TOPIC 18

# Functions and relations

## 18.1 Overview

### Why learn this?

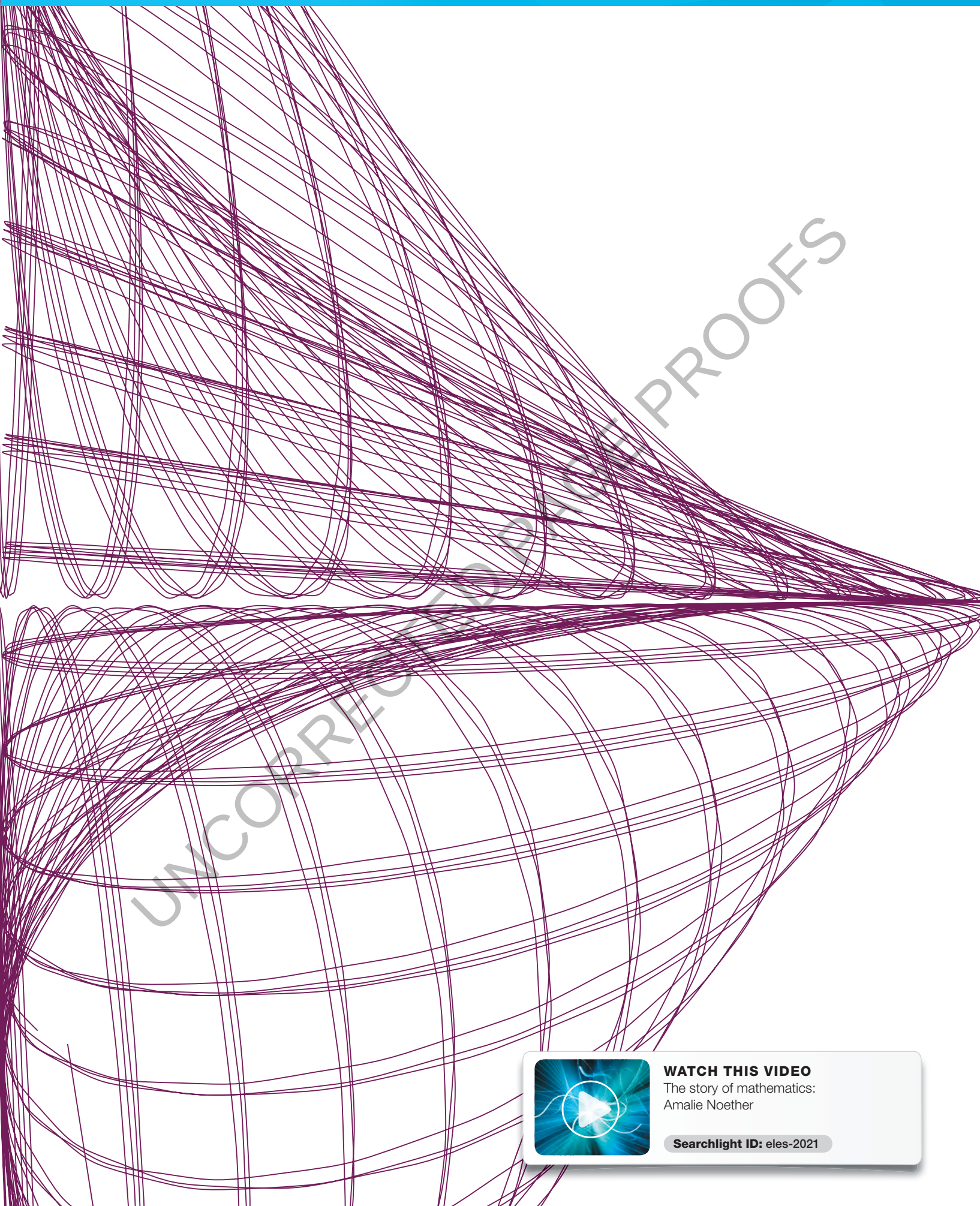
A relation is a set of ordered pairs; functions are special types of relations. It is important to understand relationships between variables in order to be able to model the relationships. Many different functions can be used to model events in the world. You have already studied lines of best fit, where straight lines are fitted to data in order to make predictions. Exponential functions are used to model growth and decay. There are many examples of functions used in engineering, science, finance, architecture and medicine.

### What do you know?

assess on

1 **THINK** List what you know about functions and relations. Use a thinking tool such as a concept map to show your list.
2 **PAIR** Share what you know with a partner and then with a small group.
3 **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of functions and relations.

### Learning sequence







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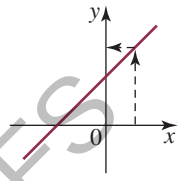
# 18.2 Functions and relations

## Relations

- A **relation** is a set of ordered pairs of values such as all the points on the circle $x^2 + y^2 = 4$ or all the points on the exponential $y = 2^x$. Relations can be grouped into the following four categories.

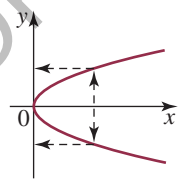
### One-to-one relations

- A **one-to-one relation** exists if for any $x$-value there is only one corresponding $y$-value and vice versa. For example:



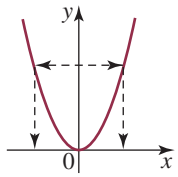

### One-to-many relations

- A **one-to-many relation** exists if for any $x$-value there is more than one $y$-value, but for any $y$-value there is only one $x$-value. For example:



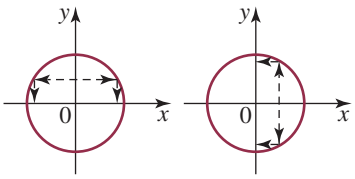

### Many-to-one relations

- A **many-to-one relation** exists if there is more than one $x$-value for any $y$-value but for any $x$-value there is only one $y$-value. For example:




### Many-to-many relations

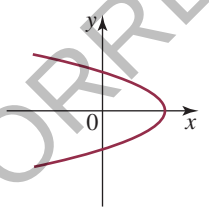
- A **many-to-many relation** exists if there is more than one $x$-value for any $y$-value and vice versa. For example:



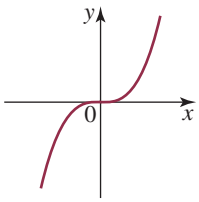

**WORKED EXAMPLE 1**

**What type of relation does each graph represent?**

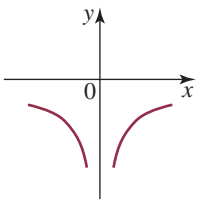
**a**



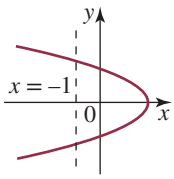

**b**



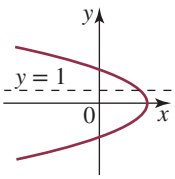

**c**




| THINK | WRITE |
|---|---|
| **a** **1** For some $x$-values there is more than one $y$-value. A line through some $x$-values shows that 2 $y$-values are available. | **a** One-to-many relation |



2 For any $y$-value there is only one $x$-value. A line through any $y$-value shows that only one $x$-value is available.




**b** 1 For any $x$-value there is only one $y$-value.

2 For any $y$-value there is only one $x$-value.

**b** One-to-one relation

**c** 1 For any $x$-value there is only one $y$-value.

2 For some $y$-values there is more than one $x$-value.

**c** Many-to-one relation

## Functions

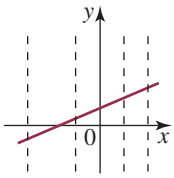
- Relations that are one-to-one or many-to-one are called **functions**. That is, a function is a relation where for any $x$-value there is at most one $y$-value.

### Vertical line test

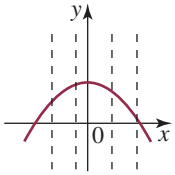
- To determine if a graph is a function, a vertical line is drawn anywhere on the graph. If it does not intersect with the curve more than once, then the graph is a function.

  For example, in each of the two graphs below, each vertical line intersects the graph only once.

1.



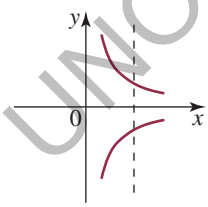

2.



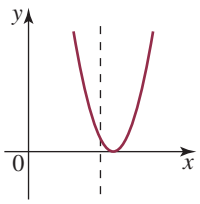

**WORKED EXAMPLE 2**

**State whether or not each of the following relations are functions.**

**a**




**b**




**THINK**

**a** It is possible for a vertical line to intersect with the curve more than once.

**b** It is not possible for any vertical line to intersect with the curve more than once.

**WRITE**

**a** Not a function

**b** Function

## Function notation

- Consider the relation $y = 2x$, which is a function.
  The $y$-values are determined from the $x$-values, so we say '$y$ is a function of $x$', which is abbreviated to $y = f(x)$.
  So, the rule $y = 2x$ can also be written as $f(x) = 2x$.

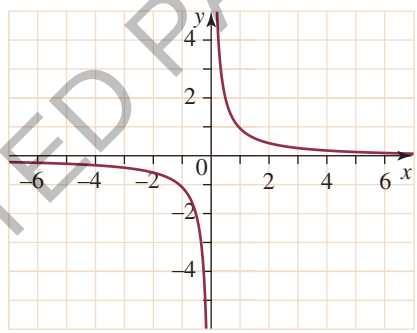
  If $x = 1$, then $y = f(1)$
  $= 2 \times 1$
  $= 2$.

  If $x = 2$, then $y = f(2)$
  $= 2 \times 2$
  $= 4$, and so on.

## Domain and range

- The **domain** of a function is the set of all allowable values of $x$. It is sometimes referred to as the **maximal domain**.
- The **range** of a function is the set of $y$-values produced by the function.
  For example, the domain of the function $f(x) = 2x + 3$ is the set of all real numbers ($x \in R$), and the range is the set of all real numbers ($y \in R$).

  The domain of the function $f(x) = \frac{1}{x}$ is the set of all real numbers apart from 0 ($x \in R\backslash 0$), and the range is the set of all real numbers apart from 0 ($y \in R\backslash 0$).




## Evaluating functions

- For a given function $y = f(x)$, the value of $y$ when $x = 1$ is written as $f(1)$, the value of $y$ when $x = 5$ is written as $f(5)$, the value of $y$ when $x = a$ as $f(a)$, etc.

**WORKED EXAMPLE 3** TI CASIO

**If $f(x) = x^2 - 3$, find:**

**a** $f(1)$ **b** $f(a)$ **c** $3f(2a)$ **d** $f(a) + f(b)$ **e** $f(a + b)$.

| THINK | WRITE |
|---|---|
| **a** 1 Write the rule. | **a** $f(x) = x^2 - 3$ |
| 2 Substitute $x = 1$ into the rule. | $f(1) = 1^2 - 3$ |
| 3 Simplify. | $= 1 - 3$<br>$= -2$ |

| | |
|---|---|
| **b** 1 Write the rule. | **b** $f(x) = x^2 - 3$ |
| 2 Substitute $x = a$ into the rule. | $f(a) = a^2 - 3$ |
| **c** 1 Write the rule. | **c** $f(x) = x^2 - 3$ |
| 2 Substitute $x = 2a$ into the rule and simplify. | $f(2a) = (2a)^2 - 3$ |
| 3 Multiply the answer by 3 and simplify. | $= 2^2a^2 - 3$<br>$= 4a^2 - 3$<br>$3f(2a) = 3(4a^2 - 3)$<br>$= 12a^2 - 9$ |
| **d** 1 Write the rule. | **d** $f(x) = x^2 - 3$ |
| 2 Evaluate $f(a)$. | $f(a) = a^2 - 3$ |
| 3 Evaluate $f(b)$. | $f(b) = b^2 - 3$ |
| 4 Evaluate $f(a) + f(b)$. | $f(a) + f(b) = a^2 - 3 + b^2 - 3$<br>$= a^2 + b^2 - 6$ |
| **e** 1 Write the rule. | **e** $f(x) = x^2 - 3$ |
| 2 Evaluate $f(a + b)$. | $f(a + b) = (a + b)^2 - 3$<br>$= (a + b)(a + b) - 3$<br>$= a^2 + 2ab + b^2 - 3$ |

## Identifying features of functions

- We can identify features of certain functions by observing what happens to the function value ($y$ value) when $x$ approaches a very small value such as 0 ($x \to 0$) or a very large value such as $\infty$ ($x \to \infty$).

**WORKED EXAMPLE 4**

**Describe what happens to these functions as the value of $x$ increases, that is, as $x \to \infty$.**

**a** $f(x) = x^2$ **b** $f(x) = 2^{-x}$ **c** $f(x) = \frac{1}{x} + 1$

| THINK | WRITE |
|---|---|
| **a** 1 Write the function. | **a** $f(x) = x^2$ |
| 2 Substitute large $x$ values into the function, such as $x = 10\,000$ and $x = 1\,000\,000$. | $f(10\,000) = 100\,000\,000$<br>$f(1\,000\,000) = 1 \times 10^{12}$ |
| 3 Write a conclusion. | As $x \to \infty$, $f(x)$ also increases; that is, $f(x) \to \infty$. |

| | |
|---|---|
| **b** 1 Write the function. | **b** $f(x) = 2^{-x}$ |
| 2 Substitute large $x$ values into the function, such as $x = 10\,000$ and $x = 1\,000\,000$. | $f(10\,000) \approx 0$<br>$f(1\,000\,000) \approx 0$ |
| 3 Write a conclusion. | As $x \to \infty$, $f(x) \to 0$. |
| **c** 1 Write the function. | **c** $f(x) = \frac{1}{x} + 1$ |
| 2 Substitute large $x$ values into the function, such as $x = 10\,000$ and $x = 1\,000\,000$. | $f(10\,000) = 1.0001$<br>$f(1\,000\,000) = 1.000\,001$ |
| 3 Write a conclusion. | As $x \to \infty$, $f(x) \to 1$. |

## Points of intersection

- If two functions are drawn on the one set of axes, there may be a point or points where the curves intersect. The function equations can be solved simultaneously to find the coordinates of these points of intersection.

**WORKED EXAMPLE 5** TI CASIO

**Find any points of intersection between $f(x) = 2x + 1$ and $g(x) = \frac{1}{x}$.**

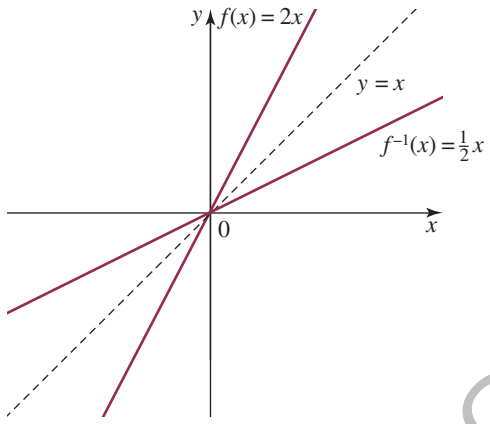
| THINK | WRITE |
|---|---|
| 1 Write the two equations. | $f(x) = 2x + 1$<br>$g(x) = \frac{1}{x}$ |
| 2 Points of intersection are common values between the two curves. To solve the equations simultaneously, equate both functions. | For points of intersection:<br>$2x + 1 = \frac{1}{x}$ |
| 3 Rearrange the resulting equation and solve for $x$. | $2x^2 + x = 1$<br>$2x^2 + x - 1 = 0$<br>$(2x - 1)(x + 1) = 0$<br>$x = \frac{1}{2}$ or $-1$ |
| 4 Substitute the $x$ values into either function to find the $y$ values. | $f(\frac{1}{2}) = 2 \times \frac{1}{2} + 1 = 2$<br>$f(-1) = 2 \times -1 + 1 = -1$ |
| 5 Write the coordinates of the two points of intersection. | The points of intersection are $\left(\frac{1}{2}, 2\right)$ and $(-1, -1)$. |

## Inverse functions

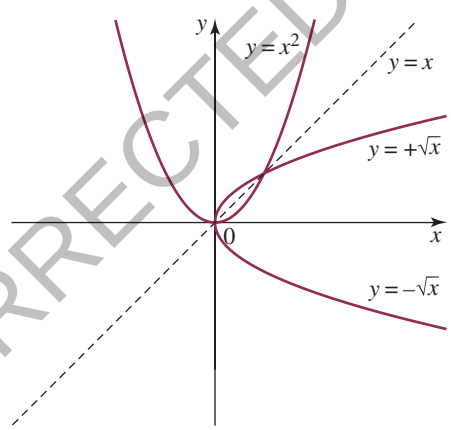
- An inverse graph is created when a graph is reflected in the line $y = x$, the 45° line. Algebraically the resulting equation is created by interchanging $x$ and $y$ in the original equation.

