $y = (x + 5)^2$
2. a. After the graph of $y = x^2$ is translated 3 units upwards and 2 units to the left, what will its equation become?
   b. A transformation of $y = \frac{1}{x}$ by a dilation factor 2 parallel to the y-axis and a horizontal translation to the right of 5 units would be represented by which equation?
   c. The graph of $y = \sqrt{x}$ is reflected in the x-axis and dilated by a factor $\frac{1}{2}$ from the x-axis. What does its equation become?
   d. After the graph of $y = x^4$ is moved downwards 4 units and then reflected in the x-axis, what will its equation become?
3. Describe the transformations required to change
   a. $y = \sqrt{x}$ into $y = \sqrt{x + 4} - 5$
   b. $y = \frac{1}{x}$ into $y = \frac{3}{x - 2} + 4$
   c. $y = x^3$ into $y = 2 - (x - 3)^3$
   d. $y = \frac{1}{x^2}$ into $y = \frac{-3}{2(x + 1)^2} + 4$
   e. $y = f(x)$ into $y = f(x + 1) + 2$
   f. $y = f(x)$ into $y = -4\ f(x)$.
4. **WE21** For the graph of $y = f(x)$ given in Worked example 21, sketch the graph of $y = f(x) - 2$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.
5. For the graph of $y = f(x)$ given in Worked example 21, sketch the graph of $y = f(x - 2) + 1$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.
6. **WE22** Consider again the graph given in Worked examples 21 and 22.
   Sketch the graph of $y = -f(x)$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.
7. a. The parabola with equation $y = (x - 1)^2$ is reflected in the x-axis followed by a vertical translation upwards of 3 units. What is the equation of its final image?
   b. Obtain the equation of the image if the order of the transformations in part **a** was reversed. Is the image the same as that in part **a**?
8. Describe the transformations that have been applied to the graph of $y = x^3$ to obtain each of the following graphs.
   a. $y = \left(\frac{x}{3}\right)^3$ b. $y = (2x)^3 + 1$ c. $y = (x - 4)^3 - 4$ d. $y = (1 + 2x)^3$
9. **WE23** For the graph in Worked example 23, sketch the graph of $y = f\left(\frac{x}{2}\right)$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.
10. For the graph of Worked example 23, sketch the graph of $y = \frac{1}{2} f(x)$ using the images of the points $(-2, 0), (0, 2), (3, 1)$.
11. Give the equation of the image of
    i. $y = \sqrt{x}$ and
    ii. $y = x^4$ if their graphs are:
    a. dilated by a factor 2 from the x-axis
    b. dilated by a factor 2 from the y-axis
    c. reflected in the x-axis and then translated 2 units vertically upwards
    d. translated 2 units vertically upwards and then reflected in the x-axis

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e. reflected in the y-axis and then translated 2 units to the right

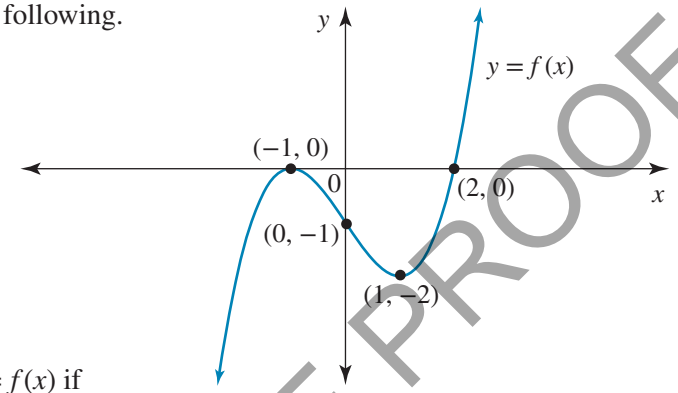
f. translated 2 units to the right and then reflected in the y-axis.

12. WE24 **a.** Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4f\left(\frac{x}{2} - 1\right) + 3$.

b. Describe the transformations applied to the graph of $y = \sqrt{x}$ to obtain $y = \sqrt{3 - \frac{x}{4}}$.

13. The graph of $y = f(x)$ is shown.

On separate diagrams sketch the graphs of the following.



a. $y = f(x - 1)$ **b.** $y = -f(x)$

c. $y = 2f(x)$ **d.** $y = f(-x)$

e. $y = f\left(\frac{x}{2}\right)$ **f.** $y = f(x) + 2$

14. Deduce the graph of $y = x^3(x - 4)^2 + 7$ from the graph of $y = x^3(x - 4)^2$.

15. a. Describe the transformations applied to $y = f(x)$ if its image is $y = 2f(-x) + 1$.

b. Describe the transformations applied to $y = f(x)$ if its image is $y = -f(3x)$.

c. What are the transformations required to change $y = \sqrt{x}$ into $y = \frac{1}{2}\sqrt{4 - x}$?

d. What is the equation of the image of $y = \frac{1}{x + 1}$ after reflection in the y-axis and a dilation of factor $\frac{1}{3}$ from the y-axis (parallel to x-axis)?

16. a. The graph of $y = \frac{1}{x}$ undergoes two transformations in the order: dilation of factor $\frac{1}{2}$ from the y-axis, followed by a horizontal translation of 3 units to the left.

What is the equation of its image?

b. Describe the sequence of transformations that need to be applied to the image to undo the effect of the transformations and revert to the graph of $y = \frac{1}{x}$.

17. Give the coordinates of the image of the point $(3, -4)$ if it is:

a. translated 2 units to the left and 4 units down

b. reflected in the y-axis and then reflected in the x-axis

c. dilated by a factor $\frac{1}{5}$ from the x-axis parallel to the y-axis

d. dilated by a factor $\frac{1}{5}$ from the y-axis parallel to the x-axis

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18. a. i. Give the equation of the image of $y = \frac{1}{x}$ after the two transformations are applied in the order given: dilation by a factor 3 from the y-axis, then reflection in the y-axis.

ii. Reverse the order of the transformations and give the equation of the image.

b. i. Give the equation of the image of $y = \frac{1}{x^2}$ after the two transformations are applied in the order given: dilation by a factor 3 from the x-axis, then vertical translation 6 units up.

ii. Reverse the order of the transformations and give the equation of the image.

c. Describe the transformations applied to $y = \frac{1}{x}$ if its image has the equation $y = -\frac{1}{2x + 2} + 1$.

d. If $f(x) = \frac{1}{x^2}$, give the equations of the asymptotes of $y = -2f(x + 1)$.

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19. Describe the transformations applied to $y = f(x)$ if its image is:

a. $y = 2f(x + 3)$ **b.** $y = 6f(x - 2) + 1$ **c.** $y = f(2x + 2)$

d. $y = f(-x + 3)$ **e.** $y = 1 - f(4x)$ **f.** $y = \frac{1}{9} f\left(\frac{x - 3}{9}\right)$.

20. Form the equation of the image after the given functions have been subjected to the set of transformations in the order specified.

a. $y = \frac{1}{x^2}$ undergoes a dilation of factor $\frac{1}{3}$ from the x-axis followed by a horizontal translation of 3 units to the left.

b. $y = x^5$ undergoes a vertical translation of 3 units down followed by reflection in the x-axis.

c. $y = \frac{1}{x}$ undergoes a reflection in the y-axis followed by a horizontal translation of 1 unit to the right.

d. $y = \sqrt[3]{x}$ undergoes a horizontal translation of 1 unit to the right followed by a dilation of factor 0.5 from the y-axis.

e. $y = (x + 9)(x + 3)(x - 1)$ undergoes a horizontal translation of 6 units to the right followed by a reflection in the x-axis.

f. $y = x^2(x + 2)(x - 2)$ undergoes a dilation of factor 2 from both the x- and y-axis.

21. **a.** The function $g : R \rightarrow R, g(x) = x^2 - 4$ is reflected in the y-axis. Describe its image.

b. Show that the image of the function $f : R \rightarrow R, f(x) = x^{\frac{1}{3}}$ when it is reflected in the y-axis is the same as when it is reflected in the x-axis.

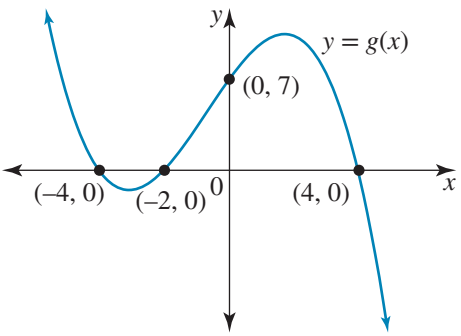
c. The function $h : [-3, 3] \rightarrow R, h(x) = -\sqrt{9 - x^2}$ is reflected in the x-axis. Describe its image. What single transformation when applied to the image would return the curve back to its original position?

d. The graph of $y = (x - 2)^2 + 5$ is reflected in both the x- and y-axis. What is the nature, and the coordinates, of the turning point of its image?

e. The graph of a relation is shifted vertically down 2 units, then reflected in the y-axis. If the equation of its image is $y^2 = (x - 3)$, undo the transformations (that is, form the inverse transformations) to obtain the equation of the original graph.

f. A curve $y = f(x)$ is dilated by a factor 2 from the x-axis, then vertically translated 1 unit up, then reflected in the x-axis. After these three transformations have been applied, the equation of its image is $y = 6(x - 2)^3 - 1$. Determine the equation of $y = f(x)$.

22. The graph of the function $y = g(x)$ is given.



a. Sketch the graph of $y = -g(2x)$.

b. Sketch the graph of $y = g(2 - x)$.

c. For what values of h will all the x-intercepts of the graph of $y = g(x + h)$ be negative?

d. Give a possible equation for the graph of $y = g(x)$ and hence find an expression for $g(2x)$.

23. Using CAS technology, on the same screen, graph $y_1 = x^2 + 5x - 6, y_2 = (2x)^2 + 5(2x) - 6$ and $y_3 = \left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right) - 6$ and compare the graphs. Which are parabolas?

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6.8 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Short answer: technology free

1. Sketch each of the following, stating the domain and range.

a. $y = -\sqrt{4 - x^2}$

b. $y = -\sqrt{4 - x}$

c. $y = 4 + \sqrt[3]{x}$

d. $y = \dfrac{4}{(4 - x)^2} + 4$

e. $y = \dfrac{6x + 5}{3x + 1}$

f. $x = (y + 2)(y - 4)$

2. **a.** Specify the centre and radius of the circle $2x^2 + 2y^2 - 12x + 8y + 3 = 0$.

b. Does the point $(-6, -3)$ lie inside, on, or outside this circle?

3. Consider the function defined by $f(x) = \begin{cases} 3, & x < 1 \\ 4 - 2x, & x \geq 1 \end{cases}$

a. Evaluate $f(0) + f(1) + f(2)$.

b. Sketch $y = f(x)$.

c. State the domain and range of the function.

d. State its type of correspondence.

4. Consider $f:[-2, 4) \to R$, $f(x) = 1 - \dfrac{x}{2}$.

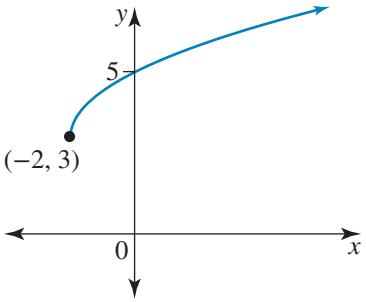
a. State the domain and determine the range of f.

b. Obtain the rule for the inverse function f^{-1}, giving its domain and range.

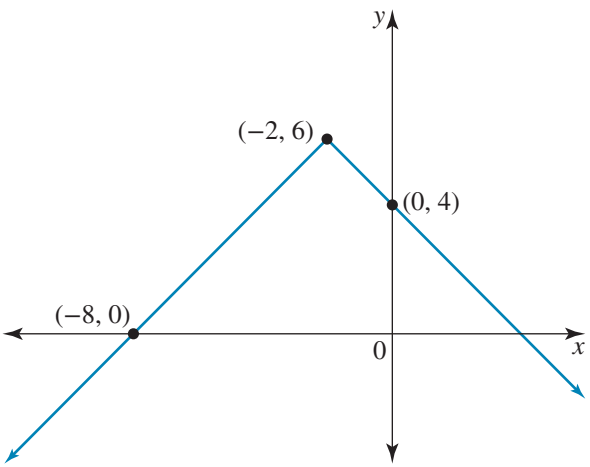
c. Sketch f and f^{-1} on the same diagram.

d. Calculate the coordinates of any point of intersection of the two graphs.

5. **a.** Form a possible equation for the square root function shown.



b. Form the rule in hybrid form for the given graph.



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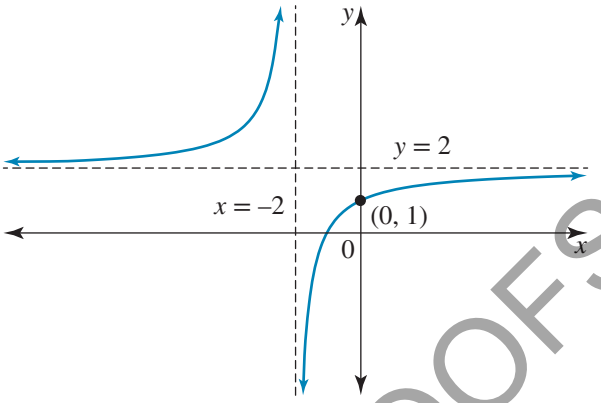
c. Obtain the equation of the hyperbola shown in the diagram.



6. a. Determine the rule for the inverse of $y = 1 - 8x^3$.

b. Determine the rule for the inverse of $y = (x + 1)^{\frac{1}{3}} + 5$.

c. i. Obtain the largest value of a so that the inverse of $f: (-\infty, a] \to R, f(x) = (x - 2)^2 + 6$, will be a function.

ii. Use the value obtained for a to determine the inverse function f^{-1}.

Multiple choice: technology active

1. MC Which of the following is not a function?

A. $\{(x, y) : y = x^2\}$ **B.** $\{(x, y) : xy = 1\}$ **C.** $\{(x, y) : y = \sqrt{-2x}\}$

D. $\{(x, y) : y = \sqrt{2 - x^2}\}$ **E.** $\{(x, y) : y^2 = x\}$

2. MC The range of the function $f: [-2, \infty) \to R, f(x) = 3[(x + 2)^2 + 5]$ is:

A. $[-2, \infty)$ **B.** $[5, \infty)$ **C.** $[6, \infty)$ **D.** $[15, \infty)$ **E.** R

3. MC The equation of the circle with centre $(2, -5)$ and radius of length 9 units is:

A. $(x - 2)^2 + (y + 5)^2 = 9$ **B.** $(x - 2)^2 + (y + 5)^2 = 81$ **C.** $(x + 2)^2 + (y - 5)^2 = 3$

D. $(x + 2)^2 + (y - 5)^2 = 9$ **E.** $(x - 2)^2 - (y + 5)^2 = 81$

4. MC The equations of the asymptotes of the truncus $y = \dfrac{-3}{(x + 2)^2} + 4$ are:

A. $x = -2, y = 4$ **B.** $x = 2, y = -4$ **C.** $x = -4, y = 4$

D. $x = 4, y = -4$ **E.** $x = -\frac{3}{4}, y = 4$

5. MC The graph of $(y - c)^2 = a(x - b)$ has a vertex at $(-2, 5)$ and passes through the point $(6, 1)$. The values of a, b and c are:

A. $a = 9, b = 2, c = -5$ **B.** $a = -9, b = 2, c = -5$ **C.** $a = 9, b = 5, c = -2$

D. $a = 2, b = -2, c = 5$ **E.** $a = -2, b = -2, c = 5$

6. MC For the function $f: [-2, 4] \to R, f(x) = ax + b, f(0) = 1$ and $f(1) = 0$. The image of -2 under the mapping is:

A. -2 **B.** -1 **C.** 1 **D.** 3 **E.** 4

7. MC Which of the following has an inverse which is a function?

A. $y = x^2 - 1$ **B.** $x^2 + y^2 = 1$ **C.** $y = \dfrac{1}{x - 1}$ **D.** $y = \sqrt{1 - x^2}$ **E.** $y = 10$

8. MC The maximal domain of $y = \dfrac{x + 6}{\sqrt{x - 2}}$ is:

A. $R \backslash \{-6\}$ **B.** $R \backslash \{2\}$ **C.** $R \backslash \{-6, 2\}$ **D.** $[-2, \infty)$ **E.** $(2, \infty)$

9. MC The graph of $y = \sqrt{x}$ is translated 4 units upwards and then reflected in the y-axis. The equation of its image is:

A. $y = -\sqrt{x} + 4$ **B.** $y = \sqrt{-x} + 4$ **C.** $y = -\sqrt{x} - 4$ **D.** $y = \sqrt{-x} - 4$ **E.** $y = -\sqrt{x - 4}$

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10. MC The graph of a function $y = f(x)$ is shown in diagram (i). The graph shown in diagram (ii) is of:

A. $y = f(-2x)$

B. $y = f\left(-\frac{1}{2}x\right)$

C. $y = -2f(x)$

D. $y = -\frac{1}{2}f(x)$

E. $y = f(x + 2)$

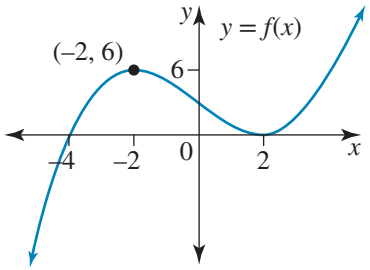


Diagram (i)

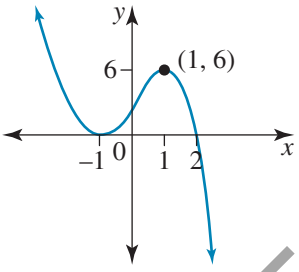


Diagram (ii)

Extended response: technology active

1. Consider the function given by $f(x) = x^2 - 10x + 21$.

a. Sketch the graph of $y = f(x)$, labelling the turning point and the intersections with the coordinate axes with their coordinates.

b. Determine the values of k so that the graph of $y = f(x) + k$ will not intersect the x-axis.

c. Determine the values of h so that the roots of the equation $f(x - h) = 0$ will always be negative.

d. Sketch the following transformations of the graph of $y = f(x)$, showing the images of its turning point and intersections with the coordinate axes.

i. $y = f(x + 3)$

ii. $y = f(-x)$

iii. $y = f(2x)$

iv. $y = f\left(4 + \frac{2x}{3}\right)$

v. $y = -0.5f(x) - 2$

vi. The reflection of $y = f(x)$ in the line $y = x$.