- For example, take the function $f(x) = 2x$.
  Let $y = 2x$.
  Interchange $x$ and $y$: $x = 2y$
  Make $y$ the subject: $y = \frac{1}{2}x$
  Write the inverse function using function notation.

  The inverse of the function $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.




- Not all functions have inverses that are functions. For example, the inverse of the function $y = x^2$ is $x = y^2$ or $y = \pm\sqrt{x}$.

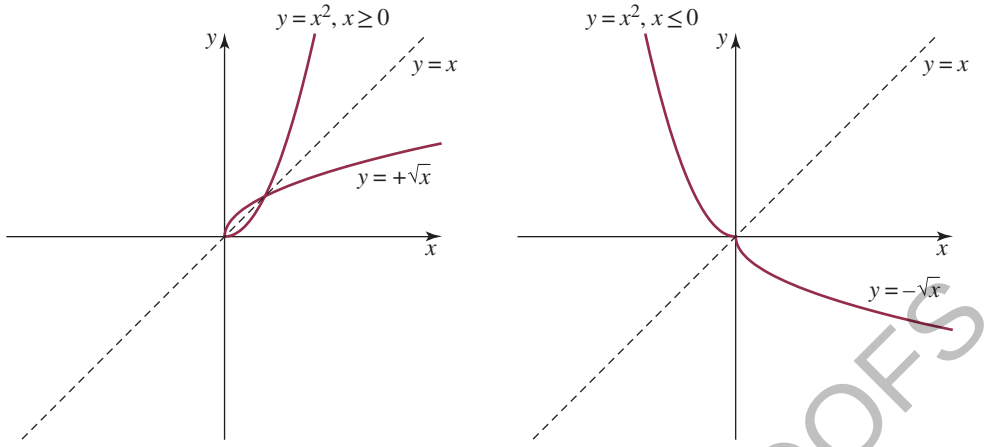



  – Notice that $y = \pm\sqrt{x}$ is not a function, as it is a one-to-many relation.
  – For this inverse to be a function, a restriction must be placed on the domain of the original quadratic function. Normally, either the part of the graph to the left of the turning point or the part to the right is chosen. The original function is re-stated as two functions:

$$y = x^2,\ x \geq 0 \text{ and } y = x^2,\ x \leq 0.$$

  Thus the inverse functions are

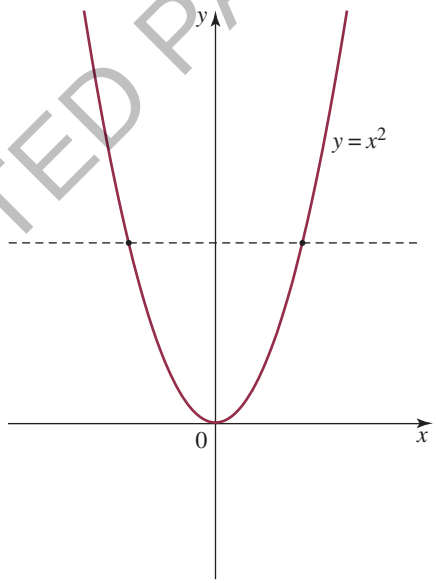
$$y = +\sqrt{x},\ x \geq 0 \text{ and } y = -\sqrt{x},\ x \geq 0.$$



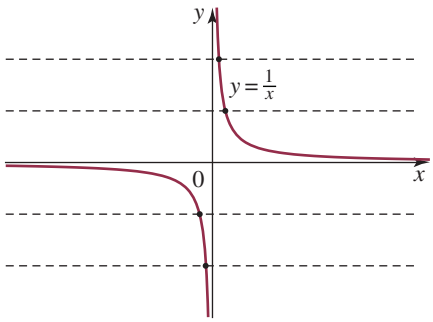

(a) (b)

## The horizontal line test

- Not all functions have inverses that are functions. For a function to have an inverse function, it must be a one-to-one function. It was shown above that $f(x) = x^2$ does not have an inverse function unless restrictions are placed.
- A function has an inverse function when a horizontal line cannot be drawn that cuts through the graph more than once.
- The inverse of the function $f(x) = x^2$ is not a function, as its graph does not satisfy the horizontal line test.




- The inverse of the function $f(x) = \dfrac{1}{x}$ is a function, as its graph satisfies the horizontal line test.



**WORKED EXAMPLE 6**

**a i Show that the function $f(x) = x(x - 5)$ will have not have an inverse function.**

**ii Suggest a restriction that would result in an inverse function.**

**b i Show that the function $f(x) = x^2 + 4$, $x \geq 0$ will have an inverse function.**

**ii Determine the equation of the inverse function.**

**THINK**

**a i 1** Sketch the graph of $f(x) = x(x - 5)$.

**2** Draw a dotted horizontal line(s) through the graph.

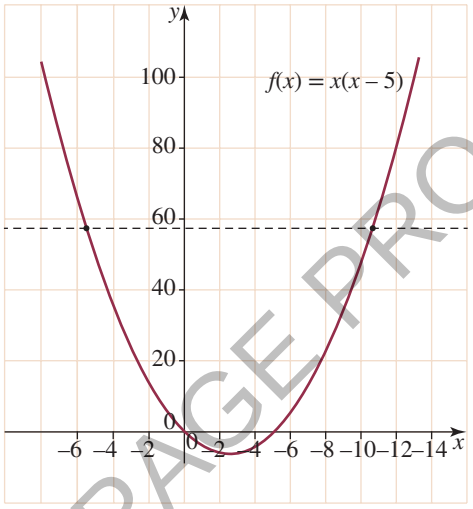
**ii** Apply a restriction to the function so that it will have an inverse.

**b i 1** Sketch the graph of $f(x) = x^2 + 4$, $x \geq 0$.

**2** Draw a dotted horizontal line through the graph.

**WRITE/DRAW**

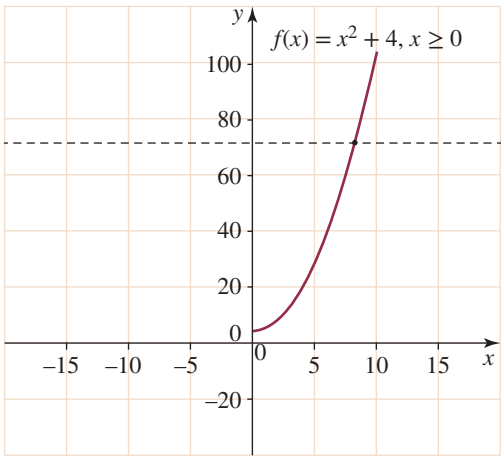
**a i**




The graph does not satisfy the horizontal line test, so the function $f(x) = x(x - 5)$ will not have an inverse function.

**ii** An inverse function will exist if $f(x) = x(x - 5)$, $x \leq 2.5$ or $f(x) = x(x - 5)$, $x \geq 2.5$.

**b i**




The graph satisfies the horizontal line test, so the function $f(x) = x^2 + 4$, $x \geq 0$ has an inverse function.

ii 1 Determine the equation of the inverse function by interchanging $x$ and $y$ and simplifying.

ii Let $y = x^2 + 4$, $x \geq 0$.
Interchange $x$ and $y$.
$x = y^2 + 4$
Make $y$ the subject.

$$x = y^2 + 4$$
$$x - 4 = y^2$$
$$\sqrt{x - 4} = y$$
$$y = \sqrt{x - 4}$$

2 Write the answer in correct form, noting the domain.

The inverse of $f(x) = x^2 + 4$ is $f^{-1}(x) = \sqrt{x - 4}$, $x \geq -4$.

assess on

## Exercise 18.2 Functions and relations

**REFLECTION**
How do you determine the difference between functions and relations?

### INDIVIDUAL PATHWAYS

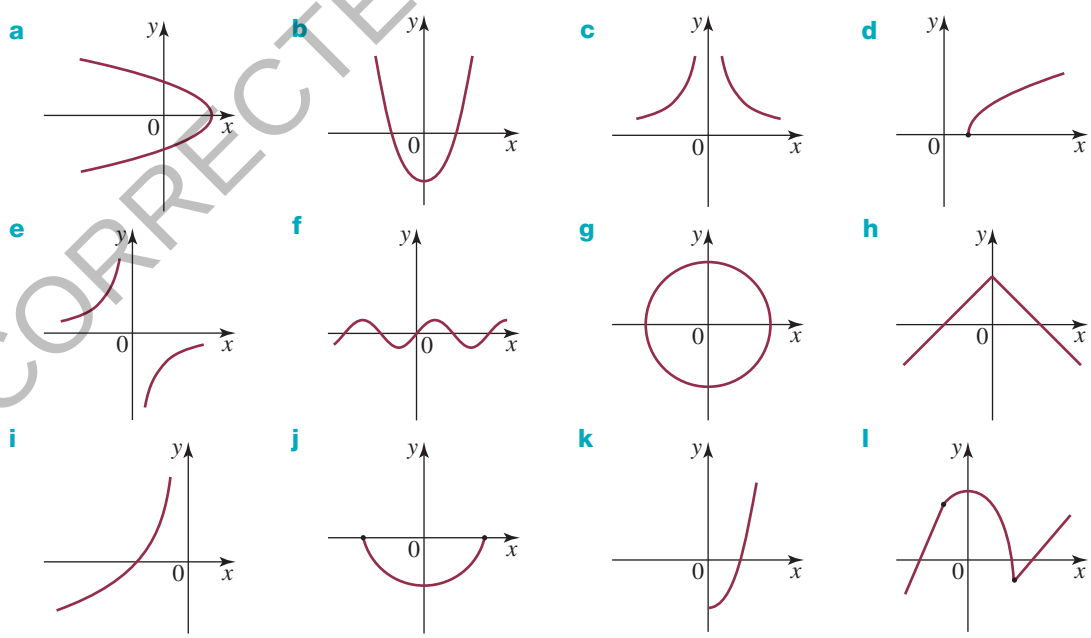
| ■ PRACTISE | ■ CONSOLIDATE | ■ MASTER |
|---|---|---|
| Questions: 1–9, 11, 12 | Questions: 1–3, 6, 8, 10, 11, 12 | Questions: 1–13 |

### FLUENCY

eBook*plus*

**Digital docs**
SkillSHEET Finding the gradient and $y$-intercept doc-5378
SkillSHEET Sketching straight lines doc-5379
SkillSHEET Sketching parabolas doc-5380
SkillSHEET Completing the square doc-5381
SkillSHEET Identifying equations of straight lines and parabolas doc-5382
SkillSHEET Finding points of intersection doc-5383
SkillSHEET Substitution into index expressions doc-5384

1 WE1 What type of relation does each graph represent?




2 WE2 a Use the vertical line test to determine which of the relations in question 1 are functions.

b Which of these functions have inverses that are also functions?

**3** WE3 **a** If $f(x) = 3x + 1$, find:

**i** $f(0)$ **ii** $f(2)$
**iii** $f(-2)$ **iv** $f(5)$.

**b** If $g(x) = \sqrt{x + 4}$, find:

**i** $g(0)$ **ii** $g(-3)$
**iii** $g(5)$ **iv** $g(-4)$.

**c** If $g(x) = 4 - \frac{1}{x}$, find:

**i** $g(1)$ **ii** $g(\frac{1}{2})$
**iii** $g(-\frac{1}{2})$ **iv** $g(-\frac{1}{5})$.

**d** If $f(x) = (x + 3)^2$, find:

**i** $f(0)$ **ii** $f(-2)$
**iii** $f(1)$ **iv** $f(a)$.

**e** If $h(x) = \frac{24}{x}$, find:

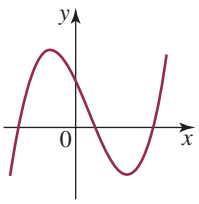
**i** $h(2)$ **ii** $h(4)$
**iii** $h(-6)$ **iv** $h(12)$.

## UNDERSTANDING

**4** MC *Note:* There may be more than one correct answer.

Which of the following relations is a function?

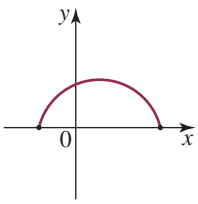
**A**




**B** $x^2 + y^2 = 9$

**C** $y = 8x - 3$

**D**




**5** Which of the following relations are functions?

**a** $y = 2x + 1$ **b** $y = x^2 + 2$
**c** $y = 2^x$ **d** $x^2 + y^2 = 25$
**e** $x^2 + 4x + y^2 + 6y = 14$ **f** $y = -4x$

**6** Given that $f(x) = \frac{10}{x} - x$, find:

**a** $f(2)$ **b** $f(-5)$
**c** $f(2x)$ **d** $f(x^2)$
**e** $f(x + 3)$ **f** $f(x - 1)$.

**7** Find the value (or values) of $x$ for which each function has the value given.

**a** $f(x) = 3x - 4$, $f(x) = 5$ **b** $g(x) = x^2 - 2$, $g(x) = 7$
**c** $f(x) = \frac{1}{x}$, $f(x) = 3$ **d** $h(x) = x^2 - 5x + 6$, $h(x) = 0$
**e** $g(x) = x^2 + 3x$, $g(x) = 4$ **f** $f(x) = \sqrt{8 - x}$, $f(x) = 3$

## REASONING

**8** **WE4** Describe what happens to:

**a** $f(x) = x^2 + 3$ as $x \to \infty$

**b** $f(x) = 2^x$ as $x \to -\infty$

**c** $f(x) = \frac{1}{x}$ as $x \to \infty$

**d** $f(x) = x^3$ as $x \to -\infty$

**e** $f(x) = -5^x$ as $x \to -\infty$.

**9** **WE5** Find any points of intersection between the following curves.

**a** $f(x) = 2x - 4$ and $g(x) = x^2 - 4$

**b** $f(x) = -3x + 1$ and $g(x) = -\frac{2}{x}$

**c** $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$

**d** $f(x) = \frac{3}{4}x - 6\frac{1}{4}$ and $x^2 + y^2 = 25$

**10** Find the equation of the inverse function of each of the following, placing restrictions on the original $x$ values as required.

**a** $f(x) = 2x - 1$

**b** $f(x) = x^2 - 3$

**c** $f(x) = (x - 2)^2 + 4$

**11** **WE6** **a** **i** Show that the function $f(x) = x(x + 2)$ will not have an inverse function.

**ii** Suggest a restriction that would result in an inverse function.

**b** **i** Show that the function $f(x) = -x^2 + 4, x \leq 0$ will have an inverse function.

**ii** Determine the equation of the inverse function.

## PROBLEM SOLVING

**12** Find the value(s) of $x$ for which:

**a** $f(x) = x^2 + 7$ and $f(x) = 16$

**b** $g(x) = \frac{1}{x - 2}$ and $g(x) = 3$

**c** $h(x) = \sqrt{8 + x}$ and $h(x) = 6$.

**13** Consider the function defined by the rule

$f: R \to R, f(x) = (x - 1)^2 + 2$.

**a** State the range of the function.

**b** Determine the type of mapping for the function.

**c** Sketch the graph of the function stating where it cuts the $y$-axis and its turning point.

**d** Select a domain where $x$ is positive such that $f$ is a one-to-one function.

**e** Determine the inverse function. Give the domain and range of the inverse function.

**f** Sketch the graph of the inverse function on the same set of axes used for part **c**.

**g** Find where $f$ and the function $g(x) = x + 3$ intersect each other.

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**CHALLENGE 18.1**

**Famous inverses**

Draw and compare the graphs of the inverse functions $y = a^x$ and $y = \log_a x$, choosing various values for $a$. Explain why these graphs are inverses.

## 18.3 Exponential functions

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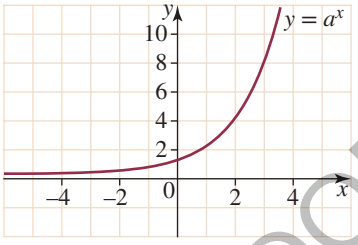
**eLesson**
Exponential growth
eles-0176

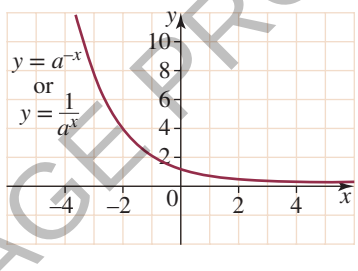
- **Exponential functions** can be used to model many real situations involving natural growth and decay.
- **Exponential growth** is when a quantity grows by a constant percentage in each fixed period of time. Examples of exponential growth include growth of investment at a certain rate of compound interest and growth in the number of cells in a bacterial colony.
- **Exponential decay** is when a quantity decreases by a constant percentage in each fixed period of time. Examples of exponential decay include yearly loss of value of an item (called depreciation) and radioactive decay.
- Both exponential growth and decay can be modelled by exponential functions of the type $y = ka^x (y = k \times a^x)$. The difference is in the value of the base $a$. When $a > 1$, there is exponential growth and when $0 < a < 1$ there is exponential decay.

  The value of $k$ corresponds to the initial quantity that is growing or decaying.

Exponential growth




Exponential decay




**WORKED EXAMPLE 7** TI CASIO

**The number of bacteria, $N$, in a Petri dish after $x$ hours is given by the equation $N = 50 \times 2^x$.**

**a Determine the initial number of bacteria in the Petri dish.**

**b Determine the number of bacteria in the Petri dish after 3 hours.**

**c Draw the graph of the function of $N$ against $x$.**

**d Use the graph to estimate the length of time it will take for the initial number of bacteria to treble.**

| THINK | WRITE/DRAW |
|---|---|
| **a** 1 Write the equation. | **a** $N = 50 \times 2^x$ |
| 2 Substitute $x = 0$ into the given formula and evaluate. (Notice that this is the value of $k$ for equations of the form $y = k \times a^x$.) | When $x = 0$, $N = 50 \times 2^0$<br>$= 50 \times 1$<br>$= 50$ |
| 3 Write the answer in a sentence. | The initial number of bacteria in the Petri dish is 50. |

**b** **1** Substitute $x = 3$ into the formula and evaluate.

**b** When $x = 3$, $N = 50 \times 2^3$
$= 50 \times 8$
$= 400$

**2** Write the answer in a sentence.

After 3 hours there are 400 bacteria in the Petri dish.

**c** **1** Draw a set of axes, labelling the horizontal axis as $x$ and the vertical axis as $N$.

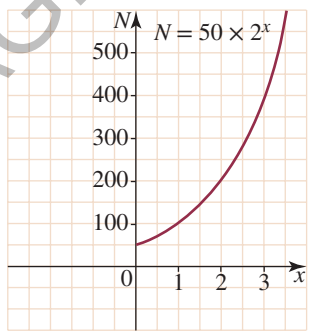
**c**

**2** Plot the points generated by the answers to parts **a** and **b**.

**3** Calculate the value of $N$ when $x = 1$ and $x = 2$ and plot the points generated.

At $x = 1$, $N = 50 \times 2^1$
$= 50 \times 2$
$= 100$
At $x = 2$, $N = 50 \times 2^2$
$= 50 \times 4$
$= 200$

**4** Join the points plotted with a smooth curve.

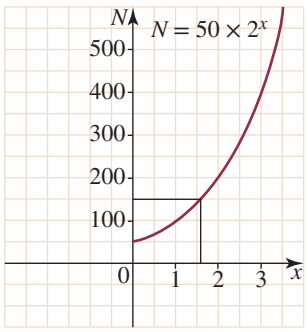



**5** Label the graph.

**d** **1** Determine the number of bacteria required.

**d** Number of bacteria $= 3 \times 50$
$= 150$

**2** Draw a horizontal line from $N = 150$ to the curve and from this point draw a vertical line to the $x$-axis.




**3** The point on the $x$-axis will be the estimate of the time taken for the number of bacteria to treble.

**4** Write the answer in a sentence.

The time taken will be approximately 1.6 hours.

**WORKED EXAMPLE 8**

**A new computer costs \$3000. It is estimated that each year it will be losing 12% of the previous year's value.**

**a Determine the value, \$$V$, of the computer after the first year.**

**b Determine the value of the computer after the second year.**

**c Determine the equation that relates the value of the computer to the number of years, $n$, it has been used.**

**d Use your equation to determine the value of the computer in 10 years' time.**

| THINK | WRITE |
|---|---|
| **a 1** State the original value of the computer. | **a** $V_0 = 3000$ |
| **2** Since 12% of the value is being lost each year, the value of the computer will be 88% or (100 – 12)% of the previous year's value. Therefore, the value after the first year ($V_1$) is 88% of the original cost. | $V_1 = 88\%$ of $3000$<br>$= 0.88 \times 3000$<br>$= 2640$ |
| **3** Write the answer in a sentence. | The value of the computer after 1 year is \$2640. |
| **b 1** The value of the computer after the second year, $V_2$, is 88% of the value after the first year. | **b** $V_2 = 88\%$ of $2640$<br>$= 0.88 \times 2640$<br>$= 2323.2$ |
| **2** Write the answer in a sentence. | The value of the computer after the second year is \$2323.20. |
| **c 1** The original value is $V_0$. | **c** $V_0 = 3000$ |
| **2** The value after the first year, $V_1$, is obtained by multiplying the original value by 0.88. | $V_1 = 3000 \times 0.88$ |
| **3** The value after the second year, $V_2$, is obtained by multiplying $V_1$ by 0.88, or by multiplying the original value, $V_0$, by $(0.88)^2$. | $V_2 = (3000 \times 0.88) \times 0.88$<br>$= 3000 \times (0.88)^2$ |
| **4** The value after the third year, $V_3$, is obtained by multiplying $V_2$ by 0.88, or $V_0$ by $(0.88)^3$. | $V_3 = 3000 \times (0.88)^2 \times 0.88$<br>$= 3000 \times (0.88)^3$ |
| **5** By observing the pattern we can generalise as follows: the value after the $n$th year, $V_n$, can be obtained by multiplying the original value, $V_0$, by 0.88 $n$ times; that is, by $(0.88)^n$. | $V_n = 3000 \times (0.88)^n$ |
| **d 1** Substitute $n = 10$ into the equation obtained in part **c** to find the value of the computer after 10 years. | **d** When $n = 10$,<br>$V_{10} = 3000 \times (0.88)^{10}$<br>$= 835.50$ |
| **2** Write the answer in a sentence. | The value of the computer after 10 years is \$835.50. |



- Sometimes the relationship between the two variables closely resembles an exponential pattern, but cannot be described exactly by an exponential function. In such cases, part of the data are used to model the relationship with exponential growth or the decay function.

**WORKED EXAMPLE 9**

**The population of a certain city is shown in the table below.**

| Year | 1985 | 1990 | 1995 | 2000 | 2005 | 2010 |
|---|---|---|---|---|---|---|
| Population (× 1000) | 128 | 170 | 232 | 316 | 412 | 549 |

**Assume that the relationship between the population, $P$, and the year, $x$, can be modelled by the function $P = ka^x$, where $x$ is the number of years after 1985. The value of $P$ must be multiplied by 1000 in order to find the actual population.**

**a State the value of $k$, which is the population, in thousands, at the start of the period.**

**b Use a middle point in the data set to find the value of $a$, correct to 2 decimal places. Hence, write the formula, connecting the population, $P$, with the number of years, $x$, since 1985.**

**c For the years given, find the size of the population using the formula obtained in part b. Compare it with the actual size of the population in those years.**

**d Predict the population of the city in the years 2015 and 2020.**

**THINK**

a From the given table, state the value of $k$ that corresponds to the population of the city in the year 1985.

b 1 Write the given formula for the population of the city.

2 Replace the value of $k$ with the value found in a.

3 Using a middle point of the data, replace $x$ with the number of years since 1985 and $P$ with the corresponding value.

4 Solve the equation for $a$.

5 Round the answer to 2 decimal places.

6 Rewrite the formula with this value of $a$.

c 1 Draw a table of values and enter the given years, the number of years since 1985, $x$, and the population for each year, $P$. Round values of $P$ to the nearest whole number.

2 Comment on the closeness of the fit.

**WRITE**

a $k = 128$

b $P = ka^x$

$P = 128 \times a^x$

Middle point is (1995, 232).
When $x = 10$, $P = 232$, so $232 = 128 \times a^{10}$.

$$a^{10} = \frac{232}{128}$$
$$a^{10} = 1.8125$$
$$a = \sqrt[10]{1.8125}$$
$$a = 1.0613\ldots$$

$a \approx 1.06$

So $P = 128 \times (1.06)^x$.

c

| Year | 1985 | 1990 | 1995 | 2000 | 2005 | 2010 |
|---|---|---|---|---|---|---|
| $x$ | 0 | 5 | 10 | 15 | 20 | 25 |
| $P$ | 128 | 171 | 229 | 307 | 411 | 549 |

The values for the population obtained using the formula closely resemble the actual data.

**d** **1** Find the value of $x$, the number of years after 1985.

**2** Substitute this value of $x$ into the formula and evaluate.

**3** Round to the nearest whole number.

**4** Answer the question in a sentence.

**5** Repeat for the year 2020.

**d** For the year 2015, $x = 30$.

$P = 128 \times (1.06)^{30}$
$= 735.166\,87\ldots$

$P \approx 735$

The predicted population for 2015 is 735 000.

For the year 2020, $x = 35$.
$P = 128 \times (1.06)^{35}$
$= 983.819\ldots$
$P \approx 984$
The predicted population for 2020 is 984 000.

## Exercise 18.3 Exponential functions

assesson

### INDIVIDUAL PATHWAYS

| PRACTISE | CONSOLIDATE | MASTER |
|---|---|---|
| Questions: 1, 3, 5, 6, 8–10, 13, 15, 16 | Questions: 1, 2, 4, 6, 7, 9, 11, 12, 14–16 | Questions: 1, 4–6, 9–17 |

**REFLECTION**
What are the main differences between a graph modelling exponential growth compared with one showing decay?

### FLUENCY

**1** WE7 The number of micro-organisms, $N$, in a culture dish after $x$ hours is given by the equation $N = 2000 \times 3^x$.

**a** Determine the initial number of micro-organisms in the dish.

**b** Determine the number of micro-organisms in a dish after 5 hours.

**c** Draw the graph of $N$ against $x$.

**d** Use the graph to estimate the number of hours needed for the initial number of micro-organisms to quadruple.

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SkillSHEET
Decreasing a quantity by a percentage
doc-5387

**2** The value of an investment (in dollars) after $n$ years is given by $A = 5000 \times (1.075)^n$.

**a** Determine the size of the initial investment.

**b** Determine the value of the investment (to the nearest dollar) after 6 years.

**c** Draw the graph of $A$ against $n$.

**d** Use the graph to estimate the number of years needed for the initial investment to double.

**3** MC **a** The function $P = 300 \times (0.89)^n$ represents an:

**A** exponential growth with the initial amount of 300

**B** exponential growth with the initial amount of 0.89

**C** exponential decay with the initial amount of 300

**D** exponential decay with the initial amount of 0.89

**E** exponential decay with the initial amount of $300 \times 0.89$

**b** The relationship between two variables, $A$ and $t$, is described by the function $A = 45 \times (1.095)^t$, where $t$ is the time, in months, and $A$ is the amount, in dollars. This function indicates:

**A** a monthly growth of \$45

**B** a monthly growth of 9.5 cents

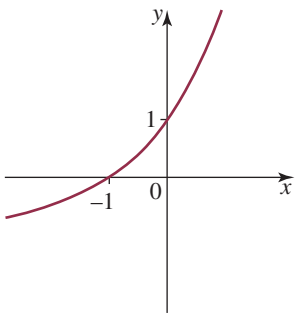
**C** a monthly growth of 1.095%

**D** a monthly growth of 9.5%

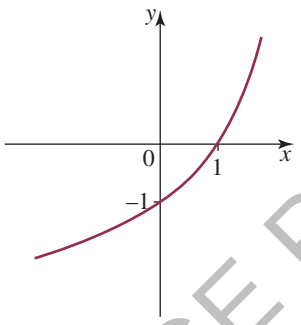
**E** a yearly growth of 9.5%

**4** **MC** The graph of $y = 2^{x+1} - 1$ is best represented by:

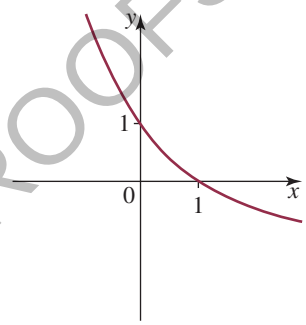
**A**



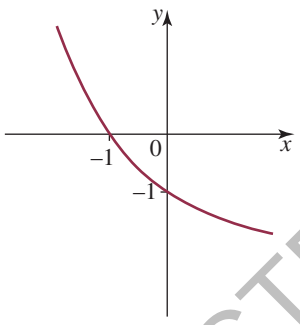

**B**



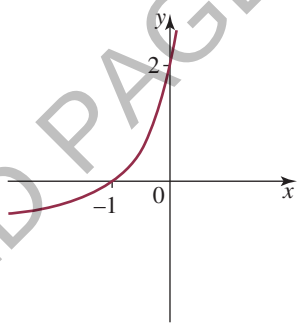

**C**




**D**




**E**




**5** **MC** The graph of $y = 3^{x-2} + 2$ has an asymptote and $y$-intercept respectively at:

**A** $y = 0,\ 2\frac{1}{9}$ **B** $y = 2,\ 2\frac{1}{9}$ **C** $y = 2,\ 2$

**D** $y = 2,\ 1\frac{8}{9}$ **E** $y = 0,\ 2$

## UNDERSTANDING

**6** **WE8** A new washing machine costs \$950. It is estimated that each year it will be losing 7% of the previous year's value.

**a** Calculate the value of the machine after the first year.

**b** Calculate the value of the machine after the second year.

**c** Determine the equation that relates the value of the machine, $\$V$, to the number of years, $n$, that it has been used.

**d** Use your equation to find the value of the machine in 12 years' time.

**7** A certain radioactive element decays in such a way that every 50 years the amount present decreases by 15%. In 1900, 120 mg of the element was present.

**a** Calculate the amount present in 1950.

**b** Calculate the amount present in the year 2000.

**c** Determine the rule that connects the amount of the element present, $A$, with the number of 50-year intervals, $t$, since 1900.

**d** Calculate the amount present in the year 2010. Round your answer to 3 decimal places.

**e** Graph the function of $A$ against $t$.

**f** Use the graph to estimate the half-life of this element (that is, the number of years needed for half the initial amount to decay).

**8** When a shirt made of a certain fabric is washed, it loses 2% of its colour.

**a** Determine the percentage of colour that remains after:

**i** two washes

**ii** five washes.

**b** Write a function for the percentage of colour, $C$, remaining after $w$ washings.

**c** Draw the graph of $C$ against $w$.

**d** Use the graph to estimate the number of washes after which there is only 85% of the original colour left.

**9** WE9 The population of a certain country is shown in the table below.

| Year | Population (in millions) |
|---|---|
| 1990 | 118 |
| 1995 | 130 |
| 2000 | 144 |
| 2005 | 160 |
| 2010 | 178 |

Assume that the relationship between the population, $P$, and the year, $n$, can be modelled by the formula $P = ka^n$, where $n$ is the number of years since 1990.

**a** State the value of $k$.

**b** Use the middle point of the data set to find the value of $a$ rounded to 2 decimal places. Hence, write the formula that connects the two variables, $P$ and $n$.

**c** For the years given in the table, find the size of the population, using your formula. Compare the numbers obtained with the actual size of the population.