2. a. If the intensity I of a sound is inversely proportional to the distance d from the source of the sound, draw a graph which illustrates this relationship.

b. If the quantity $y + 2$ is inversely proportional to the square of the quantity $x + 2$, establish the relationship between x and y given $y = -0.5$ when $x = 4$, and hence calculate a so that $y = a$ when $x = a$.

c. i. Describe a reflection and a translation with respect to the x- and y-coordinate axes that will result in the curve $y = \frac{2}{x - 2} - 1$, $x > 2$ being reflected in the line $x = 2$. Give the equation of the image.

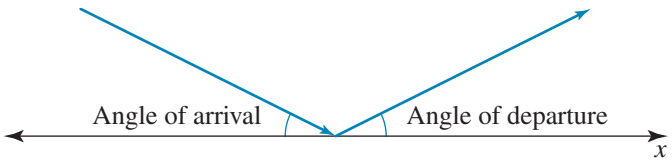
A small container with height h cm has it longitudinal cross-section in the shape of the curve with equation $h = \frac{2}{x - 2} - 1$, $2 < x \leq 4$ and its reflection in the line $x = 2$.

ii. Sketch this container and state the diameter of its circular base.

iii. If the diameter of the top of the container is 1 cm, calculate its height.

iv. If the container is filled with a powder to a height of 1.5 cm, what is the top circular surface area of the powder?

3. A ray of light comes in along the line $x + y = 2$ above the x-axis and is reflected off the axis so that the angle of departure (the angle of reflection) is equal to the angle of arrival (the angle of incidence).



a. i. Calculate the magnitude of the angle of departure. (*Hint:* $m = \tan\theta$.)

ii. Form the equation of the line along which the departing light travels.

iii. Express the path of the incoming and departing rays in terms of a hybrid function.

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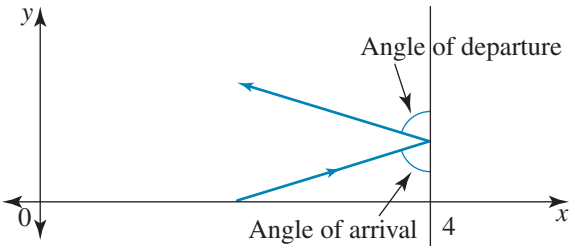
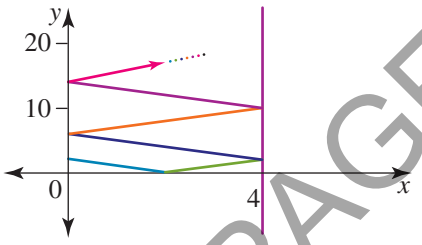
b. The reflected ray of light strikes the vertical line $x = 4$ and is reflected off this line in the same way so that the angle of departure is equal to the angle of arrival.



i. Give a reason why this section of the path of the ray of light is not a function.

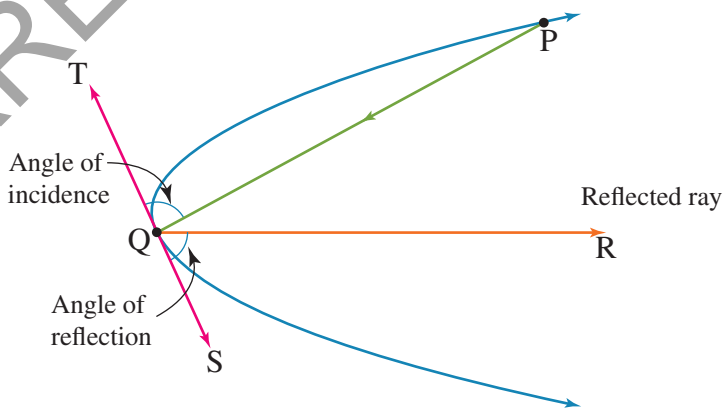
ii. Form a hybrid rule with x in terms of y, which describes both the incoming and departing paths of the ray of light for this section of its path.

c. The light rays continue to bounce back and forth between the vertical lines $x = 0$ and $x = 4$.



Predict the equation of the line the ray of light will travel along immediately after its fourth reflection on the line $x = 4$.

d. Light rays are reflected off parabolic mirrors in such a way that the angle of incidence the incoming ray of light makes with the tangent to the parabola is equal to the angle of departure made between the departing ray and the tangent. This is illustrated in the diagram where TS is a tangent to the parabola at a point Q, the incoming ray of light is PQ, the reflected ray is QR and the angles PQT and SQR are the angles of incidence and reflection respectively.



Let the equation of the parabola in the diagram be $y^2 = 4x$.

i. Show that P(9, 6) lies on the parabola.

ii. Show that the line with equation $3y + 9x + 1 = 0$ is a tangent to the parabola and give the coordinates of the point Q, its point of contact with the parabola.

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iii. Calculate the magnitude of the angle PQT, the angle of incidence (angle of arrival) between the light ray PQ and the tangent at Q.

iv. Using the fact that the angle of reflection (angle of departure) is equal to that of the angle of incidence, show that the reflected ray QR is parallel to the axis of symmetry of the parabola and give its equation.

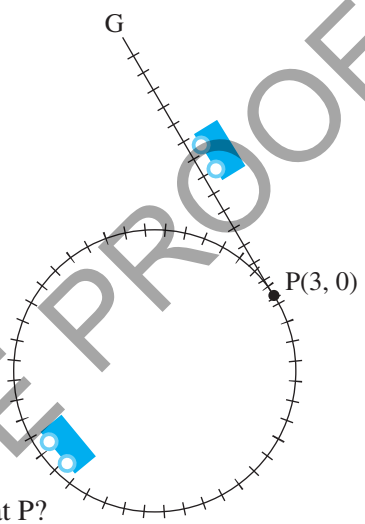
4. A circle passes through the three points $(1, 1)$, $(3, 0)$ and $(-1.5, -1.5)$.

a. Use the equation $x^2 + y^2 + ax + by + c = 0$ to calculate the values of a, b and c.

b. Hence obtain the centre and radius of the circle.

c. Sketch the circle, stating the domain and range.

A toy train runs around a circular track that passes through the points M$(-1.5, -1.5)$, N$(1, 1)$ and P$(3, 0)$. There is also a straight track tangential to the circle at point P which leads to a train depot at point G. All units are in metres.



d. Form the equation of the straight track PG.

e. A vertical line connects the depot at G with the centre of the circular track. Determine the coordinates of G.

f. One train engine continues to travel round and round the circular track at a speed of π m/s while a second train engine continues to shunt backwards and forwards between P and G at a speed of 1 m/s. If the small child playing with the trains releases both engines at the same time from point P, how long does it take before they will collide at P?

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Answers

Topic 6 Functions and relations

Exercise 6.2 Functions and relations

1. a. Domain is $\{-4, 0, 3\}$, range is $\{5, 7, 10\}$.

b. $[-10, 4)$

c. Domain $(2, 7)$, Range $\left(\frac{1}{7}, \frac{1}{2}\right)$.

d. Domain $[-2, 6]$, Range $[-3, 3]$.

2. a. $(-1, -2)$ is an open endpoint, $(3, 6)$ is a closed endpoint. Domain $(-1, 3]$; range $(-2, 6]$

b. $(3, 2)$ is an open endpoint. Domain $(3, \infty)$; range $(2, \infty)$

c. $(-5, 15)$ and $(5, -15)$ are closed endpoints. Domain $[-5, 5]$; range $[-15, 15]$

d. No endpoints. Domain $(-\infty, \infty)$; range $[0, \infty)$

e. Open endpoint at $(0, 0)$. Domain $(0, \infty)$; range $(0, \infty)$

f. $(-2, 4)$ and $(2, 4)$ are open endpoints. Domain $(-2, 2)$; range $[0, 4)$

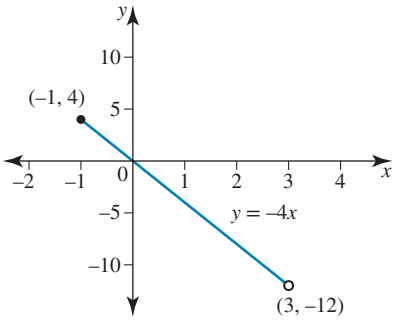
3. a. Domain $\{0, 2, 3, 4\}$; range $\{-1, 0, 3, 4\}$; a function

b. Domain $[-2, \infty)$; range R; not a function

c. Domain $[0, 3]$; range $[0, 4]$; a function

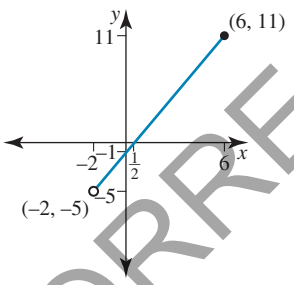
d. Maximal domain R; range $(-\infty, 4)$; a function

4.



Domain $[-1, 3)$; range $(-12, 4]$

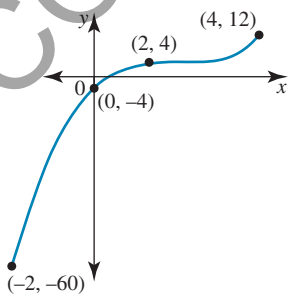
5. a.



b. Domain $(-2, 6]$, Range $(-5, 11]$

c. Function as a vertical line cuts graph exactly once.

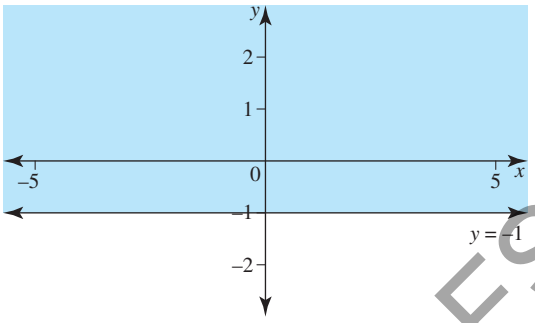
6. a.



b. Domain $[-2, 4]$, Range $[-60, 12]$

c. Function as a vertical line cuts graph exactly once.

7. a.



b. Domain R; range $[-1, \infty)$

c. Not a function as a vertical line cuts graph at more than one place.

8. a. One-to-one correspondence; a function

b. Many-to-many correspondence; not a function

9. a. Domain $[0, 5]$; range $[0, 15]$

b. Domain $[-4, 2) \cup (2, \infty)$; range $(-\infty, 10)$

c. Domain $[-3, 6]$; range $[0, 8]$

d. Domain $[-2, 2]$; range $[-4, 4]$

e. Domain $\{3\}$; range R

f. Domain R; range R

10. a. Relation **a** is one-to-one; **b** is many-to-one; **c** is many-to-one; **d** is one-to-many; **e** is one-to-many; **f** is many-to-one

b. Relations **d** and **e** are not functions.

11. a. Domain $\{-11, -3, -1, 5\}$; range $\{0, 2, 8\}$; many-to-one; function

b. Domain $\{20, 50, 60\}$; range $\{6, 10, 20\}$; many-to-many: not a function

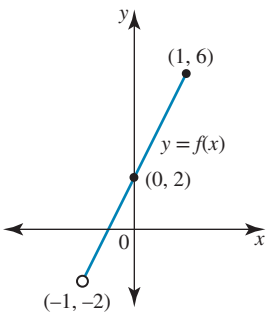
c. Domain $\{-14, 0, 14\}$; range $\{-7, 0, 2, 7\}$; one-to-many: not a function

d. Domain R; range R; one-to-one; function

e. Domain R; range $(-\infty, 4]$; many-to-one; function

f. Domain R; range $[0, \infty)$; many-to-one; function

12. a. Line; endpoints $(-1, -2)$ (open), $(1, 6)$ (closed); domain $(-1, 1]$; range $(-2, 6]$



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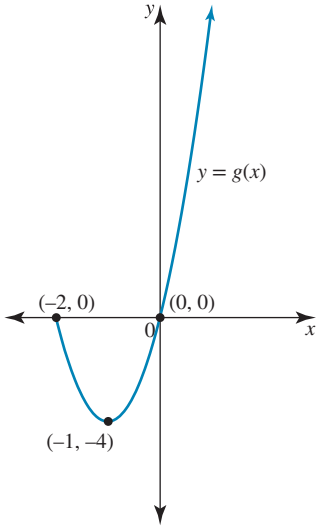
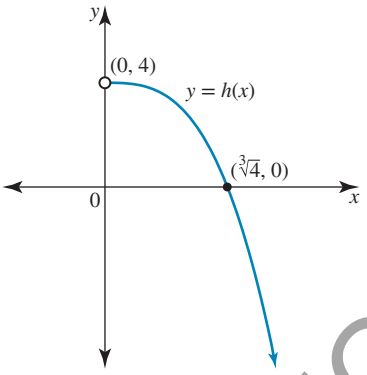
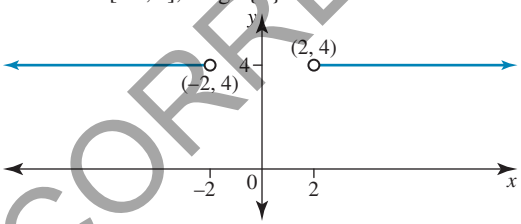
b. Parabola; endpoint $(-2, 0)$ (closed);
minimum turning point $(-1, -4)$;
y-intercept $(0, 0)$;
domain $[-2, \infty)$; range $[-4, \infty)$



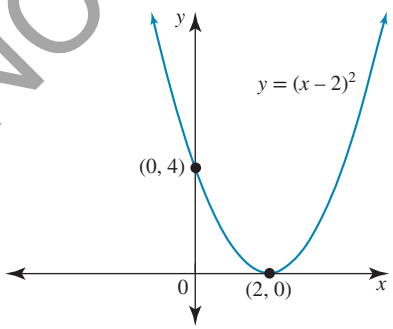
c. Cubic with stationary point of inflection;
open endpoint $(0, 4)$;
x-intercept $(\sqrt[3]{4}, 0)$; domain R^+; range $(-\infty, 4)$



d. Two sections of a horizontal line;
open endpoints at $(\pm 2, 4)$;
domain $R \setminus [-2, 2]$; range $\{4\}$



13. a.



Domain R; range $R^+ \cup \{0\}$; many-to-one correspondence

b. A possible answer is $[2, \infty)$.

14. a. $a = 2,\ b = 3; f(x) = 2x + 3$

b. $f(0) = 3$

c. $x = -1.5$

d. $f(-3) = -3$

e. $g: (-\infty, 0] \to R, g(x) = 2x + 3$

15. a. **i.** 1 **ii.** 21 **iii.** $\frac{1}{2}$

b. **i.** $\frac{1}{4}$ **ii.** $\frac{1}{2}$ **iii.** $\frac{1}{5}$

c. **i.** 45 **ii.** -5 **iii.** $4\sqrt{2} - 3$

d. $x = -2$

16. a. **i.** -3 **ii.** 96

b. **i.** $f(2a) = 4a^2 + 4a - 3$ **ii.** $f(1 - a) = a^2 - 4a$

c. $2xh + h^2 + 2h$

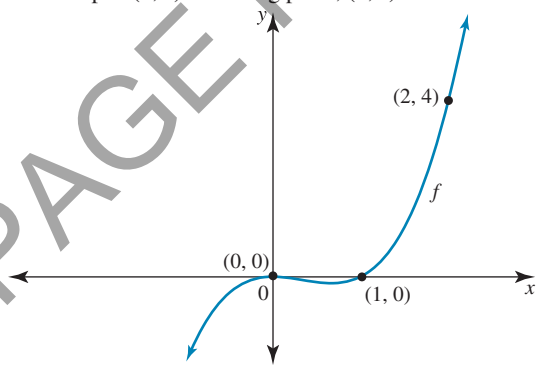
d. $\{x : x < -3\} \cup \{x : x > 1\}$

e. $x = -5, x = 3$

f. $x = -4, x = 1$

17. a. 4

b. x-intercepts: $(0, 0)$ at turning point, $(1, 0)$



c. Domain R; range R

d. Many-to-one

e. An answer is R^-.

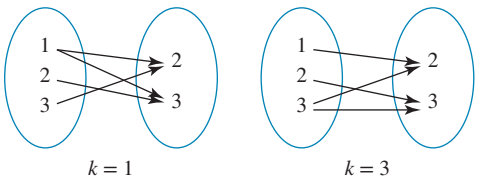
f. $\{2\}$

18. $f : [0, 7) \to R, f(x) = x^2 - 6x + 10$; domain $[0, 7)$;
range $[1, 17)$.

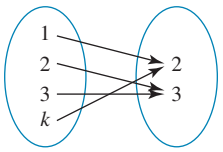
19. a. $k = 1$ or $k = 3$; many-to-many

b. $k \in R \setminus \{1, 3\}$; many-to-one

c.



d.



20. a. Function; $f: Z^+ \to R, f(x) = x^2$;
domain Z^+; range $\{1, 4, 9, 16, \ldots\}$

b. Function; $f: R \to R, f(x) = \frac{6 - 2x}{3}$; domain R; range R

c. Not a function

d. Function; $f: R \to R, f(x) = 5$; domain R; range $\{5\}$

e. Not a function

f. Function; $f: R^+ \to R, f(x) = -x^5$; domain R^+; range R^-

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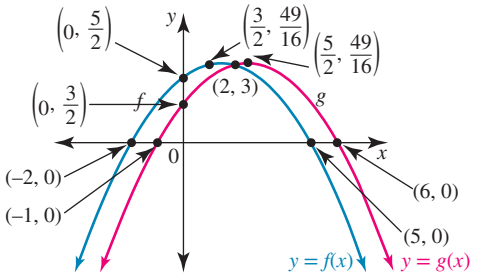
21. a. $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$

b. $g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$

c. $x = 2$

d.



Graphs intersect at (2, 3).

$y = f(x)$: maximum turning point $\left(\frac{3}{2}, \frac{49}{16}\right)$;

x-intercepts $(-2, 0)$, $(5, 0)$; y-intercept $\left(0, \frac{5}{2}\right)$

$y = g(x)$: maximum turning point $\left(\frac{5}{2}, \frac{49}{16}\right)$;

x-intercepts $(-1, 0)$, $(6, 0)$; y-intercept $\left(0, \frac{3}{2}\right)$

$y = g(x)$ has the same shape as $y = f(x)$ but it has been translated 1 unit to the right.

22. a. Domain $\{t: 0 \le t \le 5\}$; range $\{x: 4 \le x \le 29\}$; the distance travelled is 25 units.

b. 2 seconds; domain $[0, 2]$; range $[0, 5]$

c. i. Domain $[0, 2]$; range $[0.5, 2.1]$

ii. Approximately 1.4 weeks

23. Not a function; domain R; range $R \setminus (-1, 1)$

24. a. $l = 0; m = -12; n = -16; f(x) = x^3 - 12x - 16$

b. -28.672

c. 4.477

d. $f: R^+ \cup \{0\} \to R, f(x) = x^3 - 12x - 16$; domain $R^+ \cup \{0\}$; range $[-32, \infty)$

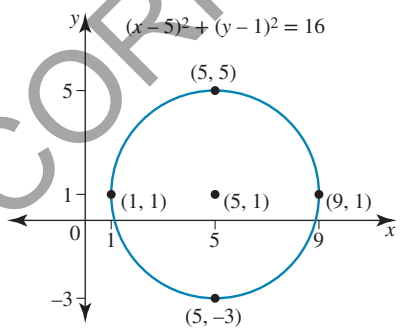
Exercise 6.3 The circle

1. a. (5, 1)

b. 4

c. Domain $[1, 9]$; range $[-3, 5]$

d.

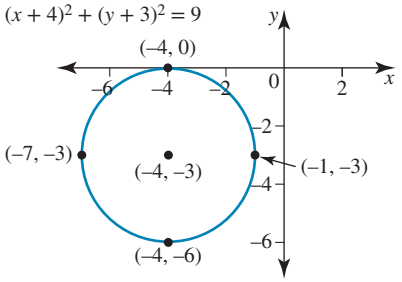


2. a. $(-4, -3)$

b. 3

c. Domain $[-7, -1]$; Range $[-6, 0]$

d. $(x+4)^2 + (y+3)^2 = 9$

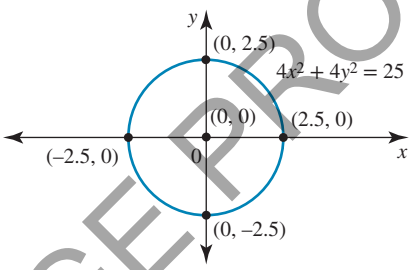


3. a. (0, 0)

b. 2.5

c. Domain $[-2.5, 2.5]$; range $[-2.5, 2.5]$.

d.



4. a. Centre $(-11, 0)$, radius 7.

b. Domain $[-18, -4]$, range $[-7, 7]$.

c. $(-8, -2\sqrt{10})$ and $(-8, 2\sqrt{10})$.

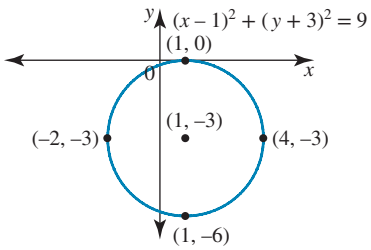
d. Functions do not have more than one point with the same x-value.

5. a. $(x-9)^2 + (y+2)^2 = 81$

b. $x^2 + (y-10)^2 = \frac{4}{9}$ or $9x^2 + 9(y-10)^2 = 4$.

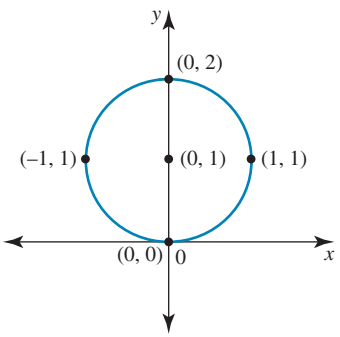
c. $(x-3)^2 + (y-3)^2 = 9$

6. a. Centre $(1, -3)$; radius 3; domain $[-2, 4]$; range $[-6, 0]$



b. Centre $(-1, -4)$; radius $\sqrt{17}$, domain $(-1-\sqrt{17}, -1+\sqrt{17}]$; range $[-4-\sqrt{17}, -4+\sqrt{17}]$

7. a. Centre $(0, 1)$; radius 1; domain $[-1, 1]$; range $[0, 2]$



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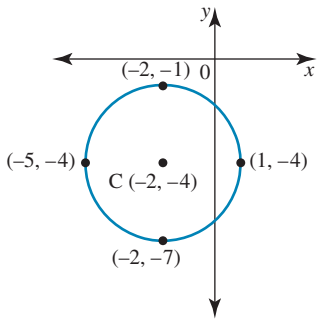
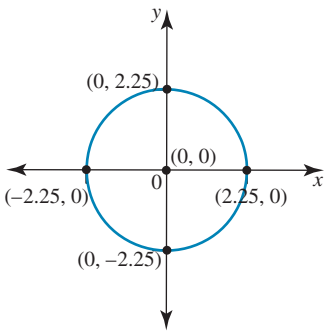
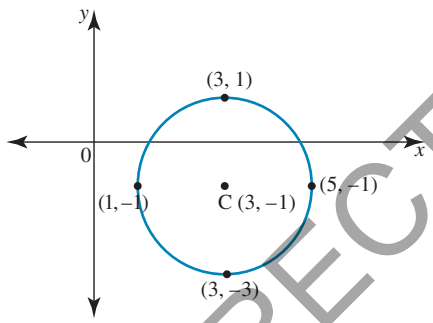
b. Centre $(-2, -4)$; radius 3; domain $[-5, 1]$; range $[-7, -1]$



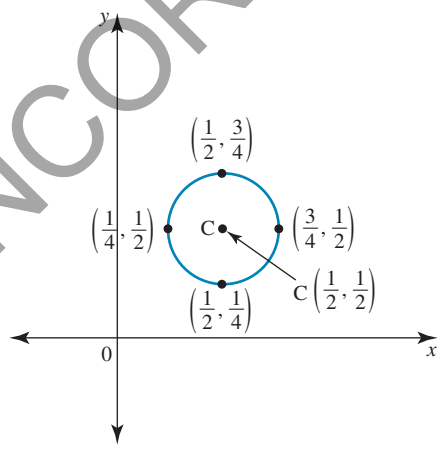
c. Centre $(0, 0)$; radius $\frac{9}{4}$; domain $\left[-\frac{9}{4}, \frac{9}{4}\right]$; range $\left[-\frac{9}{4}, \frac{9}{4}\right]$



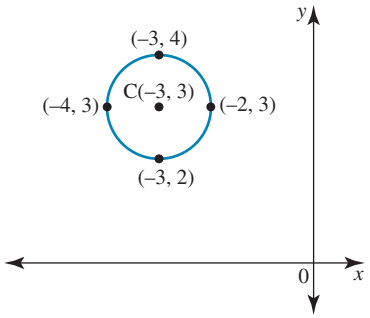
d. Centre $(3, -1)$; radius 2; domain $[1, 5]$; range $[-3, 1]$



e. Centre $\left(\frac{1}{2}, \frac{1}{2}\right)$; radius $\frac{1}{4}$; domain $\left[\frac{1}{4}, \frac{3}{4}\right]$; range $\left[\frac{1}{4}, \frac{3}{4}\right]$



f. Centre $(-3, 3)$; radius 1; domain $[-4, -2]$; range $[2, 4]$



8. a. $(x+8)^2+(y-9)^2 = 36$ **b.** $(x-7)^2+y^2 = 8$
c. $(x-1)^2+(y-6)^2 = 136$ **d.** $9x^2+9(y-2)^2 = 16$

9. $x^2 + y^2 + 10x - 33 = 0$

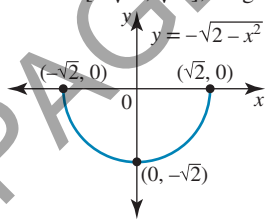
10. a. Expand $(x-3)^2 + (y-5)^2 = 5$.

b. Expand $(x+4)^2 + \left(y+\frac{1}{2}\right)^2 = 1$.

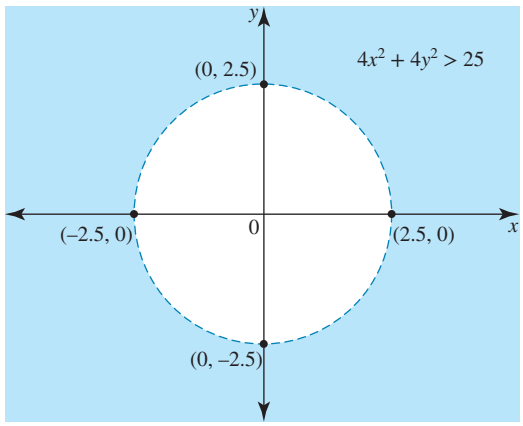
c. $(x+1)^2 + (y-2)^2 = 7$, centre $(-1, 2)$, radius $\sqrt{7}$.

d. $(x-4)^2 + y^2 = 36$, centre $(4, 0)$, radius 6.

11. a. Centre $(0, 0)$; radius $\sqrt{2}$; domain $[-\sqrt{2}, \sqrt{2}]$; range $[-\sqrt{2}, 0]$



b. Open region outside the circle with centre $(0, 0)$ and radius $\frac{5}{2}$



c. Upper semicircle $y = \sqrt{\frac{25}{4} - x^2}$; domain $\left[-\frac{5}{2}, \frac{5}{2}\right]$; range $\left[0, \frac{5}{2}\right]$

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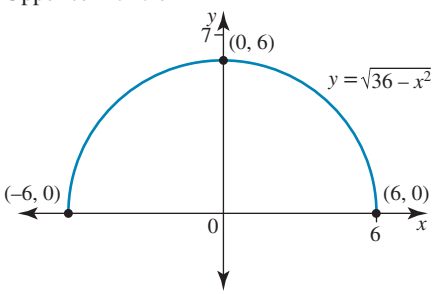
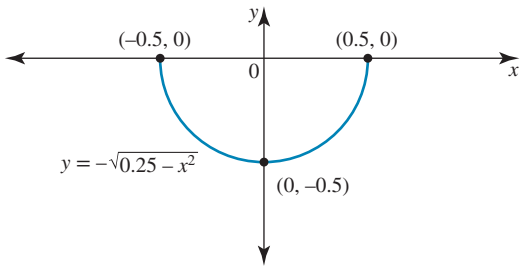
12. a. Centre (0, 0); radius 6; domain [−6, 6]; range [0, 6]; Upper semicircle



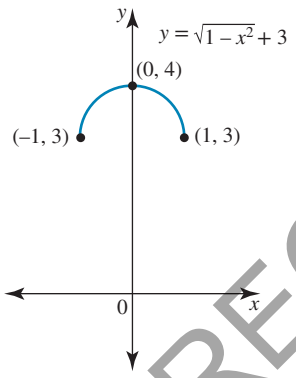
A function; f: $[-6, 6] \to R$, $f(x) = \sqrt{36 - x^2}$

b. Centre (0, 0); radius 0.5; domain [−0.5, 0.5]; range [−0.5, 0]; Lower semicircle



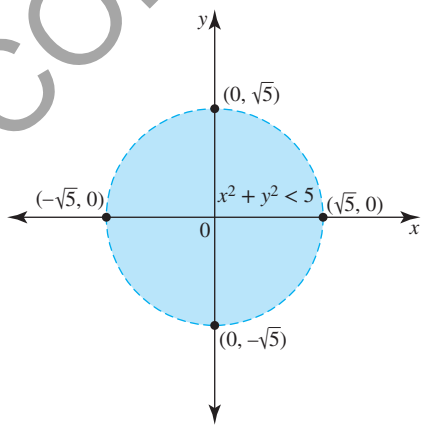
A function; g: $[-0.5, 0.5] \to R$, $g(x) = -\sqrt{0.25 - x^2}$

c. Centre (0, 3); radius 1; domain [−1, 1]; range [3, 4]; upper semicircle

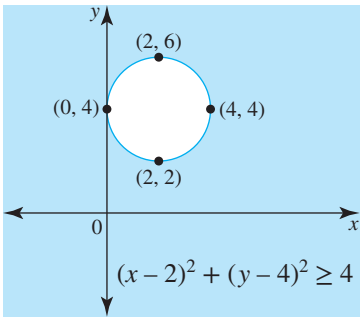


A function; h: $[-1, 1] \to R$, $h(x) = \sqrt{1 - x^2} + 3$

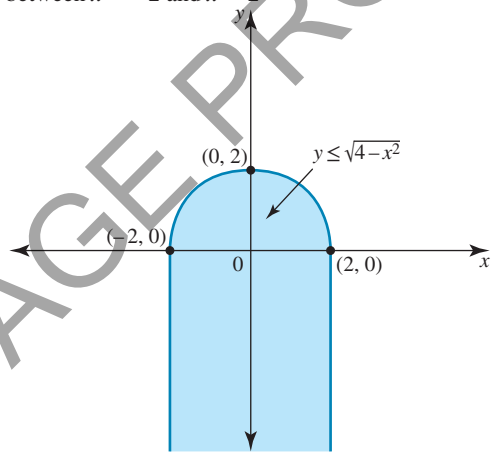
d. Centre (0, 0); radius $\sqrt{5}$; domain $(-\sqrt{5}, \sqrt{5})$; range $(-\sqrt{5}, \sqrt{5})$; Open region inside circle



e. Centre (2, 4); radius 2; domain R; range R; closed region outside circle



f. Centre (0, 0); radius 2; domain [−2, 2]; range $(-\infty, 2]$; closed region below semicircle and between $x = -2$ and $x = 2$



13. a. Outside

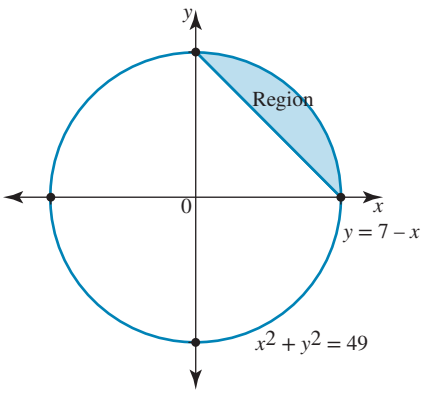
b. $a = 0, -8$; $y = 0.5\sqrt{1 - 32x - 4x^2} + 1.5$

14. a. $(x + 2)^2 + (y - 2)^2 = 12$

b. Domain $[-2 - 2\sqrt{3}, -2 + 2\sqrt{3}]$; range $[2, 2 + 2\sqrt{3}]$

15. a. (1, 2), (1.4, 2.8)

b. (0, 7), (7, 0); region is inside circle and above the line (boundaries included)



16. $m = \pm\dfrac{\sqrt{5}}{2}$

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17. a. $(2, 1)$

b. i. $k = \pm\frac{1}{2\sqrt{6}} = \pm\frac{\sqrt{6}}{12}$

ii. $-\frac{1}{2\sqrt{6}} < k < \frac{1}{2\sqrt{6}}$ or $\frac{-\sqrt{6}}{12} < k < \frac{\sqrt{6}}{12}$

iii. $k < -\frac{1}{2\sqrt{6}}$ or $k > \frac{1}{2\sqrt{6}}$

or $\left(k < -\frac{\sqrt{6}}{12} \text{ or } k > \frac{\sqrt{6}}{12}\right)$

18. a. $y = -\frac{1}{3}x - 3$

b. $\sqrt{55}$ units

c. No intersections

19. a. Centre $(3, -2)$; radius 5

b. i. Sample responses can be found in the worked solutions in the online resources.