**d** Predict the population of the country in the year 2035.

**10** The temperature in a room (in degrees Celsius), recorded at 10-minute intervals after the air conditioner was turned on, is shown in the table below.

| Time (min) | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Temperature (°C) | 32 | 26 | 21 | 18 | 17 |

Assume that the relationship between the temperature, $T$, and the time, $t$, can be modelled by the formula $T = ca^t$, where $t$ is the time, in minutes, since the air conditioner was turned on.

**a** State the value of $c$.

**b** Use the middle point in the data set to find the value of $a$ to 2 decimal places.

**c** Write the rule connecting $T$ and $t$.

**d** Using the rule, find the temperature in the room 10, 20, 30 and 40 minutes after the air conditioner was turned on and compare your numbers with the recorded temperature. Comment on your findings. (Give answers correct to 1 decimal place.)

**11** The population of a species of dogs ($D$) increases exponentially and is described by the equation $D = 60(1 - 0.6^t) + 3$, where $t$ represents the time in years.

**a** Calculate the initial number of dogs.

**b** Calculate the number of dogs after 1 year.

**c** Determine the time taken for the population to reach 50 dogs.

**12** Carbon-14 decomposes in such a way that the amount present can be calculated using the equation, $Q = Q_0(1 - 0.038)^t$, where $Q$ is measured in milligrams and $t$ in centuries.

**a** If there is 40 mg present initially, how much is present in:

**i** 10 years' time

**ii** 2000 years' time?

**b** How many years will it take for there to be less than 10 mg?

## REASONING

**13** Fiona is investing \$20 000 in a fixed term deposit earning 6% p.a. interest. When Fiona has \$30 000 she intends to put a deposit on a house.

**a** Determine an exponential function that will model the growth of Fiona's investment.

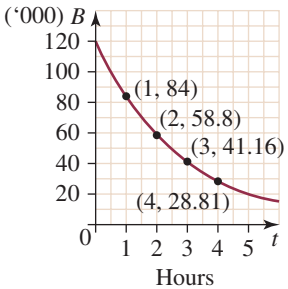
**b** Graph this function.

**c** Determine the length of time (correct to the nearest year) that it will take for Fiona's investment to grow to \$30 000.

**d** Suppose Fiona had been able to invest at 8% p.a. How much quicker would Fiona's investment have grown to the \$30 000 she needs?

**e** Alvin has \$15 000 to invest. Find the interest rate at which Alvin must invest his money, if his investment is to grow to \$30 000 in less than 8 years.

**14** A Petri dish containing a bacteria colony was exposed to an antiseptic. The number of bacteria within the colony, $B$, over time, $t$, in hours is shown in the graph at right.




**a** Using the graph, predict the number of bacteria in the Petri dish after 5 hours.

**b** Using the points from the graph, show that if $B$ can be modelled by the function $B$ (in thousands) $= ab^t$, then $a = 120$ and $b = 0.7$.

**c** After 8 hours, another type of antiseptic was added to the Petri dish. Within three hours, the number of bacteria in the Petri dish had decreased to 50. If the number of bacteria decreased at a constant rate, show that the total of number of bacteria that had decreased within two hours was approximately 6700.

**15** One hundred people were watching a fireworks display at a local park. As the fireworks were set off, more people started to arrive to see the show. The number of people, $P$, at time, $t$ minutes, after the start of the fireworks display, can be modelled by the function, $P = ab^t$.

**a** If after 5 minutes there were approximately 249 people, show that the number of people arriving at the park to watch the fireworks increased by 20% each minute.

**b** The fireworks display lasted for 40 minutes. After 40 minutes, people started to leave the park. The number of people leaving the park could be modelled by an exponential function. 15 minutes after the fireworks ceased there were only 700 people in the park.

Derive an exponential function that can determine the number of people, $N$, remaining in the park after the fireworks had finished at any time, $m$, in minutes.

## PROBLEM SOLVING

**16** A hot plate used as a camping stove is cooling down. The formula that describes this cooling pattern is $T = 500 \times 0.5^t$ where $T$ is the temperature in degrees Celsius and $t$ is the time in hours.

**a** What is the initial temperature of the stove?

**b** What is the temperature of the stove after 2 hours?

**c** Decide when the stove will be cool enough to touch and give reasons.

**17** The temperature in a greenhouse is monitored when the door is left open. The following measurements are taken.

| Time (min) | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Temperature (°C) | 45 | 35 | 27 | 21 | 16 |

**a** State the initial temperature of the greenhouse.

**b** Determine an exponential equation to fit the collected data.

**c** What will the temperature be after 30 minutes?

It is discovered that one of the temperature readings is incorrect.

**d** Recalculate all the temperatures using the exponential rule found in part **a**.

**e** If the original incorrect temperature was omitted from the data, does this change the rule?

**f** Will the temperature ever reach 0 °C? Explain.

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## 18.4 Cubic functions

- **Cubic functions** are polynomials where the highest power of $x$ is 3. These include functions such as $y = x^3$ or $y = (x + 1)(x - 2)(x + 3)$.

**WORKED EXAMPLE 10**

**Plot the graph of $y = x^3 - 1$ by completing a table of values.**

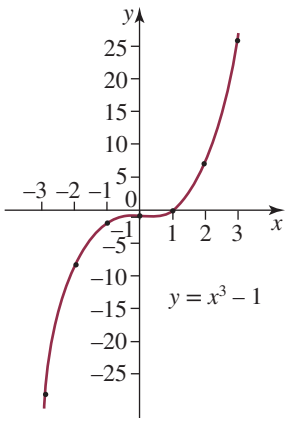
**THINK**

1. Prepare a table of values, taking $x$-values from −3 to 3. Fill in the table by substituting each $x$-value into the given equation to find the corresponding $y$-value.

2. Draw a set of axes and plot the points from the table. Join them with a smooth curve.

**WRITE/DRAW**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | −28 | −9 | −2 | −1 | 0 | 7 | 26 |



**WORKED EXAMPLE 11**

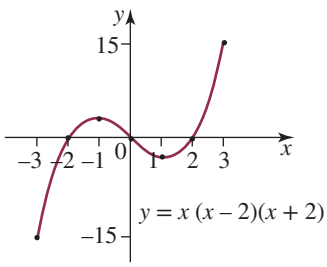
**Plot the curve of $y = x(x - 2)(x + 2)$ by completing a table of values.**

**THINK**

1. Prepare a table of values, taking $x$-values from $-3$ to 3. Fill in the table by substituting each $x$-value into the given equation.
2. Draw a set of axes and plot the points from the table. Join them with a smooth curve.

**WRITE/DRAW**

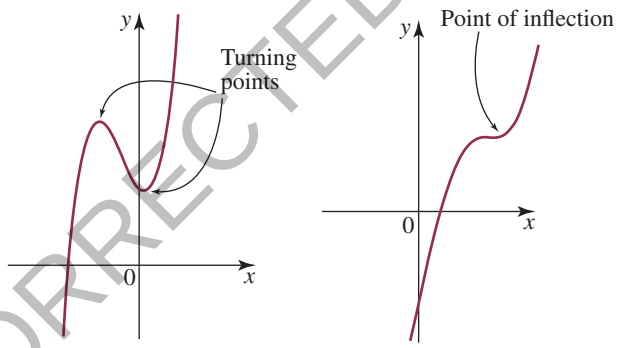
| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $-15$ | 0 | 3 | 0 | $-3$ | 0 | 15 |




- A good sketch of a cubic function shows:
  1. $x$- and $y$-intercepts
  2. the behaviour of the function at extreme values of $x$, that is, as $x$ approaches infinity ($x \to +\infty$) and as $x$ approaches negative infinity ($x \to -\infty$)
  3. the general location of turning points.

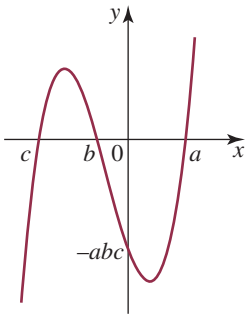
  Note that for cubic functions, 'humps' are not symmetrical as they are for parabolas, but are skewed to one side.

  The graphs below show the two main types of cubic graph.




Consider the general factorised cubic $y = (x - a)(x - b)(x - c)$.
The $x$-intercepts occur when $y = 0$, that is, when $x = a$ or $x = b$ or $x = c$.
The $y$-intercept occurs when $x = 0$, that is, the $y$-intercept is

$$\begin{aligned} y &= (0 - a)(0 - b)(0 - c) \\ &= -abc \end{aligned}$$



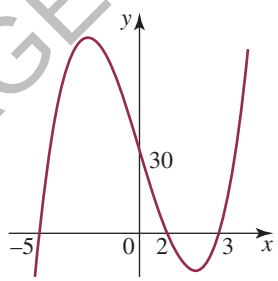
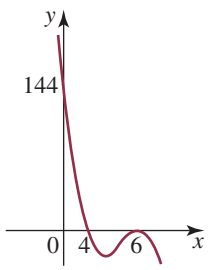
**WORKED EXAMPLE 12** TI CASIO

**Sketch the following, showing all intercepts.**

**a** $y = (x - 2)(x - 3)(x + 5)$

**b** $y = (x - 6)^2(4 - x)$

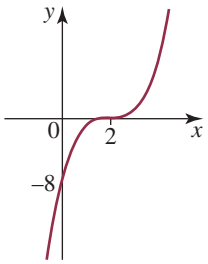
**c** $y = (x - 2)^3$

| THINK | WRITE/DRAW |
|---|---|
| **a** 1 Write the equation. | **a** $y = (x - 2)(x - 3)(x + 5)$ |
| 2 The $y$-intercept occurs where $x = 0$. Substitute $x = 0$ into the equation. | $y$-intercept: if $x = 0$,<br>$y = (-2)(-3)(5)$<br>$= 30$<br>Point: $(0, 30)$ |
| 3 Solve $y = 0$ to find the $x$-intercepts. | $x$-intercepts: if $y = 0$,<br>$x - 2 = 0$, $x - 3 = 0$ or $x + 5 = 0$<br>$x = 2$, $x = 3$ or $x = -5$<br>Points: $(2, 0)$, $(3, 0)$, $(-5, 0)$ |
| 4 Combine the above steps to sketch. |  |
| **b** 1 Write the equation. | **b** $y = (x - 6)^2(4 - x)$ |
| 2 Substitute $x = 0$ to find the $y$-intercept. | $y$-intercept: if $x = 0$,<br>$y = (-6)^2(4)$<br>$= 144$<br>Point: $(0, 144)$ |
| 3 Solve $y = 0$ to find the $x$-intercepts. | $x$-intercepts: if $y = 0$,<br>$x - 6 = 0$ or $4 - x = 0$<br>$x = 6$ or $x = 4$<br>Points: $(6, 0)$, $(4, 0)$ |
| 4 Combine all information and sketch the graph.<br>*Note:* The curve just touches the $x$-axis at $x = 6$. This occurs with a double factor such as $(x - 6)^2$. |  |
| **c** 1 Write the equation. | **c** $y = (x - 2)^3$ |
| 2 Substitute $x = 0$ to find the $y$-intercept. | $y$-intercept: if $x = 0$,<br>$y = (-2)^3$<br>$= -8$ |

**3** Solve $y = 0$ to find the $x$-intercepts.

$x$-intercept: if $y = 0$,
$x - 2 = 0$
$x = 2$

**4** Combine all information and sketch the graph.
*Note:* The point of inflection is at $x = 2$. This occurs with a triple factor such as $(x - 2)^3$.




## Exercise 18.4 Cubic functions

assess on

### INDIVIDUAL PATHWAYS

| PRACTISE | CONSOLIDATE | MASTER |
|---|---|---|
| Questions: 1a–f, 2a–f, 3–8 | Questions: 1e–h, 2e–h, 3–8, 10 | Questions: 1i–l, 2i–l, 3–11 |

**REFLECTION**
Is it possible to get symmetrical 'humps' for the graphs of cubic functions?

### FLUENCY

**1** WE10, 11, 12 Sketch the following, showing all intercepts.

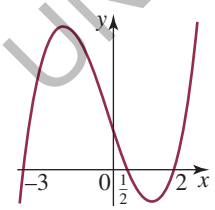
**a** $y = (x - 1)(x - 2)(x - 3)$
**b** $y = (x - 3)(x - 5)(x + 2)$
**c** $y = (x + 6)(x + 1)(x - 7)$
**d** $y = (x + 4)(x + 9)(x + 3)$
**e** $y = (x + 8)(x - 11)(x + 1)$
**f** $y = (2x - 6)(x - 2)(x + 1)$
**g** $y = (2x - 5)(x + 4)(x - 3)$
**h** $y = (3x + 7)(x - 5)(x + 6)$
**i** $y = (4x - 3)(2x + 1)(x - 4)$
**j** $y = (2x + 1)(2x - 1)(x + 2)$
**k** $y = (x - 3)^2(x - 6)$
**l** $y = (x + 2)(x + 5)^2$

**2** Sketch the following (a mixture of positive and negative cubics).

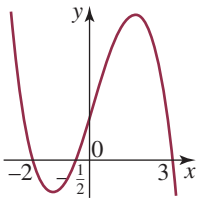
**a** $y = (2 - x)(x + 5)(x + 3)$
**b** $y = (1 - x)(x + 7)(x - 2)$
**c** $y = (x + 8)(x - 8)(2x + 3)$
**d** $y = (x - 2)(2 - x)(x + 6)$
**e** $y = x(x + 1)(x - 2)$
**f** $y = -2(x + 3)(x - 1)(x + 2)$
**g** $y = 3(x + 1)(x + 10)(x + 5)$
**h** $y = -3x(x - 4)^2$
**i** $y = 4x^2(x + 8)$
**j** $y = (5 - 3x)(x - 1)(2x + 9)$
**k** $y = (6x - 1)^2(x + 7)$
**l** $y = -2x^2(7x + 3)$

**3** MC Which of the following is a reasonable sketch of $y = (x + 2)(x - 3)(2x + 1)$?

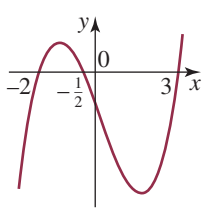
**A**



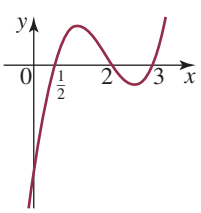

**B**



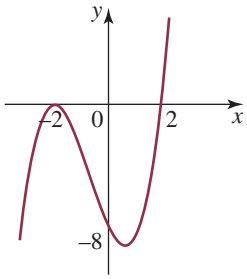

**C**




**D**

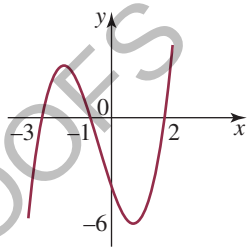


**4** **MC** The graph shown could be that of:



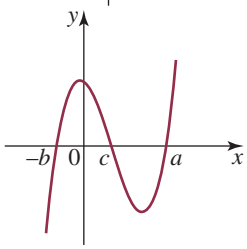

**A** $y = x^2(x + 2)$
**B** $y = (x + 2)^3$
**C** $y = (x - 2)(x + 2)^2$
**D** $y = (x - 2)^2(x + 2)$

**5** **MC** The graph at right has the equation:




**A** $y = (x + 1)(x + 2)(x + 3)$
**B** $y = (x + 1)(x - 2)(x + 3)$
**C** $y = (x - 1)(x + 2)(x + 3)$
**D** $y = (x - 1)(x + 2)(x - 3)$

**6** **MC** If $a$, $b$ and $c$ are positive numbers, the equation of the graph shown at right could be:




**A** $y = (x - a)(x - b)(x - c)$
**B** $y = (x + a)(x - b)(x + c)$
**C** $y = (x + a)(x + b)(x - c)$
**D** $y = (x - a)(x + b)(x - c)$

## UNDERSTANDING

**7** Sketch the graph of each of the following.

**a** $y = x(x - 1)^2$
**b** $y = -(x + 1)^2(x - 1)$
**c** $y = (2 - x)(x^2 - 9)$
**d** $y = -x(1 - x^2)$

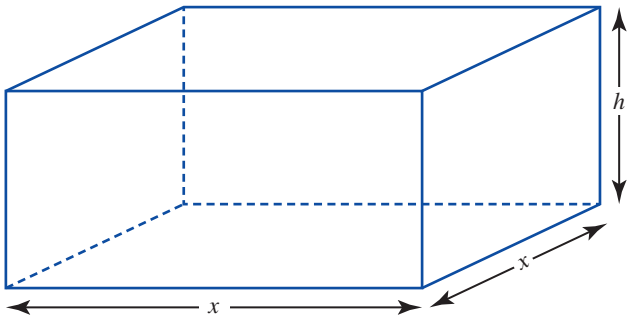
## REASONING

**8** The function $f(x) = x^3 + ax^2 + bx + 4$ has $x$-intercepts at $(1, 0)$ and $(-4, 0)$. Find the values of $a$ and $b$.