ii. $3y - 4x = 7$

c. i. Sample responses can be found in the worked solutions in the online resources.

ii. $y = 3$

d. $2\sqrt{30}$ units

e. 5 units

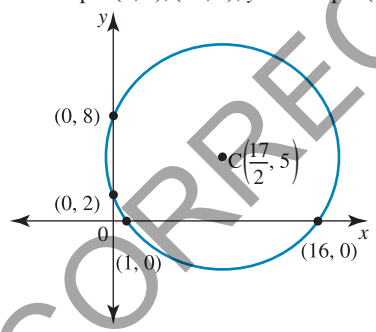
f. Either $(8, 13)$ or $(-7, -7)$

20. a. $2\sqrt{21}$ **b.** $\frac{5\sqrt{2}}{2}$

21. a. $a = -17, b = -10, c = 16$

b. Centre $\left(\frac{17}{2}, 5\right)$; radius $\frac{5\sqrt{13}}{2}$

c. x-intercepts $(1, 0), (16, 0)$; y-intercepts $(0, 2), (0, 8)$



d. 0.85 units

e. 18.88 units

22. a. Three points are required to determine a circle but only two are given, so several circles are possible. See example in worked solutions in your online resources.

b. $(x - 2)^2 + (y - 2)^2 = 8$; Sample responses can be found in the worked solutions in the online resources.

c. i. $(4, 4)$

ii. Rufus by approximately 2 minutes

23. Centre $(-2, 3.5)$; radius 3.775; x-intercepts at $x = -3.414$, $x = -0.586$; y-intercepts at $y = 6.702$, $y = 0.298$

24. Domain $[-2.124, 14.124]$; range $[-12.124, 4.124]$; tangent is vertical.

Exercise 6.4 The rectangular hyperbola and the truncus

1. a. $x = -5, y = 2$ **b.** $x = 0, y = -3$

c. $x = 0, y = 0$ **d.** $x = -14, y = -\frac{3}{4}$

2. a. i. $x = 6$ and $y = 1$.

ii. $R \setminus \{6\}$

iii. $R \setminus \{1\}$.

b. i. $x = -3$ and $y = -7$.

ii. $R \setminus \{-3\}$

iii. $R \setminus \{-7\}$

c. i. $x = 2$ and $y = 4$.

ii. $R \setminus \{2\}$

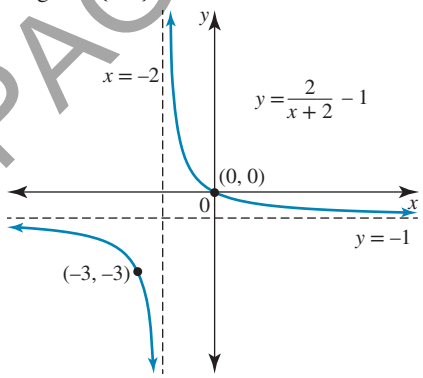
iii. $R \setminus \{4\}$

d. i. $x = 1$ and $y = -3$.

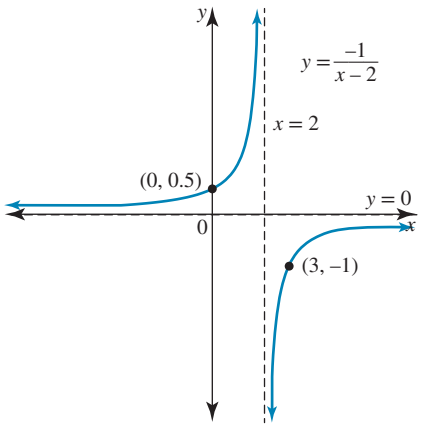
ii. $R \setminus \{1\}$

iii. $R \setminus \{-3\}$.

3. a. Vertical asymptote $x = -2$; horizontal asymptote $y = -1$; y-intercept $(0, 0)$; domain $R \setminus \{-2\}$; range $R \setminus \{-1\}$



b. Vertical asymptote $x = 2$; horizontal asymptote $y = 0$; y-intercept $\left(0, \frac{1}{2}\right)$; domain $R \setminus \{2\}$; range $R \setminus \{0\}$



4. a. $x = 1$ and $y = 5$. **b.** $(0, 4)$

c. $\left(\frac{4}{5}, 0\right)$

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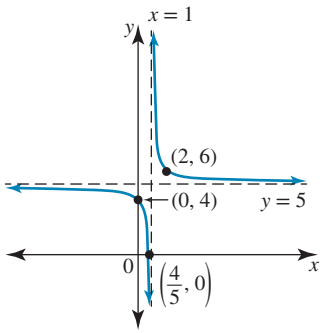
d.

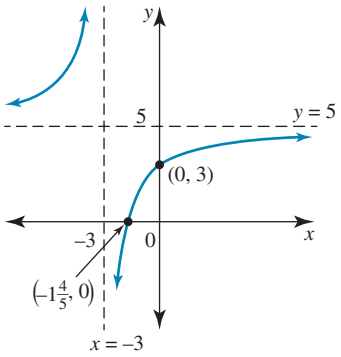


5. a. $x = -3$ and $y = 5$.

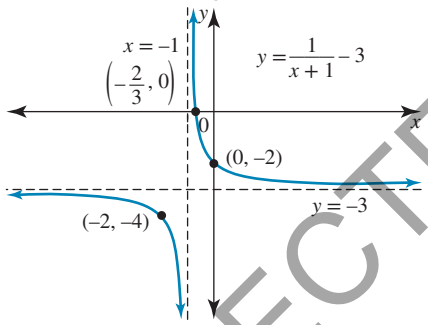
b. $(0, 3)$

c. $\left(-\frac{9}{5}, 0\right)$

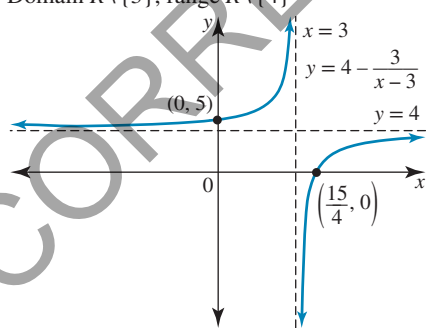
d.



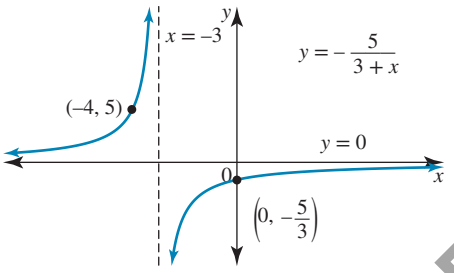
6. a. Domain $R \setminus \{-1\}$; range $R \setminus \{-3\}$



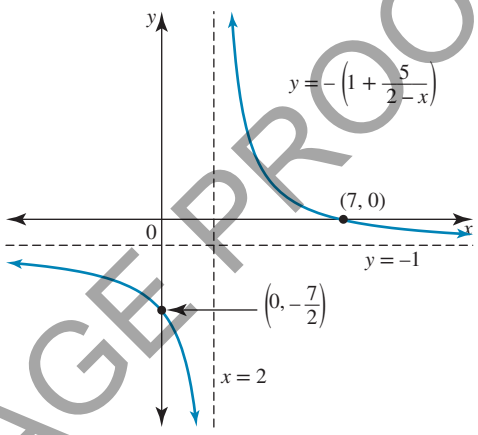
b. Domain $R \setminus \{3\}$; range $R \setminus \{4\}$



c. Domain $R \setminus \{-3\}$; range $R \setminus \{0\}$



d. Domain $R \setminus \{2\}$; range $R \setminus \{-1\}$



7. Vertical asymptote $x = \frac{1}{2}$; horizontal asymptote $y = 4$; graph lies in quadrants 1 and 3 (quadrants as defined by the asymptotes)

8. a. Vertical asymptote $x = -\frac{2}{3}$; horizontal asymptote $y = 2$

b. $y = \frac{-1}{x - 4} + \frac{1}{2}$

9. a. $y = \frac{24}{x + 4} + 2$ **b.** $y = -\frac{12}{x} + 5$

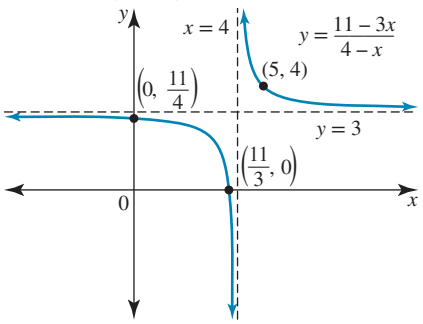
c. $y = 2 - \frac{10}{x - 5}$

d. i. Domain $(-3, 3) \setminus \{0\}$; range $R \setminus [-3, 3]$

ii. $y = \frac{1}{x} + 3,\ -3 < x < 3,\ x \neq 0$

10. a. $a = 3; b = 1$

b. Asymptotes $x = 4, y = 3$; y-intercept $\left(0, \frac{11}{4}\right)$; x-intercept $\left(\frac{11}{3}, 0\right)$; point $(5, 4)$



c. $x < \frac{11}{3}$ or $x > 4$

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11. a. $y = \dfrac{-1}{16x+4} + \dfrac{1}{4}, x = -\dfrac{1}{4}, y = \dfrac{1}{4}$

b. $y = \dfrac{4}{x-4} - 2, x = 4, y = -2$

c. $y = \dfrac{1}{x} + 2, x = 0, y = 2$

d. $y = \dfrac{-2}{2x+3}, x = -\dfrac{3}{2}, y = 0$

12. Domain $R \setminus \{4\}$; range $R \setminus \{0\}$

13. C

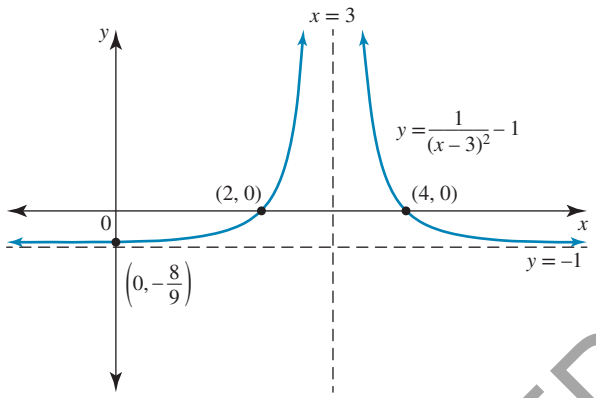
14. a. Domain $R \setminus \{-3\}$, range $(5, \infty)$

b. $y = \dfrac{20}{(x+3)^2} + 5$

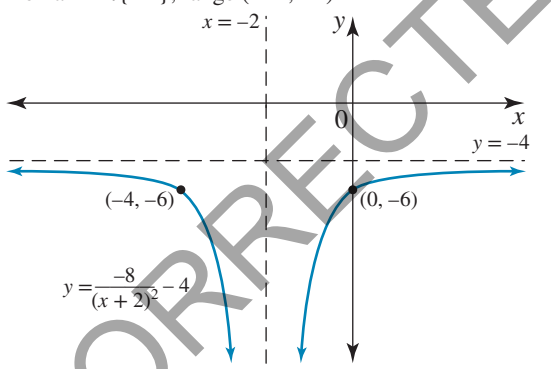
15. a. $R = \dfrac{240}{I}$

b. $333\dfrac{1}{3}$ ohm

16. a. Domain $R \setminus \{3\}$; range $(-1, \infty)$



b. Domain $R \setminus \{-2\}$; range $(-\infty, -4)$

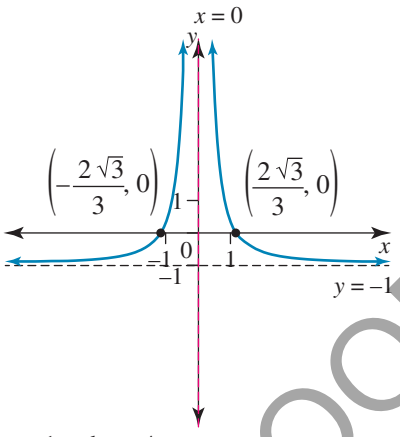


17. a. i. $x = 0$ and $y = -1$

ii. Domain $R \setminus \{0\}$, range $(-1, \infty)$.

iii. $\left(\dfrac{2\sqrt{3}}{3}, 0\right)$ and $\left(-\dfrac{2\sqrt{3}}{3}, 0\right)$, no y intercept

iv.

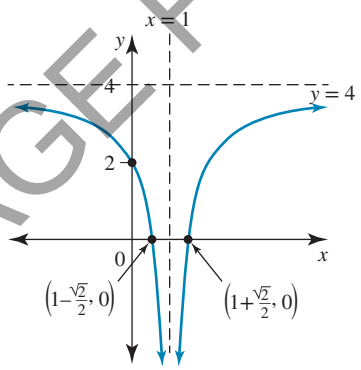


b. i. $x = 1$ and $y = 4$.

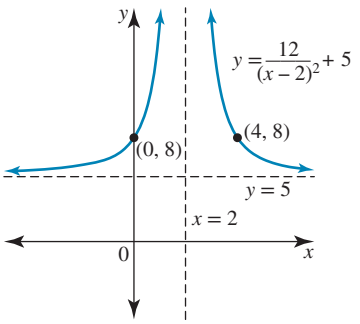
ii. Domain $R \setminus \{1\}$, range $(-\infty, 4)$.

iii. $\left(1 - \dfrac{\sqrt{2}}{2}, 0\right), \left(1 + \dfrac{\sqrt{2}}{2}, 0\right), (0, 2)$

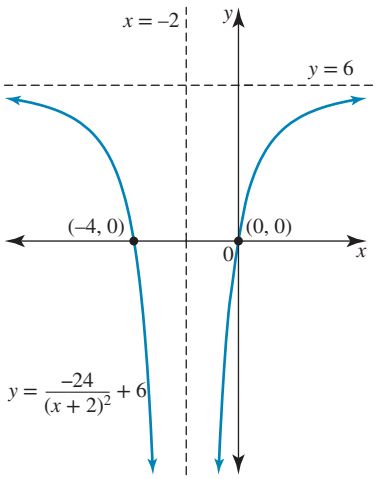
iv.



18. a. $x = 2, y = 5$



b.



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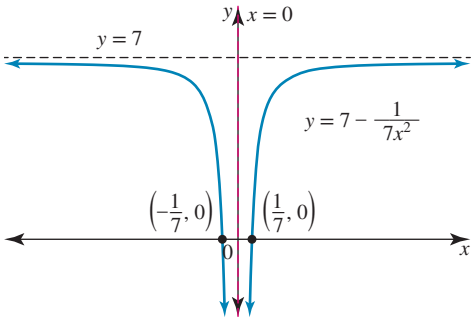
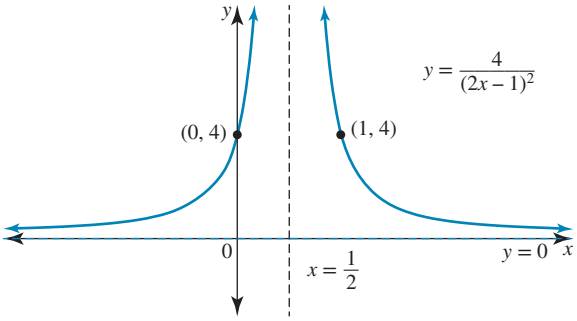
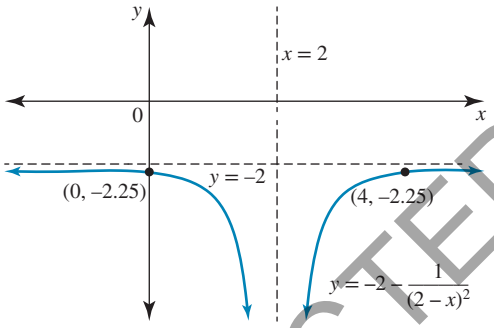
c. $x = 0, y = 7$



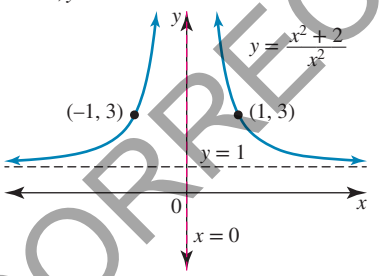
d. $x = \frac{1}{2}, y = 0$



e. $x = 2, y = -2$



f. $x = 0, y = 1$



19. Asymptotes $x = \frac{1}{5}, y = 0$; domain $R \setminus \left\{\frac{1}{5}\right\}$; range $(0, \infty)$

20. a. $y = \frac{2}{x-3} + 1$

b. $y = \frac{-1.5}{x+3} + 1$

c. $y = \frac{3}{x^2} - 2$

d. $y = 2 - \frac{9}{(x+3)^2}$

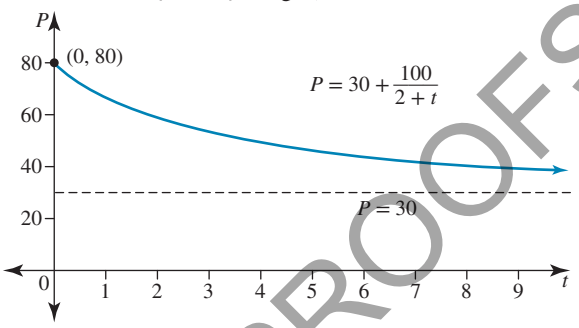
e. $y = \frac{4}{(x+2)^2}$, $f: R2 \setminus \{-2\} \to R, f(x) = \frac{4}{(x+2)^2}$

f. i. $y = \frac{-2x+2}{4x-1}$

ii. $f: R \setminus \left\{\frac{1}{4}\right\} \to R, f(x) = \frac{-2x+2}{4x-1}$

21. a. Reduced by 25 cattle

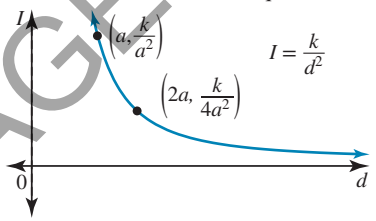
b. Domain $\{t: t \geq 0\}$; range $(30, 80]$



c. The number of cattle will never go below 30.

22. Table **b**; $y = \frac{7.2}{x}$, $y = 6.4, x = 1.125$; $x = 8, y = 0.9$

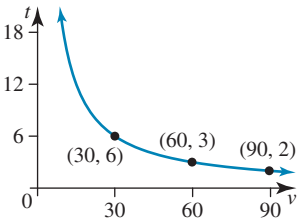
23. a. One branch of a truncus required



b. Intensity reduced by 75%

24. a. The constant is the distance; $k = 180$

b.



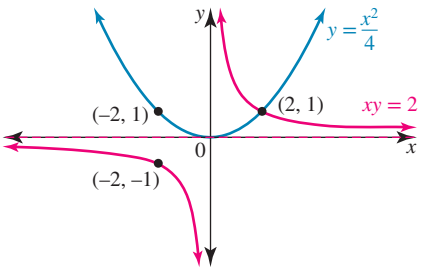
c. 80 km/h

25. a. 50 seconds

b. 25 metres above the ground

26. a. (2, 1)

b.



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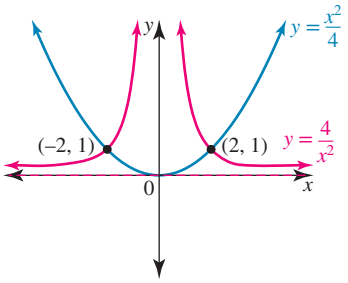
c. $(-2, 1), (2, 1)$



d. $(-\sqrt{a}, 1), (\sqrt{a}, 1)$

27. a. $a = 460; b = 40$

b. 12 years 8 months

c. Sample responses can be found in the worked solutions in the online resources.

d. Increases by approximately 4 insects in 12th year and 3 in 14th year; growth is slowing.

e. Never reaches 500 insects

f. Cannot be larger than 460

28. a. Two intersections

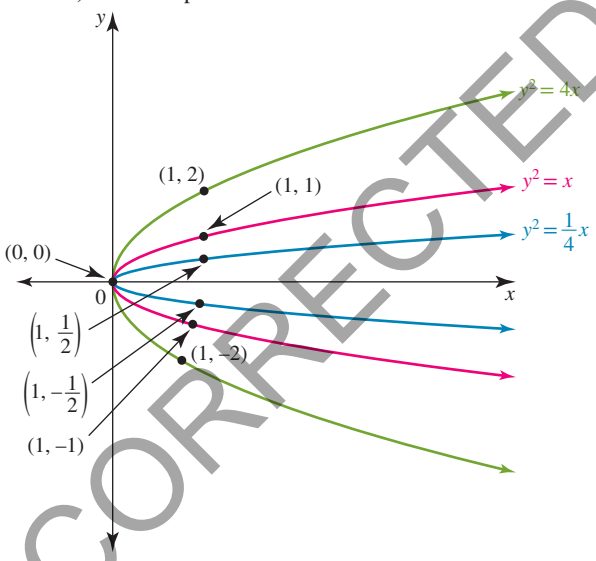
b. One intersection if $k = 1, 5$; two intersections if $k < 1$ or $k > 5$; no intersections if $1 < k < 5$

29. a. Asymptotes $x = 0, y = 0$ and $y = -x, y = x$

b. Rotate axes on $xy = 1$ graph anticlockwise by 45°

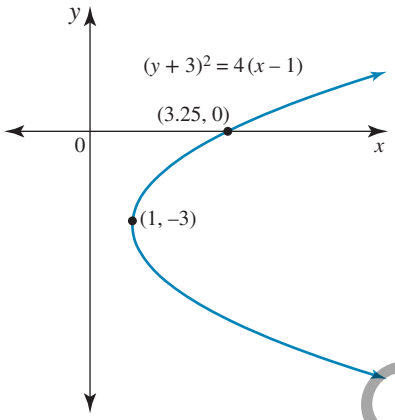
Exercise 6.5 The relation $y^2 = x$

1. Increasing the coefficient of x makes the graph wider (in the y direction) or more open.

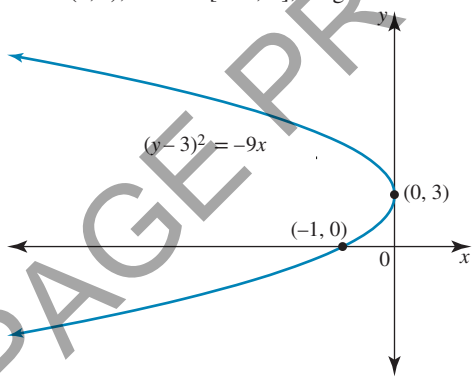


2. a. $(y + 1)^2 = -3(x - 1)$ **b.** $(y + 2)^2 = 4(x - 1)$

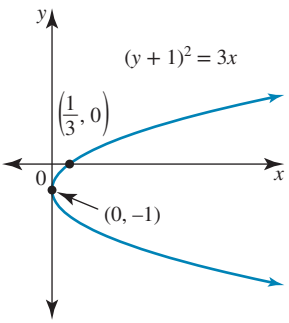
3. a. Vertex $(1, -3)$; domain $[1, \infty)$; range R



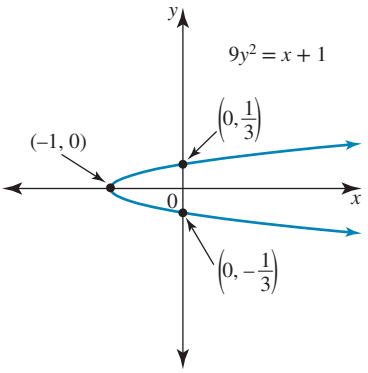
b. Vertex $(0, 3)$; domain $[-\infty, 0]$; range R



4. a. Vertex $(0, -1)$; x-intercept $\left(\frac{1}{3}, 0\right)$; y-intercept $(0, -1)$



b. Vertex $(-1, 0)$; x-intercept $(-1, 0)$; y-intercepts $\left(0, \pm\frac{1}{3}\right)$



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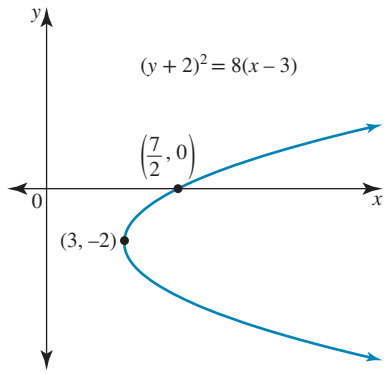
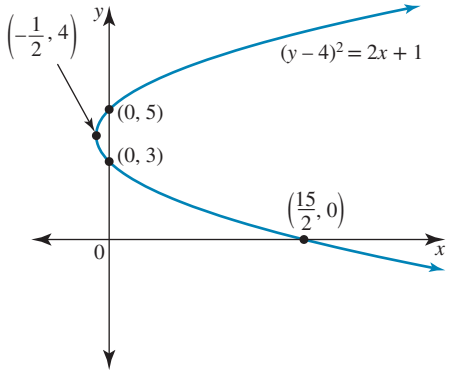
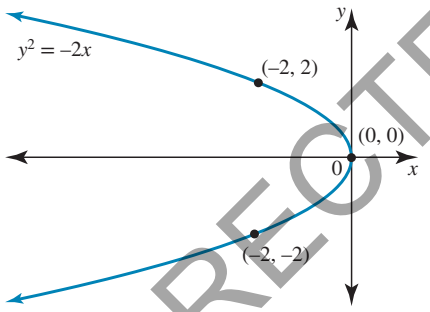
c. Vertex $(3, -2)$; x-intercept $\left(\frac{7}{2}, 0\right)$; y-intercepts: none



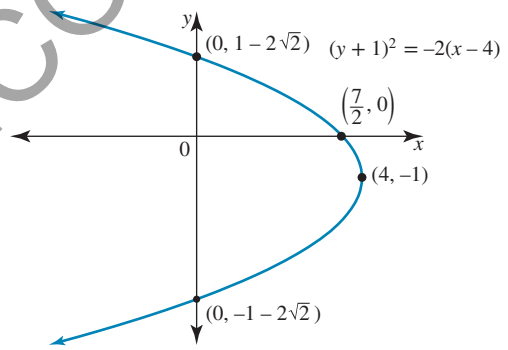
d. Vertex $\left(-\frac{1}{2}, 4\right)$; x-intercept $\left(\frac{15}{2}, 0\right)$; y-intercepts $(0, 3), (0, 5)$



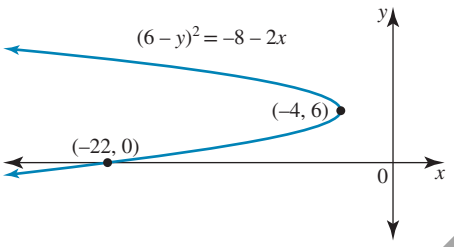
5. a. Vertex $(0, 0)$; x-intercept $(0, 0)$; y-intercept $(0, 0)$



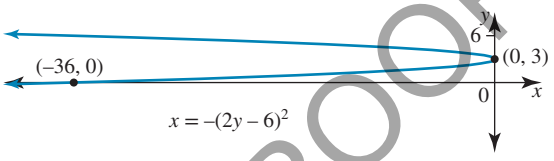
b. Vertex $(4, -1)$; x-intercept $\left(\frac{7}{2}, 0\right)$; y-intercepts $(0, -1 \pm 2\sqrt{2})$



c. Vertex $(-4, 6)$; x-intercept $(-22, 0)$; y-intercepts: none



d. Vertex $(0, 3)$; x-intercept $(-36, 0)$; y-intercept $(0, 3)$



6. a. $(y+7)^2 = -\frac{7}{2}(x-4)$

b. $(y-3)^2 = 3(x+3)$

7. Sideways parabola fails vertical line test for functions; $(y-3)^2 = \frac{9}{2}x$

8. $(y+4)^2 = 3\left(x - \frac{4}{3}\right)$; vertex $\left(\frac{4}{3}, -4\right)$; axis of symmetry $y = -4$

9. a. $(y+8)^2 = 5(x-2)$; vertex $(2, -8)$; domain $[2, \infty)$

b. $\left(y - \frac{3}{2}\right)^2 = -13\left(x - \frac{1}{4}\right)$; vertex $\left(\frac{1}{4}, \frac{3}{2}\right)$; domain $\left(-\infty, \frac{1}{4}\right]$

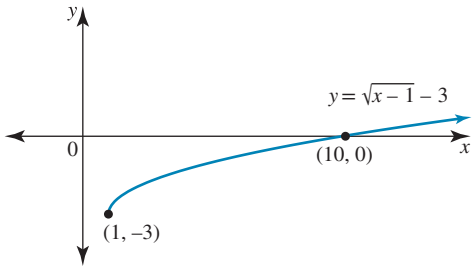
c. $\left(y + \frac{5}{2}\right)^2 = -(x-2)$; vertex $\left(2, -\frac{5}{2}\right)$; domain $(-\infty, 2]$

d. $(y-2)^2 = 5\left(x + \frac{4}{5}\right)$; vertex $\left(-\frac{4}{5}, 2\right)$; domain $\left[-\frac{4}{5}, \infty\right)$

10.

a.	**i.** $[9, \infty)$	**ii.** $(9, 0)$	**iii.** $[0, \infty)$
b.	**i.** $(-\infty, 4]$	**ii.** $(4, 0)$	**iii.** $[0, \infty)$
c.	**i.** $[-3, \infty)$	**ii.** $(-3, 0)$	**iii.** $(-\infty, 0]$
d.	**i.** $(-\infty, 0]$	**ii.** $(0, 3)$	**iii.** $[3, \infty)$
e.	**i.** $[2, \infty)$	**ii.** $(2, -7)$	**iii.** $[-7, \infty)$
f.	**i.** $\left(-\infty, \frac{1}{2}\right]$	**ii.** $\left(\frac{1}{2}, 4\right)$	**iii.** $(-\infty, 4]$

11. a. Domain $[1, \infty)$; range $[-3, \infty)$



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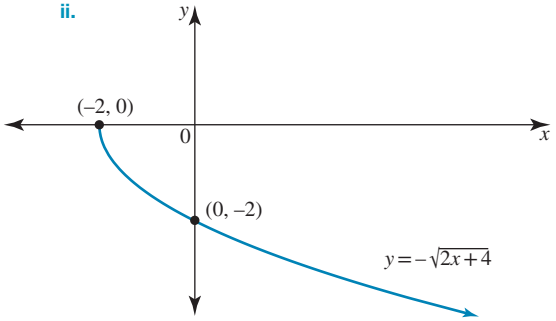
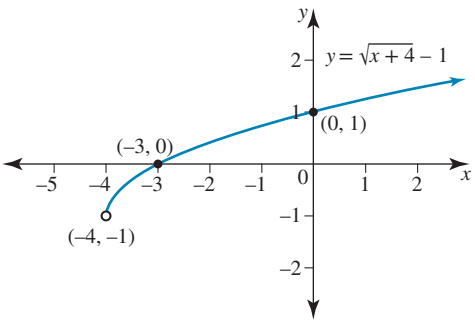
b. **i.** $[-2, \infty)$

ii.

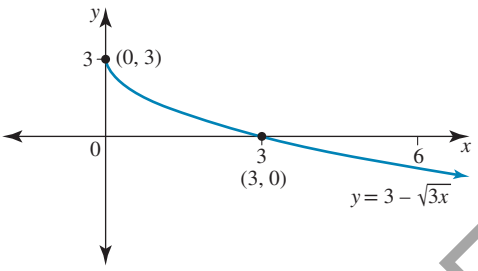


iii. $y^2 = 2(x + 2)$

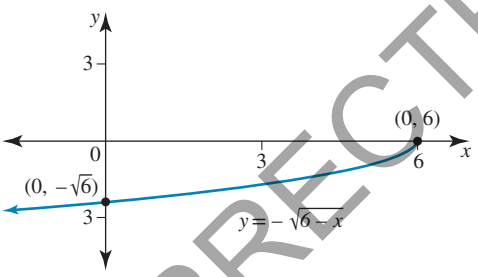
12. a.



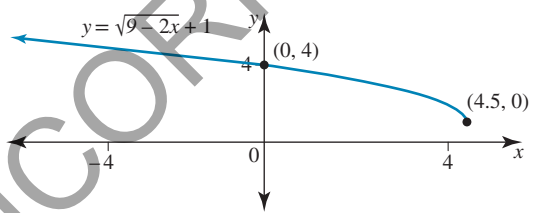
b.



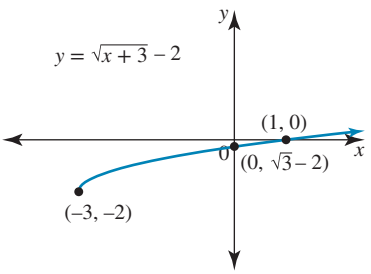
c.



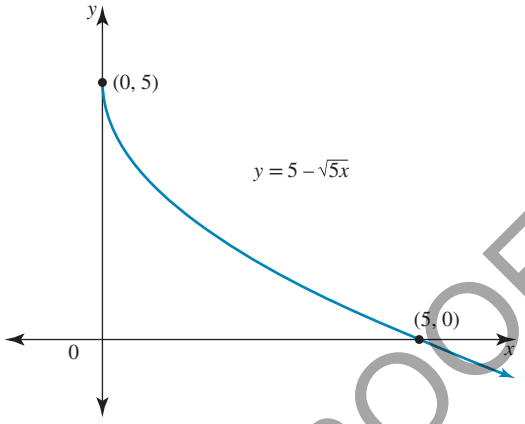
d.