**9** The graphs of the functions $f(x) = x^3 + (a + b)x^2 + 3x - 4$ and $g(x) = (x - 3)^3 + 1$ touch. Express $a$ in terms of $b$.

## PROBLEM SOLVING

**10** A girl uses 140 cm of wire to make a frame of a cuboid with a square base as shown.




The base length of the cuboid is $x$ cm and the height is $h$ cm.

**a** Explain why the volume cm$^3$ is given by $V = 35x^2 - 2x^3$.

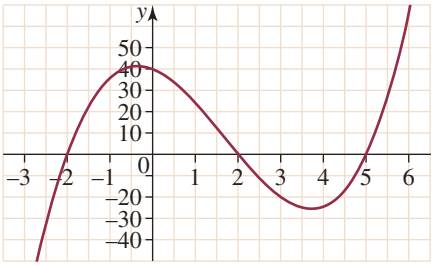
**b** What possible values can $x$ assume?

**c** Find the volume of the cuboid when the base area is 81 $cm^2$.

**d** Sketch the graph of $V$ versus $x$.

**e** Use technology to determine the coordinates of the maximum turning point. Explain what these coordinates mean.

**11** Find the rule for the cubic function shown.




## 18.5 Quartic functions

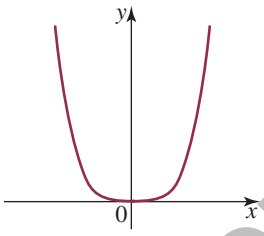
- **Quartic functions** are polynomials where the highest power of $x$ is 4. These include functions such as $y = x^4$ or $y = (x + 1)(x - 2)(x + 3)(x - 5)$.
- It is necessary, when sketching the graphs of quartic functions, to find all the intercepts on both the $x$- and $y$-axes.

### Basic shapes of quartic graphs

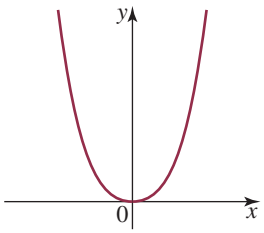
- The direction of a quartic graph is determined by the coefficient of the $x^4$ term. This is similar to the effect the coefficient of $x^2$ has on the shape of a parabola. Consider the coefficient of $x^4$ to be $a$.

**When $a$ is positive ($a > 0$)**

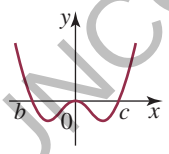
1. $y = ax^4$



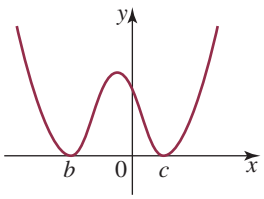

2. $y = ax^4 + cx^2, c \geq 0$



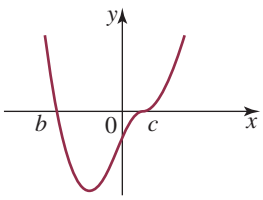

3. $y = ax^2(x - b)(x - c)$



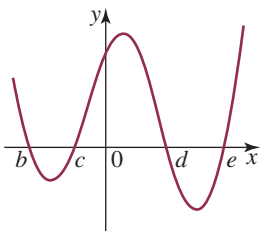

4. $y = a(x - b)^2(x - c)^2$



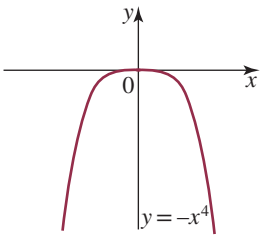

5. $y = a(x - b)(x - c)^3$




6. $y = a(x - b)(x - c)(x - d)(x - e)$

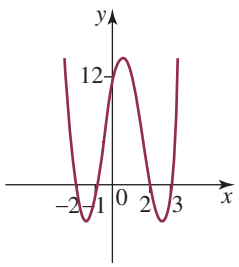


**When $a$ is negative ($a < 0$)**
If $a$ was negative in each of the previous graphs, they would be reflected in the $x$-axis.




**WORKED EXAMPLE 13** TI CASIO

**Sketch the graph of $y = x^4 - 2x^3 - 7x^2 + 8x + 12$, showing all intercepts.**

| THINK | WRITE/DRAW |
|---|---|
| 1 Find the $y$-intercept. | When $x = 0$, $y = 12$.<br>The $y$-intercept is 12. |
| 2 Let $P(x) = y$. | Let $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$. |
| 3 Find two linear factors of the quartic expressions, if possible, using the factor theorem. | $P(1) = (1)^4 - 2(1)^3 - 7(1)^2 + 8(1) + 12$<br>$= 12$<br>$\neq 0$<br>$P(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12$<br>$= 0$<br>$(x + 1)$ is a factor.<br>$P(2) = (2)^4 - 2(2)^3 - 7(2)^2 + 8(2) + 12$<br>$= 0$<br>$(x - 2)$ is a factor. |
| 4 Find the product of the two linear factors. | $(x + 1)(x - 2) = x^2 - x - 2$ |
| 5 Use long division to divide the quartic by the quadratic factor $x^2 - x - 2$. | $x^2 - x - 6$<br>$x^2 - x - 2 \overline{)x^4 - 2x^3 - 7x^2 + 8x + 12}$<br>$\underline{x^4 - x^3 - 2x^2}$<br>$-x^3 - 5x^2 + 8x$<br>$\underline{-x^3 + x^2 + 2x}$<br>$-6x^2 + 6x + 12$<br>$\underline{-6x^2 + 6x + 12}$<br>$0$ |
| 6 Express the quartic in factorised form. | $y = (x + 1)(x - 2)(x^2 - x - 6)$<br>$= (x + 1)(x - 2)(x - 3)(x + 2)$ |
| 7 To find the $x$-intercepts, solve $y = 0$. | If $0 = (x + 1)(x - 2)(x - 3)(x + 2)$<br>$x = -1, 2, 3, -2$. |
| 8 State the $x$-intercepts. | The $x$-intercepts are $-2, -1, 2, 3$. |
| 9 Sketch the graph of the quartic. |  |

# Exercise 18.5 Quartic functions

assess on

## INDIVIDUAL PATHWAYS

| PRACTISE | CONSOLIDATE | MASTER |
|---|---|---|
| Questions: 1a–d, 2a–d, 3–6, 9 | Questions: 1c–f, 2c–f, 3, 4, 6, 7, 9 | Questions: 1d–h, 2d–h, 3–10 |

**REFLECTION**
What are the basic differences between cubic and quartic functions?

## FLUENCY

**1** WE13 Sketch the graph of each of the following showing all intercepts. You may like to verify the shape of the graph using a graphics calculator or another form of digital technology.

**a** $y = (x - 2)(x + 3)(x - 4)(x + 1)$
**b** $y = (x^2 - 1)(x + 2)(x - 5)$
**c** $y = 2x^4 + 6x^3 - 16x^2 - 24x + 32$
**d** $y = x^4 + 4x^3 - 11x^2 - 30x$
**e** $y = x^4 + 4x^3 - 12x - 9$
**f** $y = x^4 - 4x^2 + 4$
**g** $y = 30x - 37x^2 + 15x^3 - 2x^4$
**h** $y = 6x^4 + 11x^3 - 37x^2 - 36x + 36$

**2** Sketch each of the following.

**a** $y = x^2(x - 1)^2$
**b** $y = -(x + 1)^2(x - 4)^2$
**c** $y = -x(x - 3)^3$
**d** $y = (2 - x)(x - 1)(x + 1)(x - 4)$
**e** $y = (x - a)(b - x)(x + c)(x + d), \quad a, b, c, d > 0$

## UNDERSTANDING

**3** MC A quartic touches the $x$-axis at $x = -3$ and $x = 2$. It crosses the $y$-axis at $y = -9$. A possible equation is:

**A** $y = \frac{1}{4}(x + 3)^2(x - 2)^2$
**B** $y = -\frac{1}{6}(x + 3)^3(x - 2)$
**C** $y = -\frac{3}{8}(x + 3)(x - 2)^3$
**D** $y = -\frac{1}{4}(x + 3)^2(x - 2)^2$

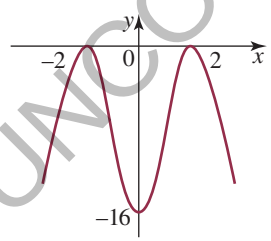
**4** MC Consider the function $f(x) = x^4 - 8x^2 + 16$.

**a** When factorised, $f(x)$ is equal to:

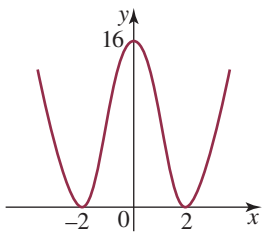
**A** $(x + 2)(x - 2)(x - 1)(x + 4)$
**B** $(x + 3)(x - 2)(x - 1)(x + 1)$
**C** $(x - 2)^3(x + 2)$
**D** $(x - 2)^2(x + 2)^2$

**b** The graph of $f(x)$ is best represented by:

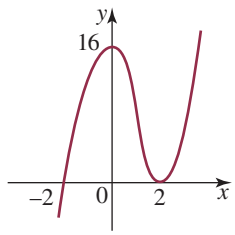
**A**



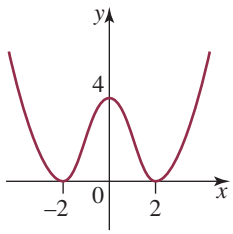

**B**




**C**




**D**



### REASONING

**5** Sketch the graph of each of the following functions.

**a** $y = x(x - 1)^3$

**b** $y = (2 - x)(x^2 - 4)(x + 3)$

**c** $y = x^4 - x^2$

**d** $y = 9x^4 - 30x^3 + 13x^2 + 20x + 4$

**e** $y = -(x - 2)^2(x + 1)^2$

**f** $y = x^4 - 6x^2 - 27$

**g** $y = (x + 2)^3(x - 3)$

**h** $y = 4x^2 - x^4$

Verify your answers using a graphics calculator.

**6** The function $f(x) = x^4 + ax^3 - 4x^2 + bx + 6$ has $x$-intercepts $(2, 0)$ and $(-3, 0)$. Find the values of $a$ and $b$.

**7** The functions $y = (a - 2b)x^4 - 3x - 2$ and $y = x^4 - x^3 + (a + 5b)x^2 - 5x + 7$ both have an $x$-intercept of 1. Find the value of $a$ and $b$.

**8** Patterns emerge when we graph polynomials with repeated factors, that is, polynomials of the form $P(x) = (x - a)^n$, $n > 1$. Describe what happens if:

**a** $n$ is even

**b** $n$ is odd.

### PROBLEM SOLVING

**9** A carnival ride has a piece of the track modelled by the rule

$$h = -\frac{1}{300}x(x - 12)^2(x - 20) + 15, \quad 0 \le x \le 20$$

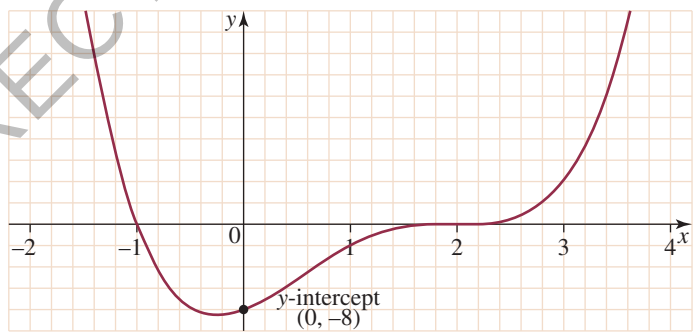
where $x$ metres is the horizontal displacement from the origin and $h$ metres is the vertical displacement of the track above the horizontal ground.

**a** How high above the ground level is the track at the origin?

**b** Use technology to sketch the function. Give the coordinates of any stationary points (that is, turning points or points of inflection).

**c** How high above ground level is the track when $x = 3$?

**10** Find the rule for the quartic function shown.

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**Digital doc**
WorkSHEET 18.3
doc-14624



### CHALLENGE 18.2

Two functions $f(x)$ and $g(x)$ are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$. The domain of $f(g(x))$ must be the domain of $g(x)$, and the domain of $g(f(x))$ must be the domain of $f(x)$. Using this information, show that $f(x) = x^2$, $x \ge 0$ and $g(x) = \sqrt{x}$, $x \ge 0$ are inverses of each other.

## 18.6 Transformations

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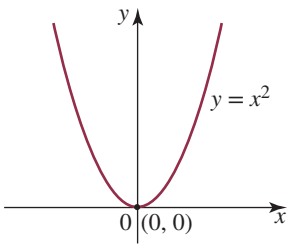
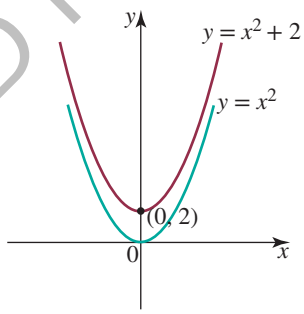
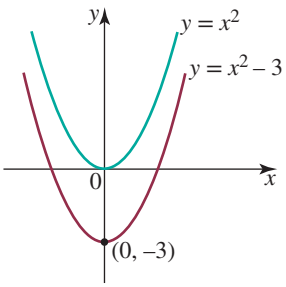
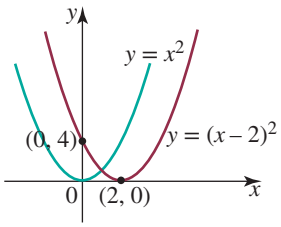
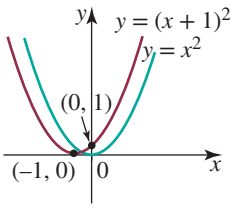
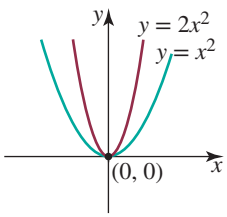
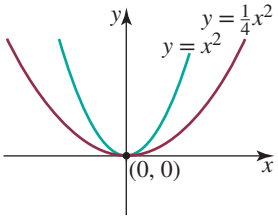
**Interactivity**
Polynomial transformations
int-2794

- Once the basic shape of the graph of a particular function or relation is known, it is not difficult to predict the shape of a related function, which is a transformation of the basic function or relation. **Transformations** of parabolas have been dealt with previously, but for the sake of comparison with other functions, they are included in this chapter. Other functions and relations considered are circles, hyperbolas, exponential functions, cubic and quartic functions. Below is a summary of transformations of functions discussed previously.

### Quadratic functions

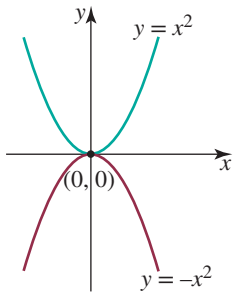
- The basic quadratic function is $y = x^2$. The shape of its graph is:




| Transformation | Description | Graph |
|---|---|---|
| Vertical translation | Adding or subtracting a constant to $y = x^2$ moves the curve up or down the $y$-axis. |     |
| Horizontal translation | If the graph of $y = x^2$ is translated $b$ units horizontally, the equation becomes $y = (x - b)^2$. |     |
| Dilation | If the graph of $y = x^2$ is dilated by a factor of $a$, the graph becomes narrower if $a > 1$ and wider if $0 < a < 1$. |     |

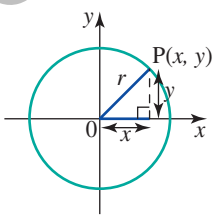
*(continued)*

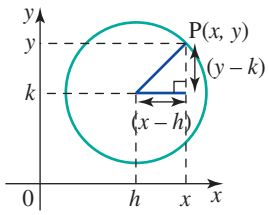
*(continued)*

| Transformation | Description | Graph |
|---|---|---|
| Reflection | If the $x^2$ term is positive, the graph is concave up, while if there is a negative sign in front of the $x^2$ term, the graph is concave down. |  |

## Circles

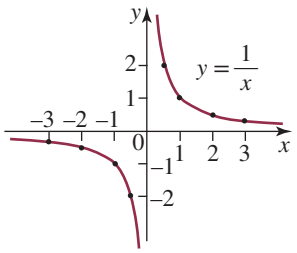
- The equation of a circle (relation) with centre (0, 0) and radius $r$ is $x^2 + y^2 = r^2$.