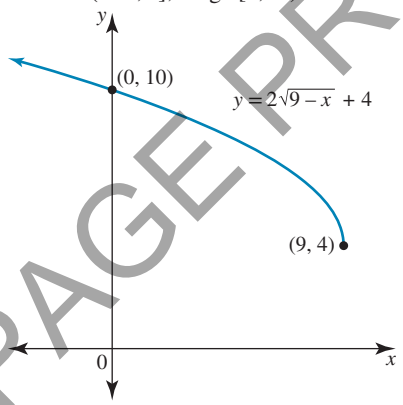
13. a. Domain $[-3, \infty)$; range $[-2, \infty)$



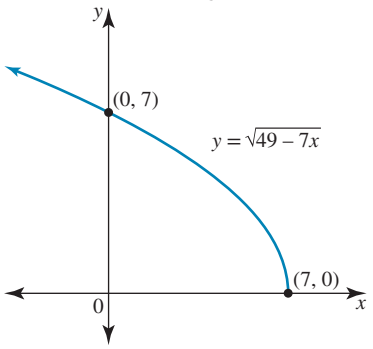
b. Domain $[0, \infty)$; range $(-\infty, 5]$



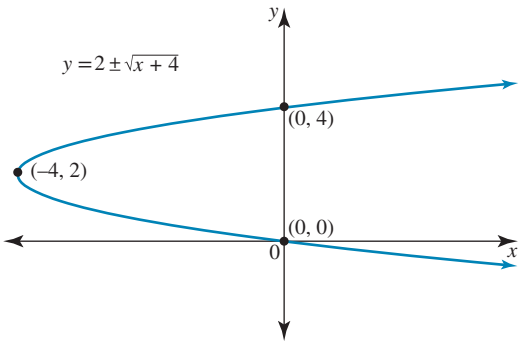
c. Domain $(-\infty, 9]$; range $[4, \infty)$



d. Domain $(-\infty, 7]$; range $[0, \infty)$



e. Domain $[-4, \infty)$; range R



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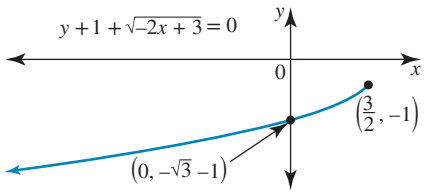
f. Domain $\left(-\infty, \frac{3}{2}\right]$ range $(-\infty, -1]$



14. $y = \sqrt{2}\sqrt{x+2} + 1 \Rightarrow y = \sqrt{2(x+2)} + 1$

15. a. $y = \frac{-3}{\sqrt{3}}\sqrt{x+2} + 2$

b. $y = 5\sqrt{4-x} - 1; \left(\frac{99}{25}, 0\right)$

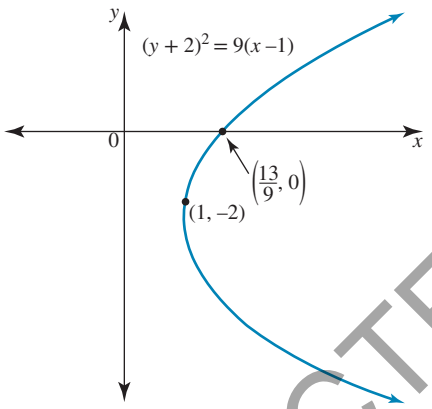
c. $y = \sqrt{4-x} - 4$

16. a. $y = 4\sqrt{x-3} + 2$

b. $y = \frac{1}{2}\sqrt{x+4} - 1$

c. $a = -2, b = 2, c = 1$, $y = \sqrt{-2(x-2)} + 1$ or $y = \sqrt{4-2x} + 1$

17. a. Vertex $(1, -2)$; x-intercept $\left(\frac{13}{9}, 0\right)$



b. $y = -\sqrt{9(x-1)} - 2$

18. $t = \sqrt{\frac{h}{4.9}}$

19. a. i. $y = 4$ **ii.** $(y-4)^2 = 64x$

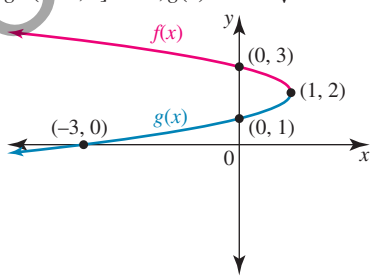
b. $y^2 = \frac{121}{48}x$

20. a. $y = 2 \pm \sqrt{1-x}$

b. $f: (-\infty, 1] \to R, f(x) = 2 + \sqrt{1-x}$

$g: (-\infty, 1] \to R, g(x) = 2 - \sqrt{1-x}$

c.



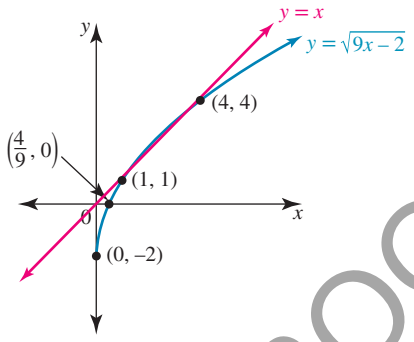
d. $f(-8) = 5; g(-8) = -1$.

21. $a = 5, b = -2, c = 2$; $(y-5)^2 = -2(x-2)$; domain $(-\infty, 2]$; range R

22. a. $m = 9; n = -2$

b. $x = \frac{4}{9}$

c. $\{x: 1 < x < 4\}$



d. $(y+2)^2 = 9x$

23. a. Domain $(-\infty, 0]$; P lies on upper branch

b. V is 2 units from both F and line D and P is 5 units from F and line D.

c. $(a, -\sqrt{-8a})$; Q is $2 - a$ units from F and line D

d. $a = -\frac{4}{3}$

24. a. f is a square root function with domain $[-1, \infty)$; g is a semicircle function with domain $[-2, 2]$

b. $(-0.6, 3.9)$

c. A has the rule $(y-2)^2 = 9(x+1)$; B has the rule $x^2 + (y-2)^2 = 4$.

d. $(-0.6, 3.9), (-0.6, 0.1)$

Exercise 6.6 Other functions and relations

1. a. $x \leq \frac{3}{2}$ **b.** $R \setminus \frac{1}{2}$ **c.** R

2. a. $R \setminus \{\pm 4\}$ **b.** R **c.** $(-1, \infty)$
d. $R \setminus (-2, 2)$ **e.** $R \setminus \{\pm 2\}$ **f.** $(-\infty, 4)$
g. $[0, 2]$ **h.** R^+ **i.** R

3. a. $R \setminus \{-4, -1\}$ **b.** $[-3, \infty)$ **c.** $[-3, \infty) \setminus \{-1\}$

4. a. Domain $\{2, 3, 4, 5\}$, range $\{9, 10, 11, 12\}$.

b. $\{(9, 2), (10, 3), (11, 4), (12, 5)\}$, domain $\{9, 10, 11, 12\}$, range $\{2, 3, 4, 5\}$.

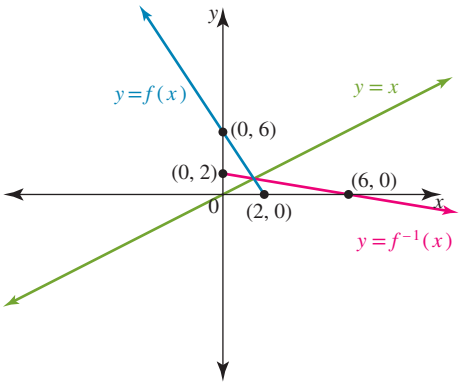
c. $y = x + 7$ **d.** $y = x - 7$

5. a. Domain $(-\infty, 2]$; range $R^+ \cup \{0\}$

b. Domain $R^+ \cup \{0\}$; range $(-\infty, 2]$

c. $f^{-1}: R^+ \cup \{0\} \to R, f^{-1}(x) = \frac{6-x}{3}$

d.



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6. a. i. Domain R, range R.

ii. $y = 4x - 3$

iii. Domain R , range R.

b. i. Domain $R \setminus \{0\}$, range $R \setminus \{2\}$.

ii. $y = -\dfrac{5}{x-2}$

iii. Domain $R \setminus \{2\}$, range $R \setminus \{0\}$.

c. i. Domain $[-1, 2]$, range $[6, 12]$.

ii. $y = 5 - \dfrac{x}{2}$, $x \in [6, 12]$

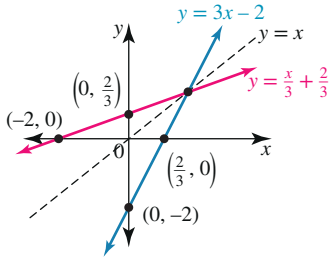
iii. Domain $[6, 12]$, range $[-1, 2]$.

d. i. Domain $[2, \infty)$, range $[0, \infty)$.

ii. $y = x^2 + 2$, $x \geq 0$

iii. Domain $[0, \infty)$, range $[2, \infty)$.

7. a.



b. see diagram in part **a**

c. $(1, 1)$

d. $f^{-1}: R \to R, f^{-1}(x) = \dfrac{x}{3} + \dfrac{2}{3}$.

8. a. $4y - 8x = 1$

b. $y = -\dfrac{3x}{2} - 6$

c. $y = \dfrac{x^2}{4}$

d. $y^2 = \dfrac{x}{4}$ $\left(\text{or } y = \pm\dfrac{\sqrt{x}}{2}\right)$

e. $(x-3)^2 + y^2 = 1$

f. $y = \dfrac{x^2 - 1}{2}, x \geq 0$

9. a. Maximal domain is $R \setminus \{2\}$; asymptote equations: $x = 2, y = 0$

b. $f^{-1}(x) = \dfrac{1}{x} + 2$

c. $x = 0, y = 2$

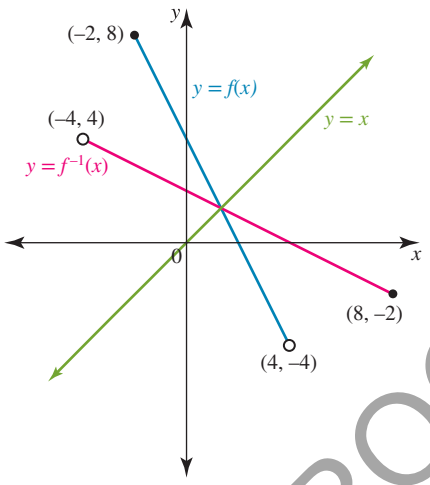
d. $x = b; y = a; y = \dfrac{1}{x-b} + a$

10. a. Domain $(-4, 8]$; range $[-2, 4)$

b. $f^{-1}(x) = \dfrac{4-x}{2}, -4 < x \leq 8$;

$f^{-1}: (-4, 8] \to R, f^{-1}(x) = \dfrac{4-x}{2}$

c.



d. $x = \dfrac{4}{3}$

11. a. R, many-to-one correspondence, not one-to-one

b. inverse domain $[2, 18]$, range $[-4, 0]$. f is one-to-one so inverse is a function.

c. $f^{-1}(x) = \sqrt{x-2} - 4$

12. a. g has one-to-one correspondence.

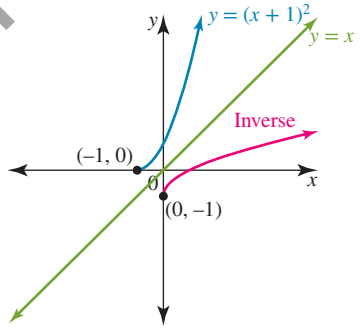
b. $g^{-1}: R \to R, g^{-1}(x) = \sqrt[3]{x-2} + 1$, domain R, range R.

13. a. domain R, range $(-\infty, 9]$

b. $k = 3$.

14. a. $(y+1)^2 = x$ or $y = \pm\sqrt{x} - 1$; one-to-many correspondence so not a function

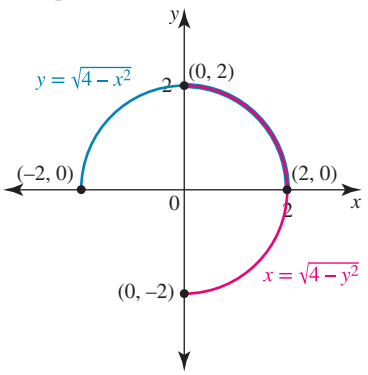
b.



c. $y = \sqrt{x} - 1$

d. No intersection

15. a. Upper semicircle has many-to-one correspondence; semicircle to the right of the y-axis has one-to-many correspondence.



b. Not a function

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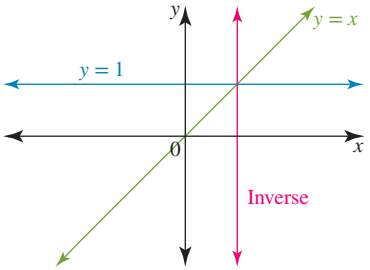
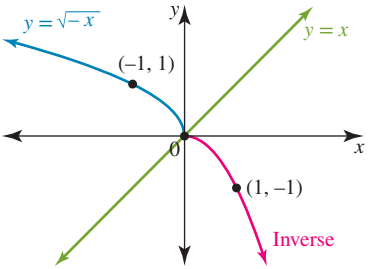
16. $a = 2$

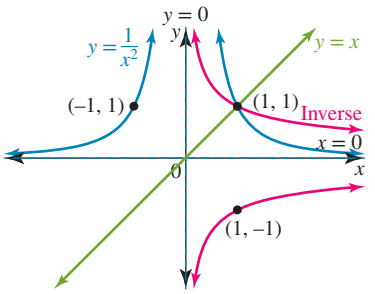
17. a. i.



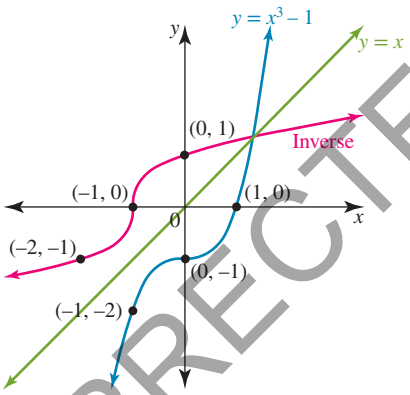
ii.



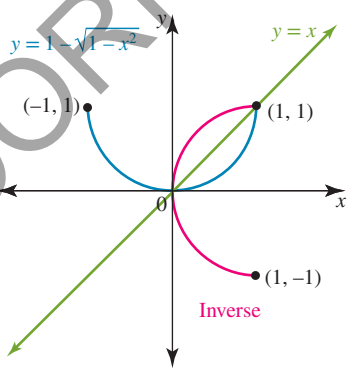
iii.



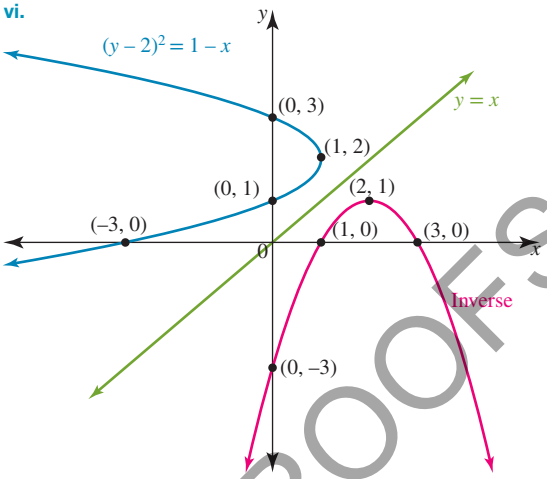
iv.



v.

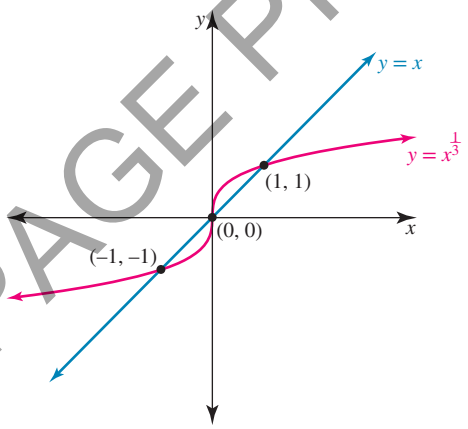


vi.



b. Functions in **ii** and **iv** have inverses which are also functions; $y = -x^2,\ x \geq 0, y = \sqrt[3]{x+1}$

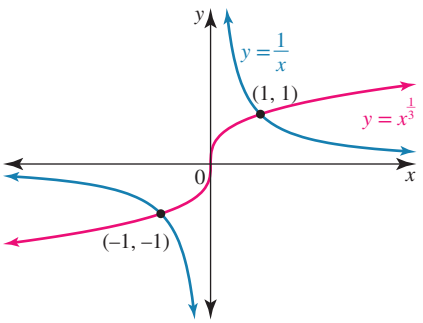
18.



$\{x: x < -1\} \cup \{x: 0 < x < 1\}$

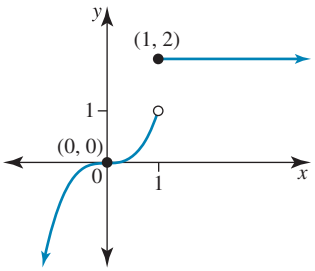
19. a. $(-1, -1),\ (1, 1)$

b.