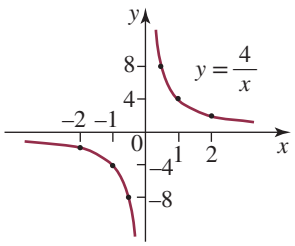


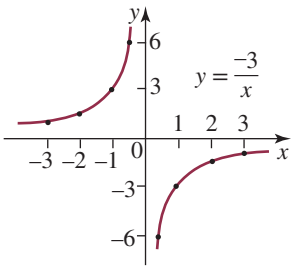

| Transformation | Description | Graph |
|---|---|---|
| Translation | If the circle is translated $h$ units to the right, parallel to the $x$-axis, and $k$ units upwards, parallel to the $y$-axis, the equation of the circle, centre $(h, k)$ becomes $(x - h)^2 + (y - k)^2 = r^2$. |  |

## Hyperbolas

- The hyperbola is a function of the form $xy = k$ or $y = \frac{k}{x}$.
- The graph of $y = \frac{1}{x}$ has the shape

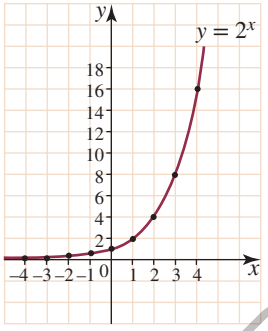


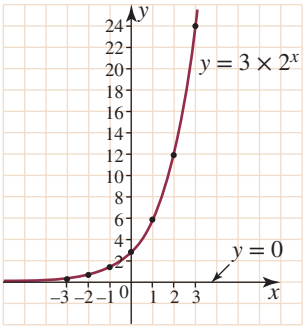
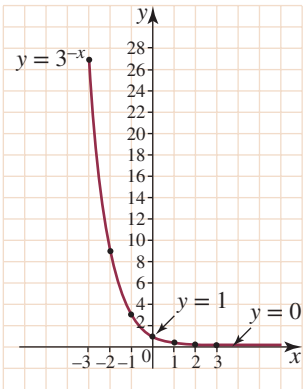

| Transformation | Description | Graph |
|---|---|---|
| Dilation | Graphs of the form $y = \frac{k}{x}$ are the same basic shape as $y = \frac{1}{x}$, with $y$-values dilated by a factor of $k$. |  |

| | | |
|---|---|---|
| Negative values of $k$ | Negative values of $k$ cause the graph to be reflected across the $y$-axis. |  |

## Exponential functions

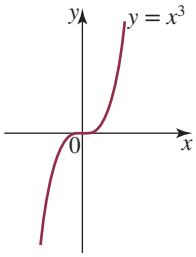
- These functions are of the form $y = a^x$, where $a \neq 1$. The basic shape has a $y$-intercept of 1.

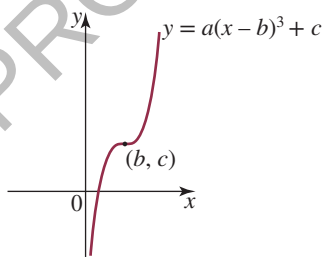
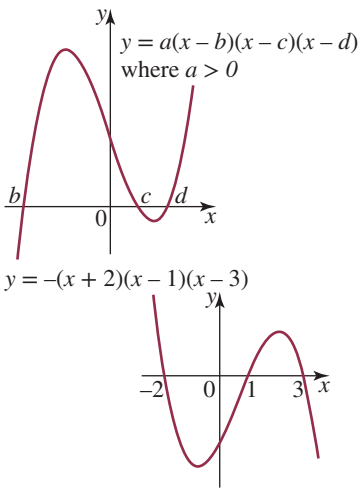



| Transformation | Description | Graph |
|---|---|---|
| Functions for the form $y = k \times ax$ | Multiplying by a factor of $k$ causes the $y$-intercept to move to the point $(0,\ k)$. |  |
| Functions with a negative exponent | This causes the graph to be reflected in the $y$-axis. |  |

## Cubic functions

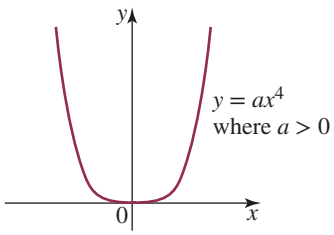
- The basic form of a cubic function is $y = x^3$. This can also be expressed in the form $y = a(x - b)^3 + c$, where $a = 1,\ b = 0$ and $c = 0$.

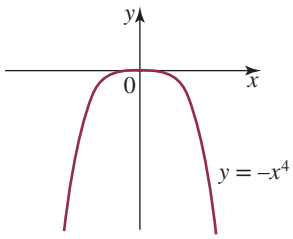



| Transformation | Description | Graph |
|---|---|---|
| Translation | If $a \neq 1,\ b \neq 0$ and $c \neq 0$, the graph is translated $+b$ units in the $x$ direction, $+c$ units in the $y$ direction, and dilated by a factor of $a$ in the $y$ direction. |   |
| Reflection | The cubic function can be expressed in factor form as $y = a(x - b)(x - c)(x - d)$, where $b$, $c$ and $d$ are the $x$-intercepts. If the value of $a$ is negative, this causes the curve to be reflected in the $x$-axis. |   |

## Quartic functions

- The basic form of the quartic function $y = ax^4$, when $a$ is positive, has the following shape.



| Transformation | Description | Graph |
|---|---|---|
| Reflection | Negative values of $a$ cause the graph to be reflected in the $x$-axis. |  |

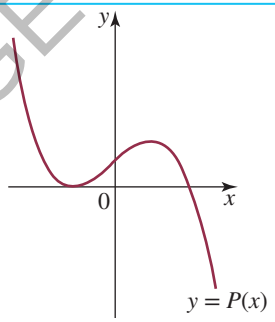
## Transformation in general polynomials

- With knowledge of the transformations that occur in the functions just discussed, it is possible to generate many other graphs without knowing the equation of the original function. Consider a basic polynomial $y = P(x)$ and what happens to the shape of the curve as the function is changed.

**WORKED EXAMPLE 14**

**Use the sketch of $y = P(x)$ shown at right to sketch:**

**a** $y = P(x) + 1$
**b** $y = P(x) - 1$
**c** $y = -P(x)$.




**THINK**

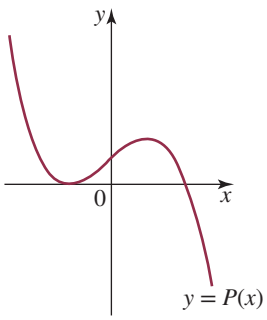
**a** **1** Sketch the original $y = P(x)$.

**2** Consider the $x$-values. They remain unchanged — there is no horizontal translation.

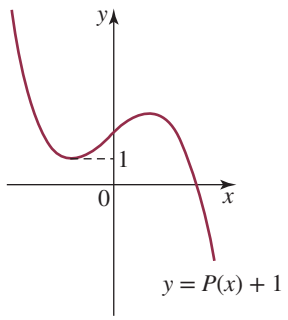
**3** Consider the $y$-values. They are increased by 1 — the curve is shifted up 1 unit.

**WRITE/DRAW**

**a**

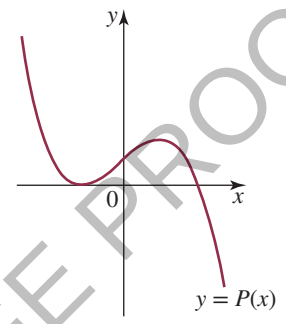


**4** Sketch the graph of $y = P(x) + 1$ using a similar scale to the original.




**b** **1** Sketch the original $y = P(x)$.

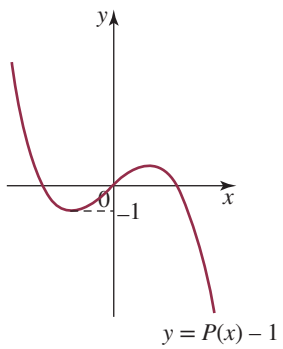
**b**




**2** Consider the $x$-values. They remain unchanged — there is no horizontal translation.

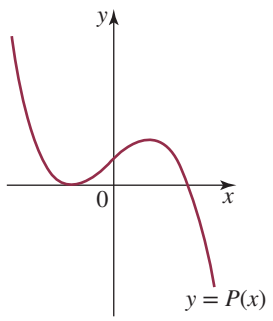
**3** Consider the $y$-values. They are decreased by 1 — the curve is shifted down 1 unit.

**4** Sketch the graph of $y = P(x) - 1$ using a similar scale to the original.




**c** **1** Sketch the original $y = P(x)$.

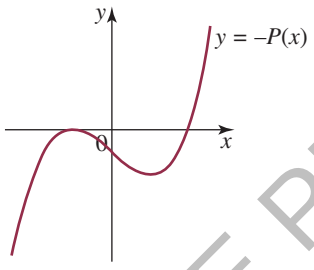
**c**



2 Consider the $x$-values. They remain unchanged — there is no horizontal translation.

3 Consider the $y$-values. They will all change sign — the curve will be reflected in the $x$-axis. That is, negative becomes positive and positive becomes negative.

4 Sketch the graph of $y = -P(x)$ using a similar scale to the original.




## Exercise 18.6 Transformations

assess on

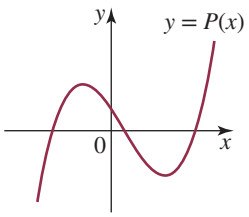
### INDIVIDUAL PATHWAYS

| PRACTISE | CONSOLIDATE | MASTER |
|---|---|---|
| Questions: 1–3, 5, 7 | Questions: 1–4, 6, 7, 9 | Questions: 1–10 |

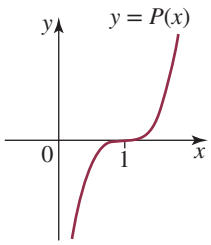
**REFLECTION**
Why is it important to understand how transformations can affect the shape of a graph?

### FLUENCY

1 WE14 Use the sketch of $y = P(x)$ shown at right to sketch:
- a $y = P(x) + 1$
- b $y = P(x) - 2$
- c $y = -P(x)$
- d $y = 2P(x)$.




2 Consider the sketch of $y = P(x)$ shown at right. Sketch:
- a $y = P(x) + 1$
- b $y = -P(x)$
- c $y = P(x + 2)$.



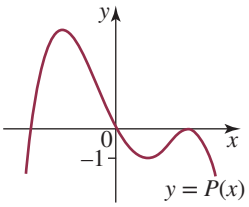

### UNDERSTANDING

3 Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = -P(x)$.

4 Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = 2P(x)$.

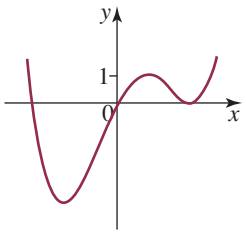
5 Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = P(x) - 2$.

**6** Consider the sketch of $y = P(x)$ shown below.

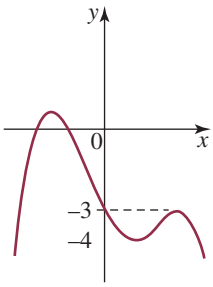



Give a possible equation for each of the following in terms of $P(x)$.

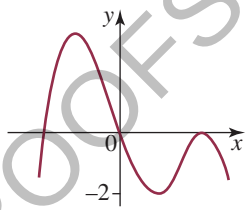
**a**




**b**




**c**




## REASONING

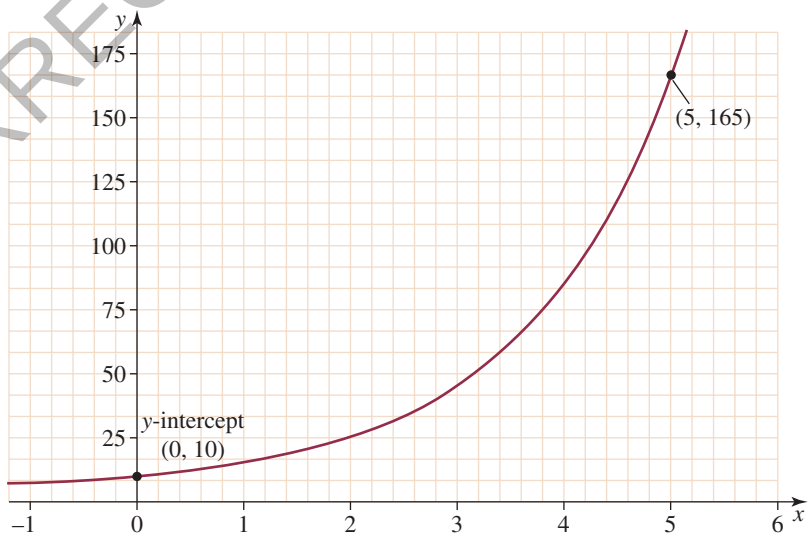
**7** $y = x(x - 2)(x - 3)$ and $y = -2x(x - 2)(x - 3)$ are graphed on the same set of axes. Describe the relationship between the two graphs using the language of transformations.

**8** If $y = -hr^{-q(x+p)} - r$, what translations take place from the original graph, $y = r^x$?

## PROBLEM SOLVING

**9** The graph of $y = \dfrac{1}{x}$ is reflected in the $x$-axis, dilated by a factor of 2 parallel to the $y$-axis, translated 2 units to the left and up 1 unit. Find the equation of the resultant curve. Give the equations of any asymptotes.

**10** The graph of an exponential function is shown.




Its general rule is given by $y = a(2^x) + b$.

**a** Find the values of $a$ and $b$.

**b** Describe any transformations that had to be applied to the graph of $y = 2^x$ to achieve this graph.

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# 18.7 Review

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The Maths Quest Review is available in a customisable format for students to demonstrate their knowledge of this topic.

The Review contains:

- **Fluency** questions — allowing students to demonstrate the skills they have developed to efficiently answer questions using the most appropriate methods
- **Problem Solving** questions — allowing students to demonstrate their ability to make smart choices, to model and investigate problems, and to communicate solutions effectively.

A summary of the key points covered and a concept map summary of this topic are available as digital documents.

## Review questions

Download the Review questions document from the links found in your eBookPLUS.

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**Interactivities**

Word search
**int-2877**

Crossword
**int-2878**

Sudoku
**int-3893**

## Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

cubic functions
dilation
domain
exponential functions
function
function notation
horizontal line test
hyperbola
inflection point
inverse
many-to-many
many-to-one
one-to-many
one-to-one
quadratic functions
quartic functions
reflection
relation
transformations
translations
vertical line test

Link to assessON for questions to test your readiness **FOR** learning, your progress **AS** you learn and your levels **OF** achievement.

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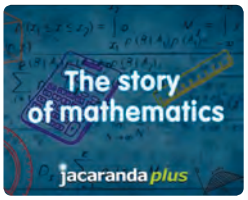
**The story of mathematics** is an exclusive Jacaranda video series that explores the history of mathematics and how it helped shape the world we live in today.