20. a. **i.** $f(-2) = -8$ **ii.** $f(1) = 2$ **iii.** $f(2) = 2$

b. Domain R; range $(-\infty, 1) \cup \{2\}$



c. Not continuous at $x = 1$

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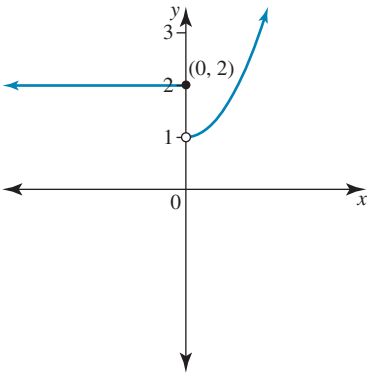
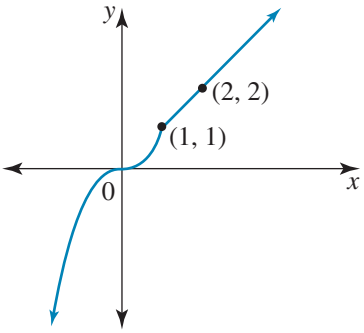
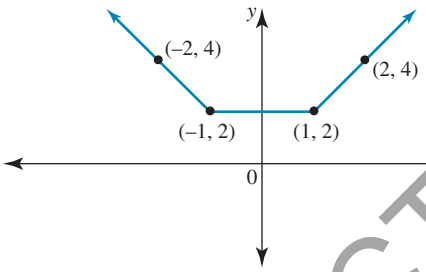
21. a. Domain R; range $(1, \infty)$



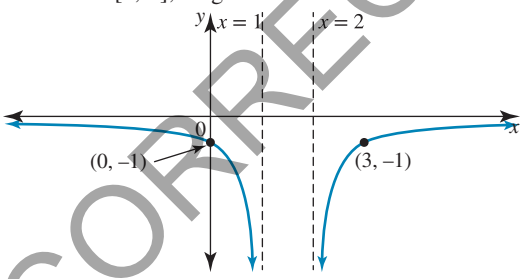
b. Domain R; range R



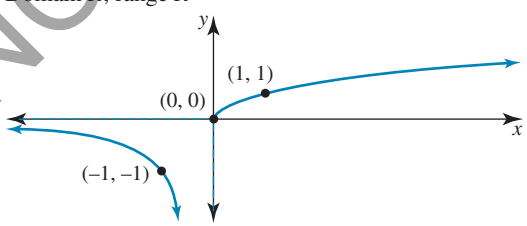
c. Domain R; range $[2, \infty)$



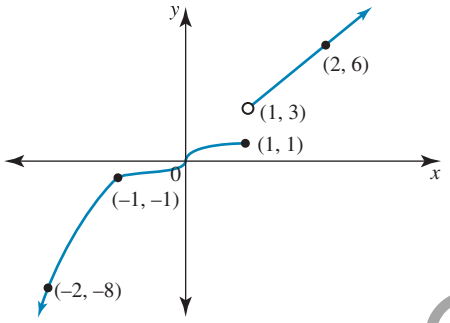
d. Domain $R \setminus [1, 2]$; range R^-



e. Domain R; range R



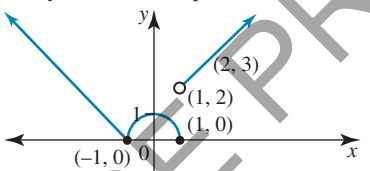
f. Domain R; range $R \setminus [1, 3]$



22. a. **i.** $f(0) = 1$ **ii.** $f(3) = 4$
iii. $f(-2) = 1$ **iv.** $f(1) = 0$

b. Sample responses can be found in the worked solutions in the online resources.

c. Many-to-one correspondence



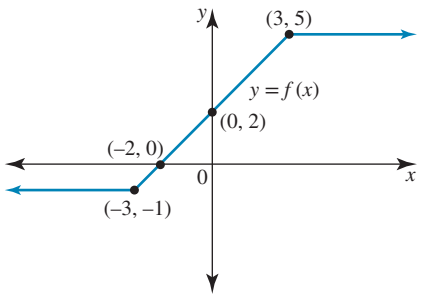
d. $a = \frac{\sqrt{2}}{2}$

23. a. $y = \begin{cases} x + 1, & x \leq 0 \\ -x + 1, & x > 0 \end{cases}$

b. $y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \geq 2 \end{cases}$

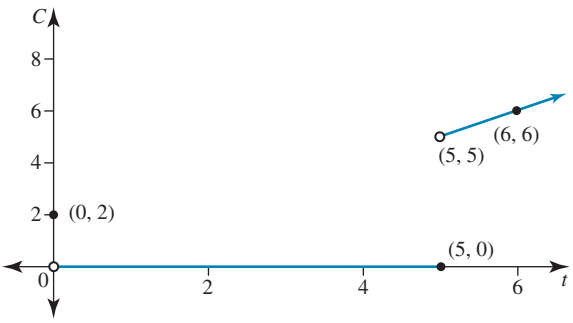
24. a. **i.** $a = -2$
ii. Continuous

b. $a = -1, b = 5$



25. With cost C dollars and time t minutes:

$$C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \leq 5 \\ t, & t > 5 \end{cases}$$



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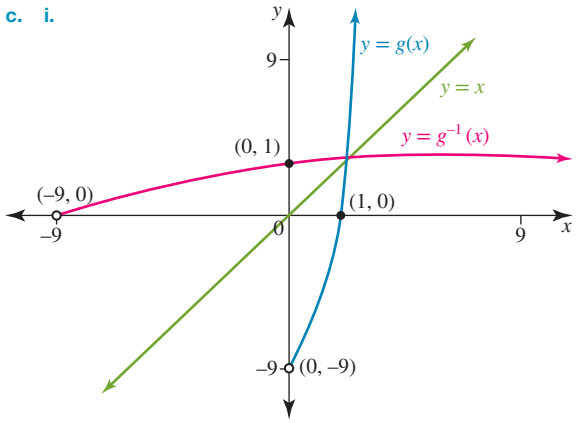
26. a. Either $D = [-4, \infty)$ or $D = (-\infty, -4]$ (Other answers are possible.)

b. $g^{-1}: (-9, \infty) \to R, g^{-1}(x) = \sqrt{x + 25} - 4$; range R^+

c. i.

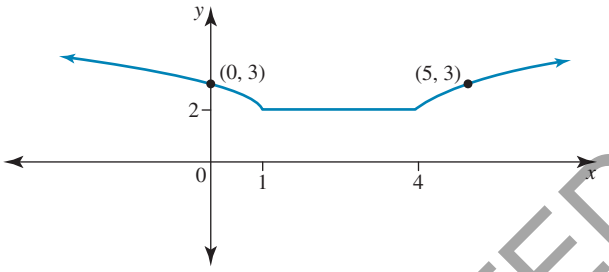


ii. $\left(\frac{-7 + \sqrt{85}}{2}, \frac{-7 + \sqrt{85}}{2}\right)$

d. $\frac{-7 + \sqrt{85}}{2}$

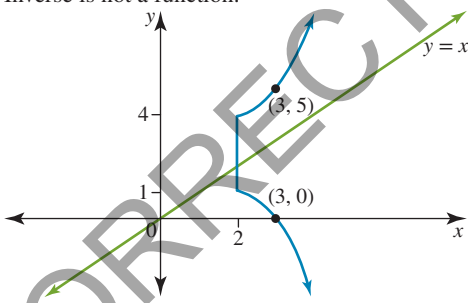
27. a. $f(0) = 3 = f(5)$

b. Domain R; range $[2, \infty)$



c. $x = -35, x = 40$

d. Inverse is not a function.



28. a. $y = 0.5\left((4x + 33)^{0.5} - 5\right), y = -0.5\left((4x + 33)^{0.5} + 5\right)$

b. $(-4.45, -4.45), (0.45, 0.45), (-0.76, -5.24), (-5.24, -0.76)$

29. a. $x = -2, x = 2$

b. The graph can be found in the worked solutions in your online resources. Range $\left[-\frac{4\sqrt{6}}{9}, \infty\right)$

Exercise 6.7 Transformations of functions

1. a. Dilation factor 3 from x-axis

b. Reflection in x-axis

c. Translation up 5

d. Translation 5 to the left

2. a. $y = (x + 2)^2 + 3$

b. $y = \frac{2}{x - 5}$

c. $y = -\frac{1}{2}\sqrt{x}$

d. $y = 4 - x^4$

3. a. Horizontal translation 4 units left, vertical translation 5 units downwards.

b. Dilation of factor 3 from the x-axis (parallel to y-axis) followed by a horizontal translation 2 units right and vertical translation 4 units upwards.

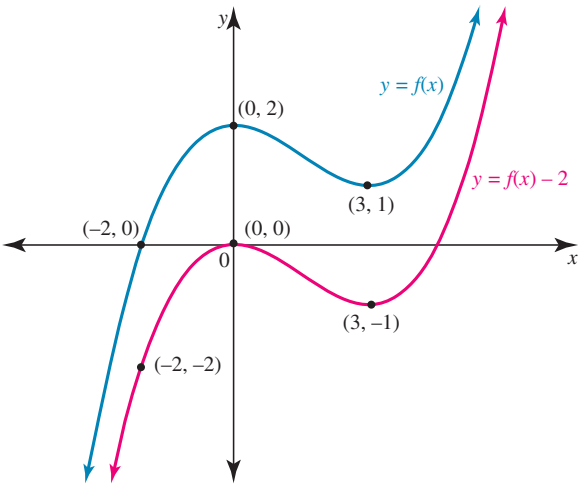
c. Reflection in the x-axis followed by a horizontal translation 3 units right and vertical translation 2 units upwards.

d. Reflection in x-axis and dilation by factor $\frac{3}{2}$ from the x-axis (parallel to y-axis), then followed by horizontal translation 1 unit left and vertical translation 4 units upwards.

e. Horizontal translation 1 unit left and vertical translation 2 units upwards.

f. Reflection in x-axis and dilation of factor 4 from the x-axis (parallel to y-axis).

4.



$(-2, 0) \to (-2, -2)$

$(0, 2) \to (0, 0)$

$(3, 1) \to (3, -1)$

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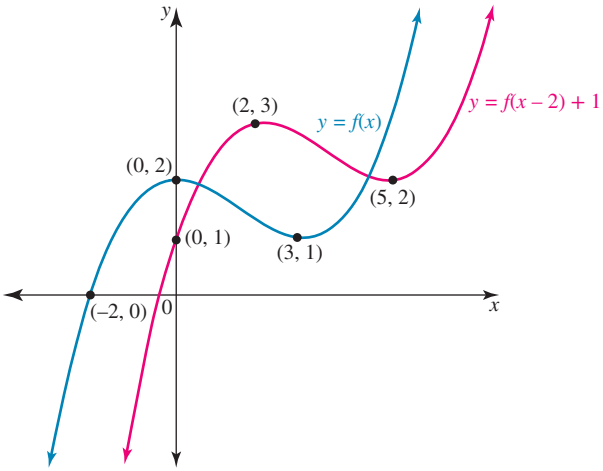
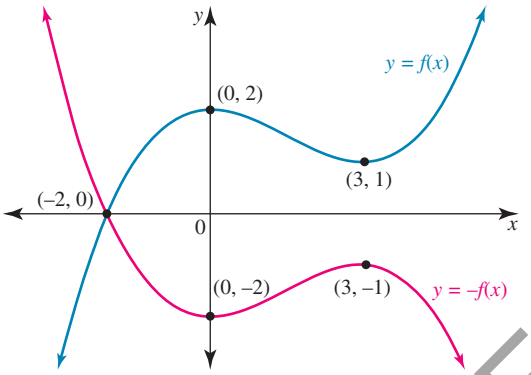
5.



$(-2, 0) \rightarrow (0, 1)$
$(0, 2) \rightarrow (2, 3)$
$(3, 1) \rightarrow (5, 2)$

6.



$(-2, 0) \rightarrow (-2, 0)$
$(0, 2) \rightarrow (0, -2)$
$(3, 1) \rightarrow (3, -1)$

7. a. $y = -(x - 1)^2 + 3$

b. $y = -(x - 1)^2 - 3$; a different image

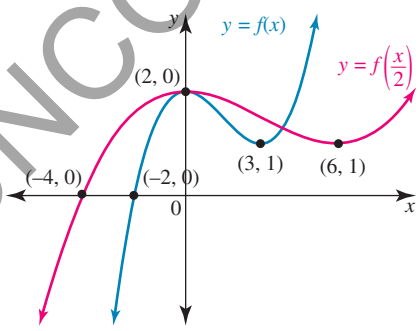
8. a. Dilation factor 3 from y-axis

b. Dilation factor $\frac{1}{2}$ from y-axis and translation 1 up

c. Horizontal translation 4 to the right; vertical translation 4 down

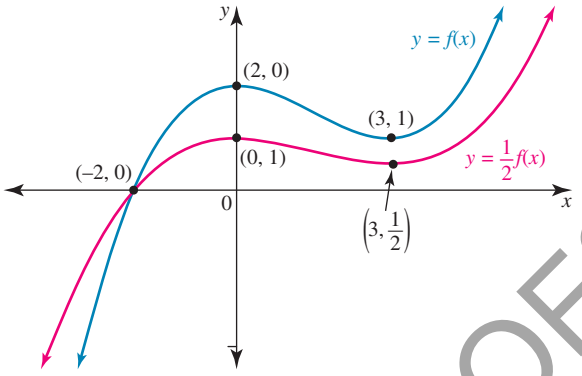
d. Dilation factor $\frac{1}{2}$ from y-axis; translation $\frac{1}{2}$ left; or dilation factor 8 from x-axis, translation $\frac{1}{2}$ left

9.



$(-2, 0) \rightarrow (-4, 0)$
$(0, 2) \rightarrow (0, 2)$
$(3, 1) \rightarrow (6, 1)$

10.



$(-2, 0) \rightarrow (-2, 0)$
$(0, 2) \rightarrow (0, 1)$
$(3, 1) \rightarrow \left(3, \frac{1}{2}\right)$

11. i. a. $y = 2\sqrt{x}$ **b.** $y = \sqrt{\frac{x}{2}}$

c. $y = -\sqrt{x} + 2$ **d.** $y = -\sqrt{x} - 2$

e. $y = \sqrt{2 - x}$ **f.** $y = \sqrt{-x - 2}$

ii. a. $y = 2x^4$ **b.** $y = \frac{1}{16}x^4$

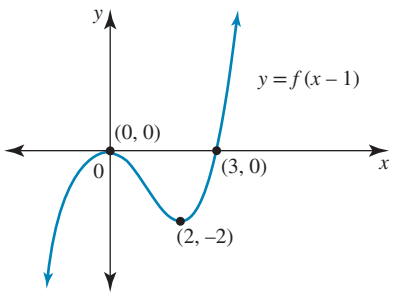
c. $y = -x^4 + 2$ **d.** $y = -x^4 - 2$

e. $y = (x - 2)^4$ **f.** $y = (x + 2)^4$

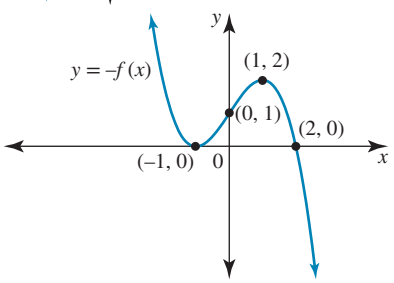
12. a. Dilation of factor 4 from the x-axis; dilation of factor 2 from the y-axis; horizontal translation 2 units to the right; vertical translation 3 units upwards

b. Either reflection in y-axis, dilation of factor 4 from the y-axis followed by horizontal translation 12 units to the right, or reflection in y-axis, dilation of factor $\frac{1}{2}$ from the x-axis followed by horizontal translation 12 units to the right.

13. a.



b.



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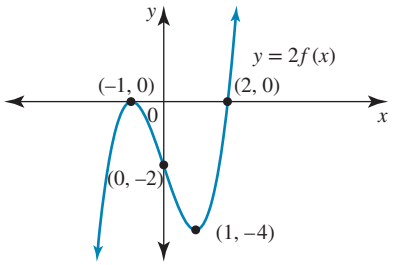
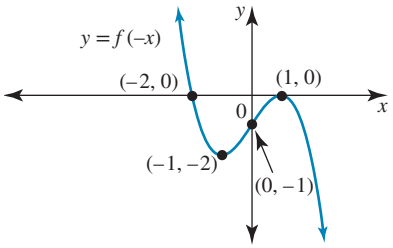
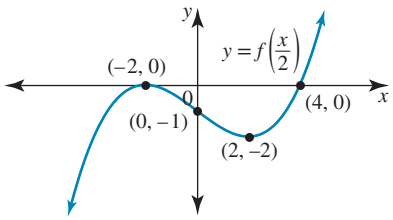
c.



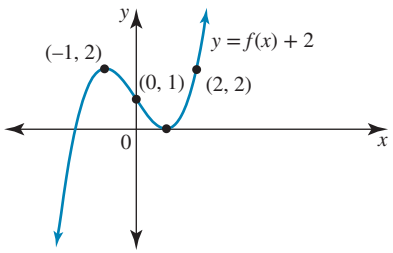
d.



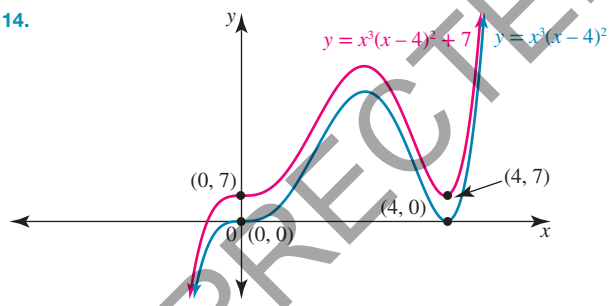
e.



f.



14.



15. a. A dilation of factor 2 from the x-axis (parallel to the y-axis) and a reflection in the y-axis followed by a vertical translation of 1 unit upwards.

b. Reflection in the x-axis and dilation of factor $\frac{1}{3}$ from the y-axis.

c. Dilation of factor $\frac{1}{2}$ from the x-axis (parallel to the y-axis), reflection in the y-axis followed by a horizontal translation of 4 units to the right.

d. $y = \dfrac{1}{1 - 3x}$.

16. a. $y = \dfrac{1}{2(x + 3)}$

b. Horizontal translation 3 units to the right followed by dilation of factor 2 from the y-axis.

17. a. $(1, -8)$ **b.** $(-3, 4)$

c. $(3, -0.8)$ **d.** $(0.6, -4)$

18. a. i. $y = -\dfrac{3}{x}$ **ii.** $y = -\dfrac{3}{x}$

b. i. $y = \dfrac{3}{x^2} + 6$ **ii.** $y = \dfrac{3}{x^2} + 18$

c. Reflection in x-axis; dilation factor $\frac{1}{2}$ from x-axis (or factor $\frac{1}{2}$ from y-axis); translation 1 unit left and 1 unit up

d. $x = -1$, $y = 0$

19. a. Dilation factor 2 from x-axis; translation 3 left

b. Dilation factor 6 from x-axis; translation 2 right, 1 up

c. Dilation factor 0.5 from y-axis; translation 1 left

d. Reflection in y-axis; translation 3 right

e. Reflection in x-axis; dilation factor 0.25 from y-axis; translation 1 up

f. Dilation factor $\frac{1}{9}$ from x-axis; dilation factor 9 from y-axis; translation 3 right

20. a. $y = \dfrac{1}{3(x + 3)^2}$

b. $y = -x^5 + 3$

c. $y = \dfrac{1}{1 - x}$

d. $y = \sqrt[3]{2x - 1}$

e. $y = -(x + 3)(x - 3)(x - 7)$

f. $y = \frac{1}{8}x^2(x + 4)(x - 4)$

21. a. Image is the same parabola g.

b. Image is $y = -x^{\frac{1}{3}}$ for either reflection.

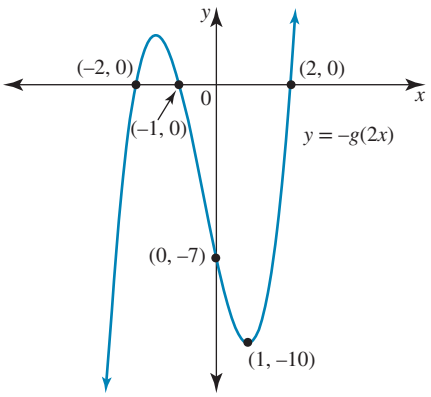
c. Upper semicircle; reflect in the x-axis again

d. Maximum turning point $(-2, -5)$

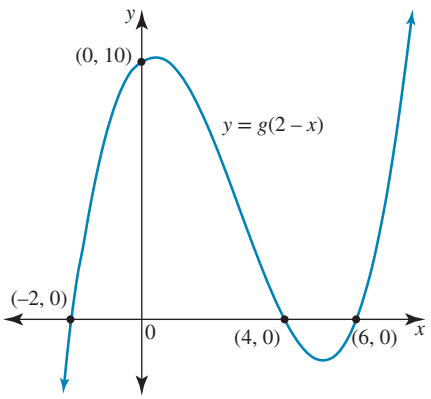
e. $(y - 2)^2 = -(x + 3)$

f. $f(x) = -3(x - 2)^3$

22. a.



b.



c. $h > 4$

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d. $g(x) = -\frac{7}{32}(x+4)(x+2)(x-4)$;

$g(2x) = -\frac{7}{4}(x+2)(x+1)(x-2)$

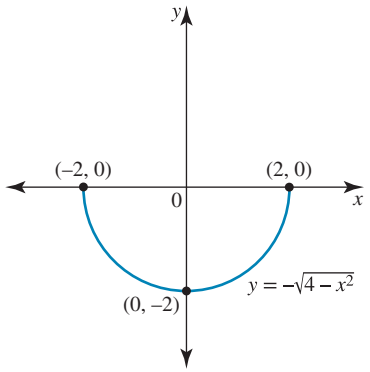
23. y_2 is a dilation of y_1 factor $\frac{1}{2}$ from y-axis; y_3 is a dilation of y_1 factor 2 from y-axis. All are parabolas.

6.8 Review: exam practice

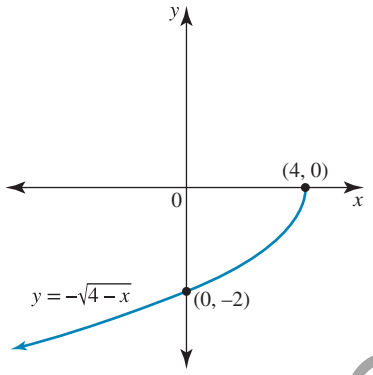
Short answer

1. See the table at the foot of the page.*

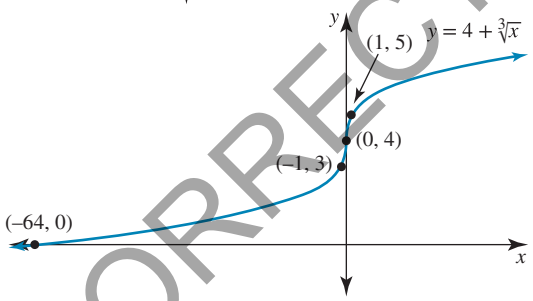
a.



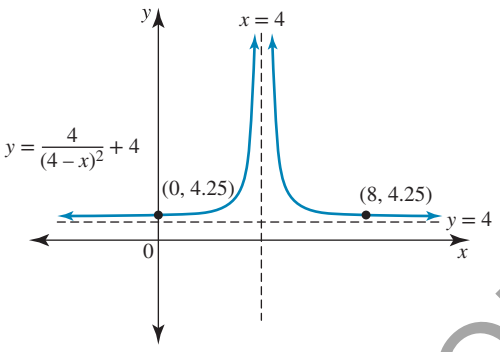
b.



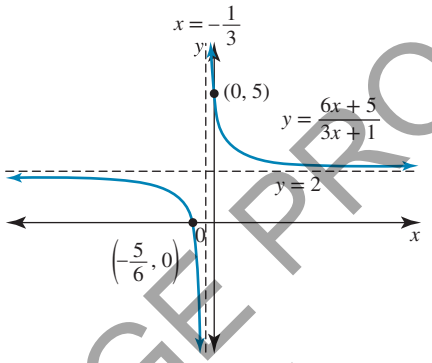
c.



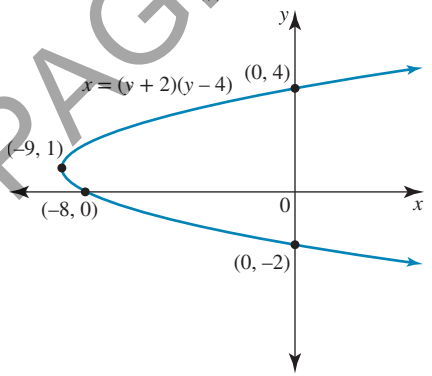
d.



e.



f.



2. a. Centre $(3, -2)$; radius $\frac{\sqrt{46}}{2}$ units

b. Outside

*1.

	Shape	x-intercepts	y-intercepts	Domain	Range
a	Semicircle, lower half, centre $(0, 0)$, radius 2	$(-2, 0), (2, 0)$	$(0, -2)$	$[-2, 2]$	$[-2, 0]$
b	Square root function, endpoint $(4, 0)$	$(4, 0)$	$(0, -2)$	$(-\infty, 4]$	$(-\infty, 0]$
c	Cube root function, point of inflection $(0, 4)$	$(-64, 0)$	$(0, 4)$	R	R
d	Truncus, asymptotes $x = 4, y = 4$	none	$\left(0, 4\frac{1}{4}\right)$	$R \setminus \{4\}$	$(4, \infty)$
e	Hyperbola, asymptotes $x = -\frac{1}{3}, y = 2$	$\left(-\frac{5}{6}, 0\right)$	$(0, 5)$	$R \setminus \left\{-\frac{1}{3}\right\}$	$R \setminus \{2\}$
f	Sideways parabola, axis of symmetry $y = 1$, vertex $(-9, 1)$	$(-8, 0)$	$(0, -2), (0, 4)$	$[-9, \infty)$	R

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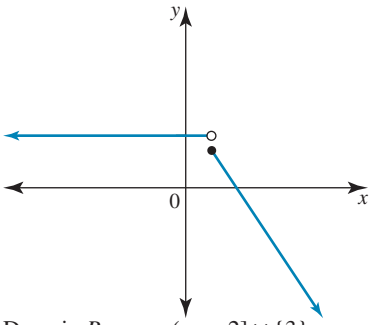
3. a. $f(0)+f(1)+f(2)=3+2+0=5$

b.



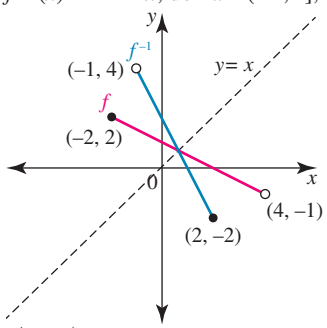
c. Domain R, range $(-\infty, 2] \cup \{3\}$

d. Many-to-one correspondence

4. a. Domain $[-2, 4)$, range $(-1, 2]$

b. $f^{-1}(x) = 2 - 2x$; domain $(-1, 2]$, range $[-2, 4)$

c.



d. $\left(\frac{2}{3}, \frac{2}{3}\right)$

5. a. $y = \sqrt{2x+4} + 3$

b. $y = \begin{cases} x+8, & x \leq -2 \\ 4-x, & x > -2 \end{cases}$

c. $y = 2 - \frac{2}{x+2}$

6. a. $y = \frac{1}{2}(1-x)^{\frac{1}{3}}$

b. $y = (x-5)^3 - 1$

c. i. $a = 2$

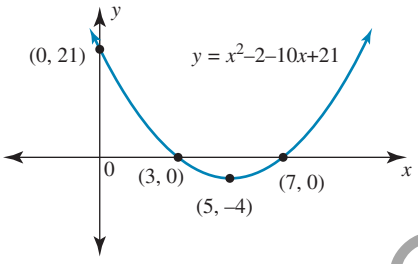
ii. $f^{-1}: [6, \infty) \to R, f^{-1}(x) = 2 - \sqrt{x-6}$

Multiple choice

1. E	**2.** D	**3.** B	**4.** A	**5.** D
6. D	**7.** C	**8.** E	**9.** B	**10.** A

Extended response

1. a. Minimum turning point (5, −4), y-intercept (0, 21), x-intercept (3, 0), (7, 0)

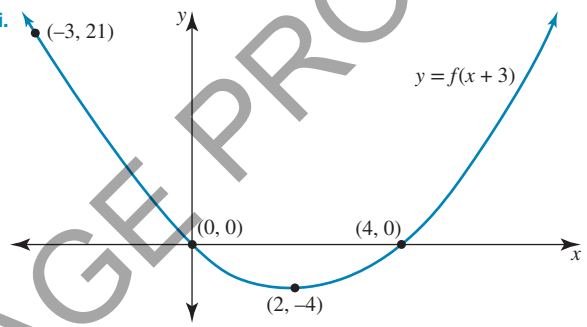


b. $k > 4$

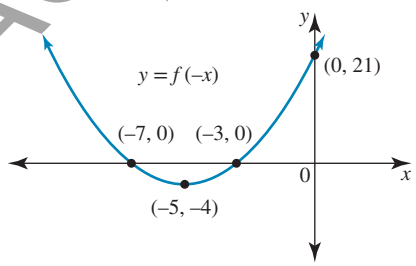
c. $h < -7$

d. See the table at the foot of the page.*

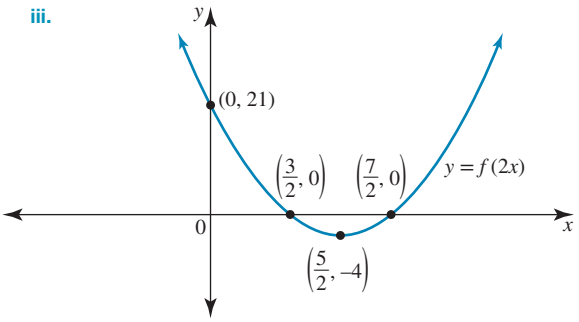
i.



ii.



iii.



*1. d.

	Transformation	Turning point	x-intercepts	y-intercept	Other
i	Translation 3 units left	$(2, -4)$	$(0, 0), (4, 0)$	$(0, 0)$	$(-3, 21)$
ii	Reflection in y-axis	$(-5, -4)$	$(-7, 0), (-3, 0)$	$(0, 21)$	
iii	Dilation factor of $\frac{1}{2}$ from the y-axis	$\left(\frac{5}{2}, -4\right)$	$\left(\frac{3}{2}, 0\right), \left(\frac{7}{2}, 0\right)$	$(0, 21)$	
iv	Dilation factor $\frac{3}{2}$ from the y-axis, translation 6 left	$\left(\frac{3}{2}, -4\right)$	$\left(-\frac{3}{2}, 0\right), \left(\frac{9}{2}, 0\right)$	$(0, -3)$	$(-6, 21)$
v	Reflection in the x-axis, dilation factor of 0.5 from the x-axis, translation 2 down	$(5, 0)$	$(5, 0)$	$(0, -12.5)$	$(3, -2), (7, -2)$
vi	Reflection in $y = x$	$(-4, 5)$	$(21, 0)$	$(0, 3), (0, 7)$	

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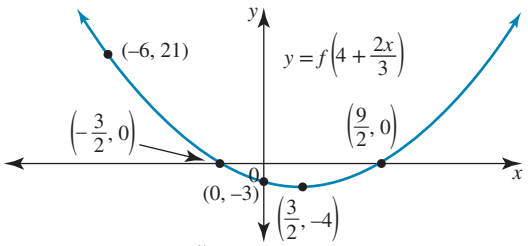
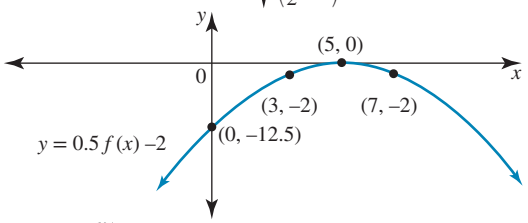
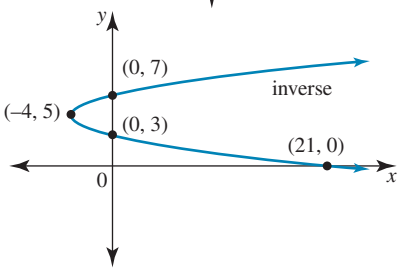
iv.



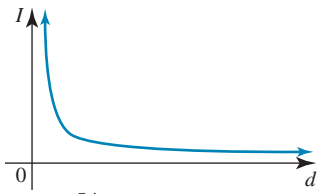
v.



vi.



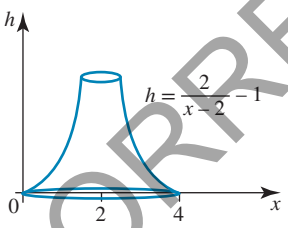
2. a. $I = \frac{k}{d}$



b. $y = \frac{54}{(x+2)^2} - 2$, $a = \sqrt[3]{54} - 2$

c. i. Reflect in the y-axis, then translate 4 right; or translate 4 left, then reflect in the y-axis. $y = \frac{-2}{x-2} - 1$, $x < 2$

ii. Diameter 4 cm



iii. 3 cm

iv. 0.8 cm, 0.64π cm^2

3. a. i. 45°

ii. $y = x - 2$

iii. $y = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

b. i. Fails the vertical line test

ii. $x = \begin{cases} y + 2, & 0 < y < 2 \\ 6 - y, & y \geq 2 \end{cases}$

c. $y = -x + 30$, $0 \leq x \leq 4$

d. i. Let $y = 6$

$36 = 4x$

$x = 9$

Hence, P(9, 6) lies on the sideways parabola.

ii. One point of intersection, Q $\left(\frac{1}{9}, -\frac{2}{3}\right)$

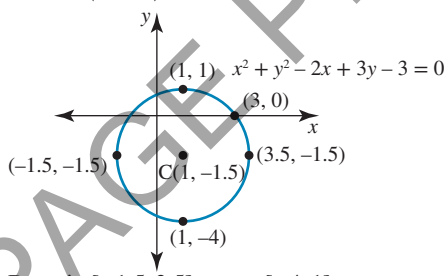
iii. 71.6°

iv. $y = -\frac{2}{3}$, $x \geq \frac{1}{9}$

4. a. $a = -2$, $b = 3$, $c = -3$

b. Centre $\left(1, -\frac{3}{2}\right)$; radius $\frac{5}{2}$

c.



Domain $[-1.5, 3.5]$; range $[-4, 1]$

d. $y = -\frac{4}{3}x + 4$

e. $\left(1, \frac{8}{3}\right)$

f. 20 seconds

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