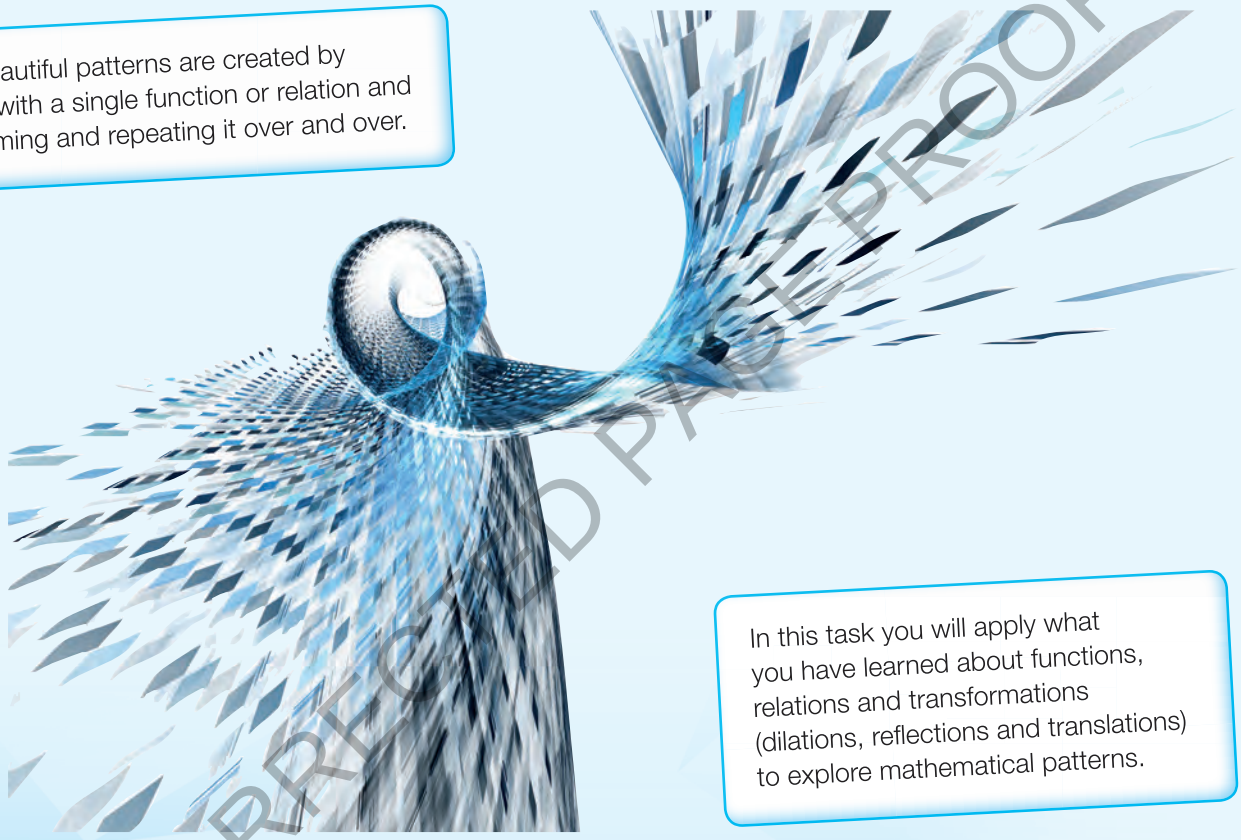

*Amalie Noether* (eles-2021) explores the life of Amalie Emmy Noether, who faced many challenges in her life, ranging from gender discrimination to racial persecution. Despite this she became one of the most influential mathematicians of the 20th Century.

INVESTIGATION

## RICH TASK

# Shaping up!

Many beautiful patterns are created by starting with a single function or relation and transforming and repeating it over and over.



In this task you will apply what you have learned about functions, relations and transformations (dilations, reflections and translations) to explore mathematical patterns.



### Exploring patterns using transformations

**1 a** On the same set of axes, draw the graphs of:

**i** $y = x^2 - 4x + 1$ **ii** $y = x^2 - 3x + 1$ **iii** $y = x^2 - 2x + 1$

**iv** $y = x^2 + 2x + 1$ **v** $y = x^2 + 3x + 1$ **vi** $y = x^2 + 4x + 1$

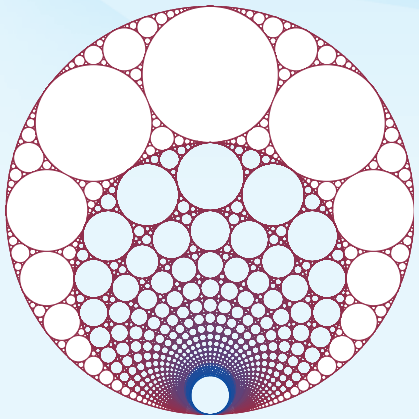
**b** Describe the pattern formed by your graphs. Use mathematical terms such as intercepts, turning point, shape and transformations.

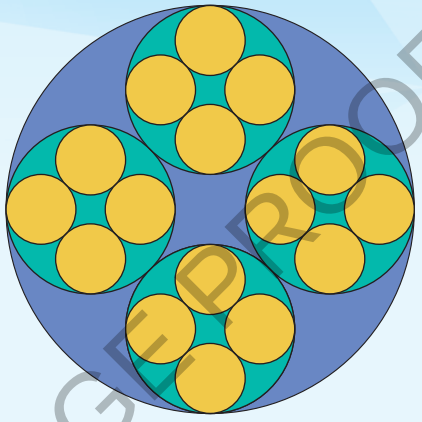
What you have drawn is referred to as a **family of curves** — curves in which the shape of the curve changes if the values of $a$, $b$ and $c$ in the general equation $y = ax^2 + bx + c$ change.

**c** Explore the family of parabolas formed by changing the values of $a$ and $c$. Comment on your findings.

**d** Explore exponential functions belonging to the family of curves with equation $y = ka^x$, families of cubic functions with equations $y = ax^3$ or $y = ax^3 + bx^2 + cx + d$, and families of quartic functions with equations $y = ax^4$ or $y = ax^4 + bx^3 + cx^2 + dx + e$. Comment on your findings.

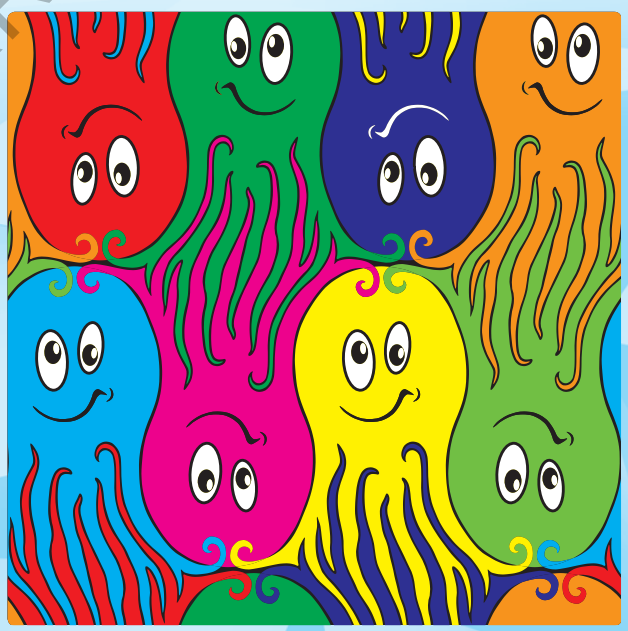
**e** Choose one of the designs shown below and recreate it (or a simplified version of it). Record the mathematical equations used to complete the design.





### Coming up with your design

**2** Use what you know about transformations to functions and relations to create your own design from a basic graph. You could begin with a circle, add some line segments and then repeat the pattern with some change.
Record all the equations and restrictions you use.
It may be helpful to apply your knowledge of inverse functions too.
A digital technology will be very useful for this task.
Create a poster of your design to share with the class.



## CODE PUZZLE

# 2 important events of 1973

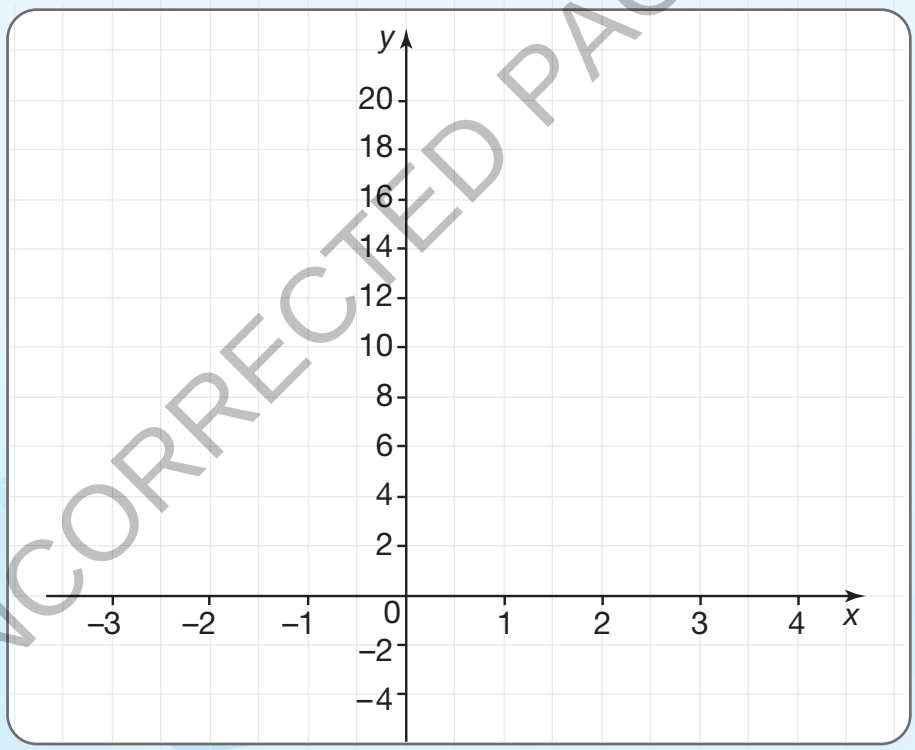

Complete the table of values for the two exponential functions and plot their graphs. The values of the functions and the letters beside the $x$-values give the puzzle's code.

$y = 2^x$:

| $x$ | –3 A | –2 D | –1 E | 0 G | 1 H | 2 I | 3 Y | 4 L |
|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | |

$y = (\frac{1}{2})^x + 5$:

| $x$ | –3 N | –2 O | –1 P | 0 R | 1 S | 2 T | 3 U | 4 V |
|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | |




| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 4 | 1 | 2 | $5\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 13 | 8 | $\frac{1}{2}$ | $\frac{1}{8}$ | 6 | 9 | 16 | $\frac{1}{4}$ | $5\frac{1}{2}$ |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | $5\frac{1}{16}$ | $\frac{1}{2}$ | 13 | $5\frac{1}{4}$ | 2 | $\frac{1}{2}$ | $5\frac{1}{16}$ | 9 | $5\frac{1}{4}$ | $\frac{1}{2}$ |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $5\frac{1}{2}$ | 8 | $\frac{1}{4}$ | 13 | $\frac{1}{2}$ | 8 | 9 | 7 | $\frac{1}{2}$ | 6 | $\frac{1}{8}$ | 2 | 9 | $5\frac{1}{8}$ | $5\frac{1}{2}$ | $\frac{1}{2}$ | 9 | 7 | $\frac{1}{2}$ | 13 | $\frac{1}{2}$ | $\frac{1}{4}$ |

# eBookplus Activities

### 18.1 Overview

**Video**

- The story of mathematics (eles-2021)

### 18.2 Functions and relations

**Digital docs**

- SkillSHEET (doc-5378): Finding the gradient and *y*-intercept
- SkillSHEET (doc-5379): Sketching straight lines
- SkillSHEET (doc-5380): Sketching parabolas
- SkillSHEET (doc-5381): Completing the square
- SkillSHEET (doc-5382): Identifying equations of straight lines and parabolas
- SkillSHEET (doc-5383): Finding points of intersection
- SkillSHEET (doc-5384): Substitution into index expressions
- WorkSHEET 18.1 (doc-14622): Functions and relations

### 18.3 Exponential functions

**eLesson**

- Exponential growth (eles-0176)

**Digital docs**

- SkillSHEET (doc-5386): Converting a percentage to a decimal
- SkillSHEET (doc-5387): Decreasing a quantity by a percentage
- WorkSHEET 18.2 (doc-14623): Exponential growth and decay

### 18.5 Quartic functions

**Digital doc**

- WorkSHEET 18.3 (doc-14624): Cubic and quartic functions

### 18.6 Transformations

**Interactivity**

- Polynomial transformations (int-2794)

### 18.7 Review

**Interactivities**

- Word search (int-2877)
- Crossword (int-2878)
- Sudoku (int-3893)

**Digital docs**

- Chapter summary (doc-14625)
- Concept map (doc-14626)

**To access eBookPLUS activities, log on to** **www.jacplus.com.au**

# Answers

## TOPIC 18 Functions and relations

### Exercise 18.2 — Functions and relations

**1 a** One-to-many **b** Many-to-one
**c** Many-to-one **d** One-to-one
**e** One-to-one **f** Many-to-one
**g** Many-to-many **h** Many-to-one
**i** One-to-one **j** Many-to-one
**k** One-to-one **l** Many-to-one

**2 a** b, c, d, e, f, h, i, j, k, l **b** d, e, i

**3 a** i 1 ii 7 iii −5 iv 16
**b** i 2 ii 1 iii 3 iv 0
**c** i 3 ii 2 iii 6 iv 9
**d** i 9 ii 1
iii 16 iv $a^2 + 6a + 9$
**e** i 12 ii 6 iii −4 iv 2

**4** A, C, D

**5** a, b, c, f

**6 a** 3 **b** 3 **c** $\frac{5}{x} - 2x$
**d** $\frac{10}{x^2} - x^2$ **e** $\frac{10}{x+3} - x - 3$ **f** $\frac{10}{x-1} - x + 1$

**7 a** 3 **b** −3 or 3 **c** $\frac{1}{3}$
**d** 2 or 3 **e** −4 or 1 **f** −1

**8 a** $f(x) \to \infty$ **b** $f(x) \to 0$ **c** $f(x) \to 0$
**d** $f(x) \to -\infty$ **e** $f(x) \to 0$

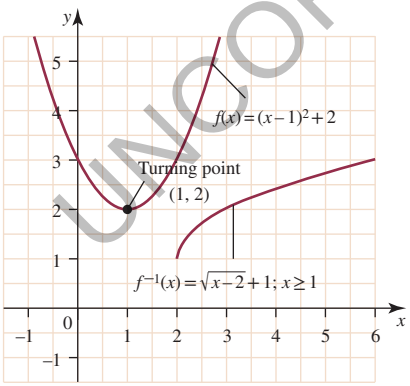
**9 a** (0, −4), (2, 0) **b** (1, −2), $(-\frac{2}{3}, 3)$
**c** (2, 0), (−2, 0) **d** (3, −4)

**10 a** $f^{-1}(x) = \frac{x+1}{2}$
**b** $f^{-1}(x) = \sqrt{x+3}$ or $f^{-1}(x) = -\sqrt{x+3}$
**c** $f(x)^{-1} = \sqrt{x-4} + 2$ or $f(x)^{-1} = -\sqrt{x-4} + 2$

**11 a** i The horizontal line test fails.
ii An inverse function will exist for $f(x) = x(x-2),\ x \le 1$ or $f(x) = x(x-2),\ x \ge 1$.
**b** i The horizontal line test is upheld.
ii $f^{-1}(x) = -\sqrt{4-x},\ x \le 4$.

**12 a** $x = \pm 3$ **b** $x = 2\frac{1}{3}$ **c** $x = 28$

**13 a** Ran = $[2, \infty)$ **b** Many-to-one **c** and f



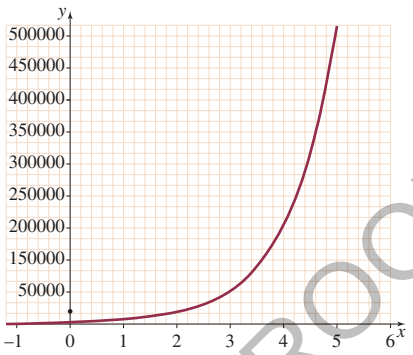

**d** Dom = $[1, \infty)$
**e** $f^{-1}(x) = \sqrt{x-2} + 1$, Dom = $[2, \infty)$, Ran = $[1, \infty)$
**g** (0, 3) and (3, 6)

### Challenge 18.1

These graphs are inverse because they are the mirror images of each other through the line $y = x$.

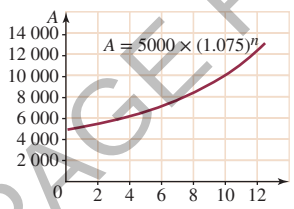
### Exercise 18.3 — Exponential functions

**1 a** 2000 **b** 486 000
**c**




**d** 1.26 h

**2 a** \$5000 **b** \$7717
**c**




**d** 10 years

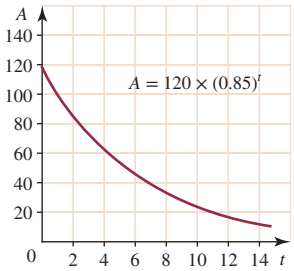
**3 a** C **b** D

**4** A

**5** B

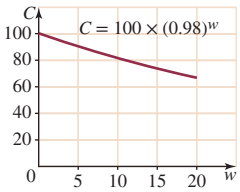
**6 a** \$883.50 **b** \$821.66
**c** $V = 950 \times (0.93)^n$ **d** \$397.67

**7 a** 102 mg **b** 86.7 mg
**c** $A = 120 \times (0.85)^t$ **d** 83.927 mg
**e**




**f** Approximately 210 years

**8 a** i 96.04% ii 90.39%
**b** $C = 100(0.98)^w$
**c**



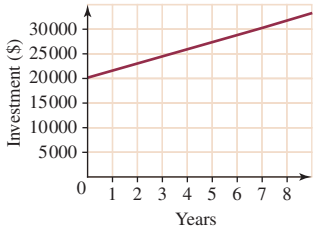

**d** 8 washings

**9 a** 118 (million)
**b** $a = 1.02$; $P = 118 \times (1.02)^n$
**c**

| Year | 1990 | 1995 | 2000 | 2005 | 2010 |
|---|---|---|---|---|---|
| Population | 118 | 130 | 144 | 159 | 175 |

Calculated population is less accurate after 10 years.
**d** 288 (million)

**10 a** 32 **b** 0.98 **c** $T = 32 \times (0.98)^t$
**d** 26.1, 21.4, 17.5, 14.3; values are close except for $t = 40$.
**11 a** 3 dogs **b** 27 dogs **c** 3 years
**12 a** **i** 39.85 mg **ii** 18.43 mg
**b** More than 35.78 centuries
**13 a** $A = 20\,000 \times 1.06^x$
**b**



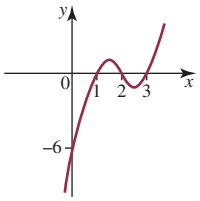

**c** 7 years **d** 6 years — 1 year quicker
**e** 9.05% p.a.
**14 a** Approximately 20 200 **b, c** Teacher to check.
**15 a** $a = 100,\ b = 1.20,$ increase = 20%/min
**b** $N = 146\,977 \times 0.70^m$
**16 a** 500 °C **b** 125 °C
**c** Between 5 and 6 hours once it has cooled to below 15 °C
**17 a** 45 °C **b** $T = 45 \times 0.95^t$ **c** 10 °C
**d**

| Time (min) | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Temperature (°C) | 45 | 35 | 27 | 21 | 16 |

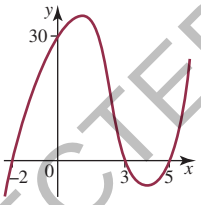
**e** No
**f** No. The line $T = 0$ is an asymptote.

## Exercise 18.4 — Cubic functions

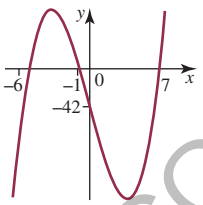
**1 a**



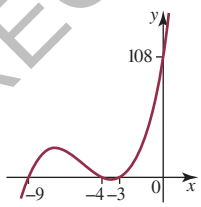

**b**



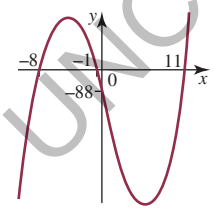

**c**



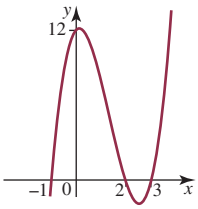

**d**



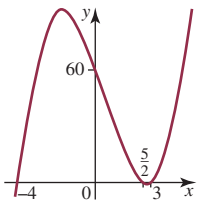

**e**



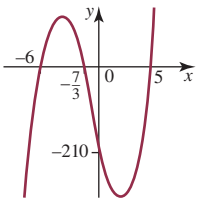

**f**



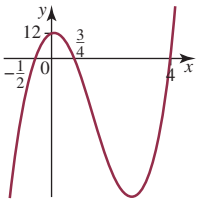

**g**



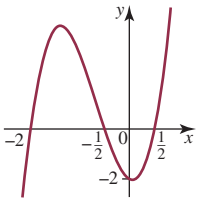

**h**



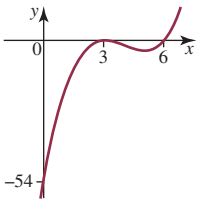

**i**



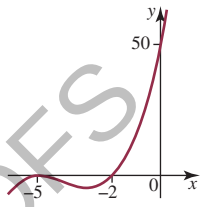

**j**



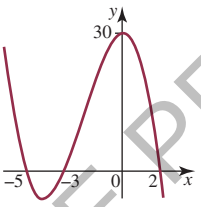

**k**



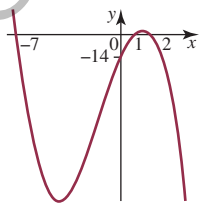

**l**



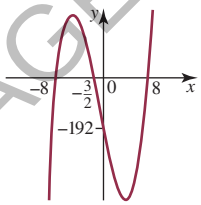

**2 a**



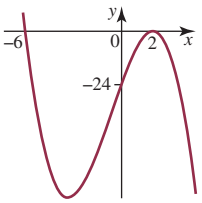

**b**



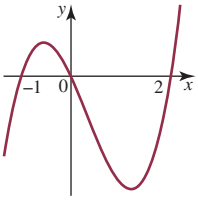

**c**



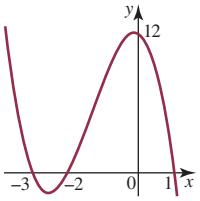

**d**



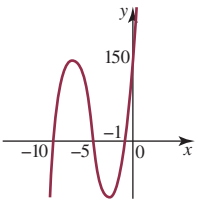

**e**



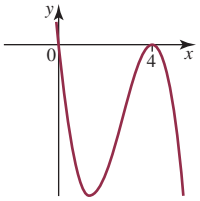

**f**



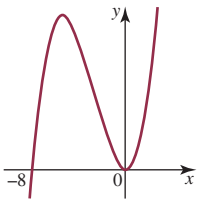

**g**



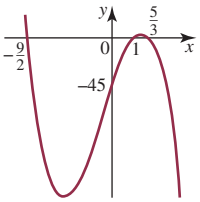

**h**




**i**




**j**



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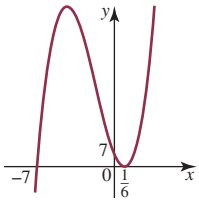
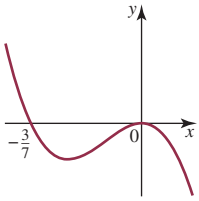
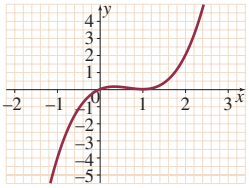



**l**

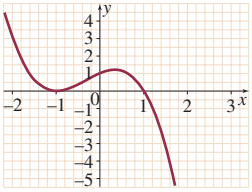



**3** C **4** C **5** B **6** D

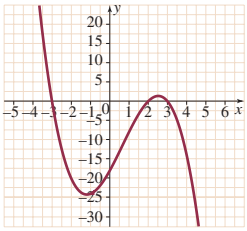
**7 a**



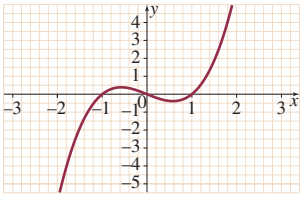
**b**



**c**



**d**



**8** $a = 2,\ b = -7$

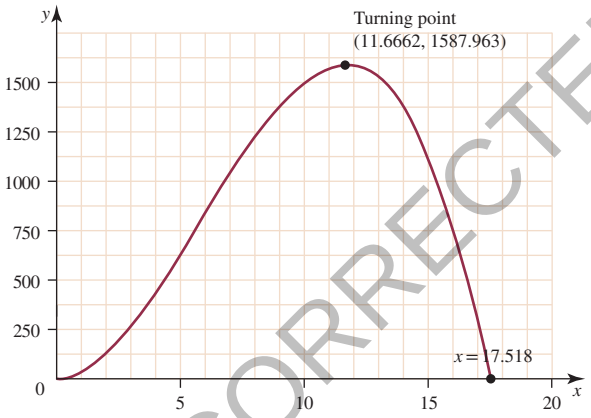
**9** $a = \dfrac{-(27 + 11b)}{11}$

**10 a** Check with your teacher.

**b** $0 < x < 17.5$

**c** 1377 cm$^3$

**d**

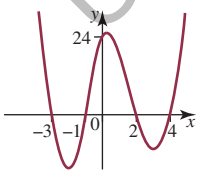



**e** (11.6662, 1587.963); this is the value of $x$ which creates the maximum volume.

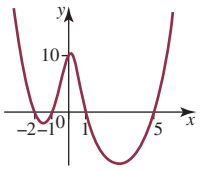
**11** $y = 2(x + 2)(x - 2)(x - 5)$

## Exercise 18.5 — Quartic functions

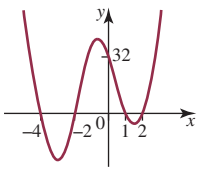
**1 a**



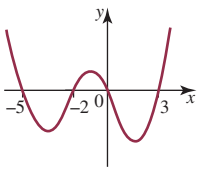
**b**



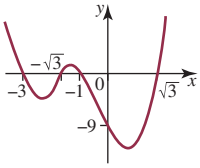
**c**



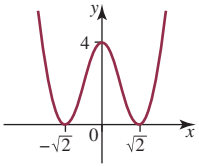
**d**



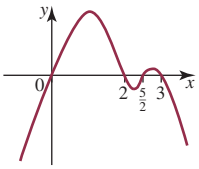
**e**



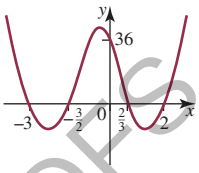
**f**



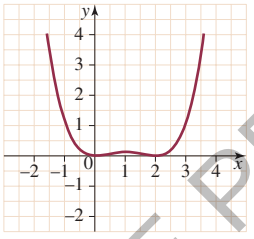
**g**



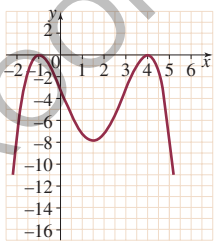
**h**



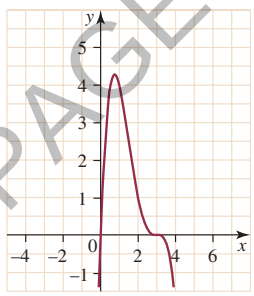
**2 a**



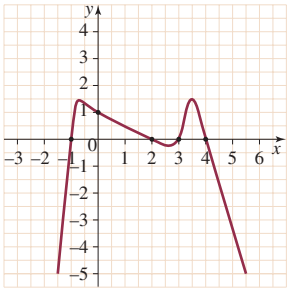
**b**



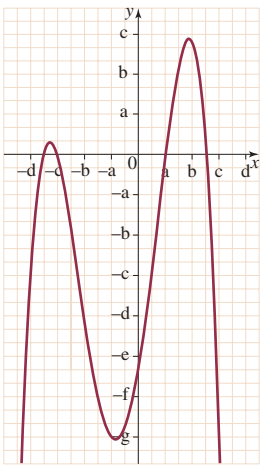
**c**



**d**



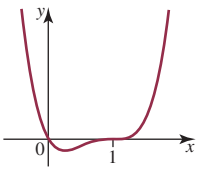
**e**



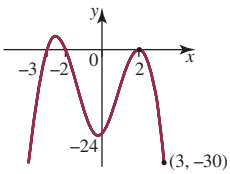
**3** D

**4 a** D **b** B

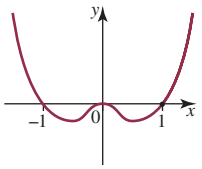
**5 a**



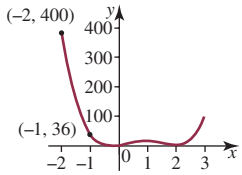
**b**



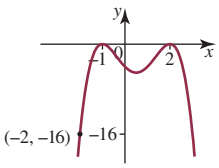
**c**



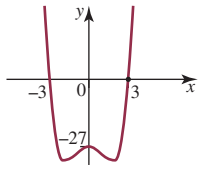

**d**



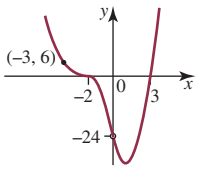

**e**



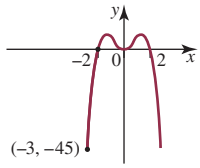

**f**




**g**




**h**




**6** $a = 4,\ b = -19$

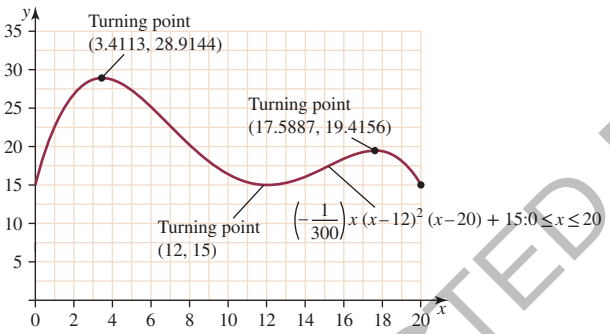
**7** $a = 3,\ b = -1$

**8** **a** If $n$ is even, the graph touches the $x$-axis.

**b** If $n$ is odd, the graph cuts the $x$-axis.

**9** **a** 15 m

**b**




**c** 28.77 m

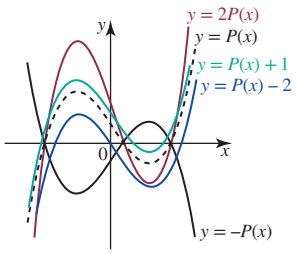
**10** $y = (x + 1)(x - 2)^3$

## Challenge 18.2

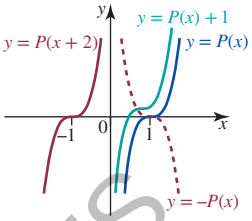
Check with your teacher.

## Exercise 18.6 — Transformations

**1**




**2**




**3** They have the same $x$-intercepts, but $y = -P(x)$ is a reflection of $y = P(x)$ in the $x$-axis.

**4** They have the same $x$-intercepts, but the $y$-values in $y = 2P(x)$ are all twice as large.

**5** The entire graph is moved down 2 units. The shape is identical.

**6** **a** $y = -P(x)$ **b** $y = P(x) - 3$ **c** $y = 2P(x)$

**7** The original graph has been reflected in the $x$-axis and dilated by a factor of 2 in the $y$ direction. The location of the intercepts remains unchanged.

**8** Dilation by a factor of $h$ from the $x$-axis, reflection in the $x$-axis, dilation by a factor of $\frac{1}{q}$ from the $y$-axis, reflection in the $y$-axis, translation of $p$ units left, translation of $r$ units down.

**9** $y = -\dfrac{2}{(x + 2)} + 1,\ x = -2, y = 1$

**10** **a** $y = 5(2^x) + 5$

**b** Dilation by a factor of 5 parallel to the $y$-axis and translation of 5 units up. Graph asymptotes to $y = 5$.

## Investigation — Rich task

Check with your teacher.

## Code puzzle

Eighteen year olds given the vote.
Sydney Opera House opened.