TOPIC 20

Trigonometry II

20.1 Overview

Why learn this?

The ability to calculate distances using angles has long been critical. Early explorers, using rudimentary calculations, were able to navigate their way around the planet and were even able to map coastlines along the way. It is essential to be able to calculate distances that can't be physically measured. Our explorations have now turned towards the skies. By applying trigonometry we can approximate the distances to other planets and beyond.

What do you know?

assessON

1 **THINK** List what you know about trigonometry. Use a thinking tool such as a concept map to show your list.
2 **PAIR** Share what you know with a partner and then with a small group.
3 **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of trigonometry.

Learning sequence

20.1 Overview
20.2 The sine rule
20.3 The cosine rule
20.4 Area of triangles
20.5 The unit circle
20.6 Trigonometric functions
20.7 Solving trigonometric equations
20.8 Review eBookplus ONLINE ONLY

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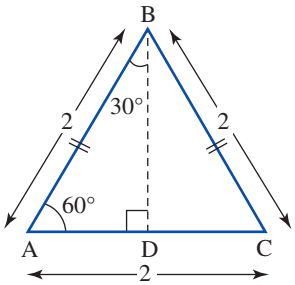
20.2 The sine rule

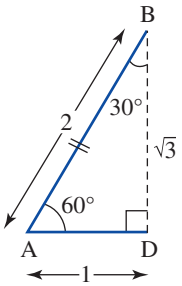
Exact values

- Most of the trigonometric values that we will deal with in this chapter are approximations.
- However, angles of 30°, 45° and 60° have exact values of sine, cosine and tangent.



- Consider an equilateral triangle, ABC, of side length 2 units.
  If the triangle is perpendicularly bisected, then two congruent triangles, ABD and CBD, are obtained. From triangle ABD it can be seen that altitude BD creates a right-angled triangle with angles of 60° and 30° and base length (AD) of 1 unit. The altitude BD is obtained using Pythagoras' theorem.

$$\begin{aligned}(AB)^2 &= (AD)^2 + (BD)^2\\ 2^2 &= 1^2 + (BD)^2\\ 4 &= 1 + (BD)^2\\ 4 - 1 &= (BD)^2\\ (BD)^2 &= 3\\ BD &= \sqrt{3}\end{aligned}$$



- Using triangle ABD and the three trigonometric ratios the following exact values are obtained:

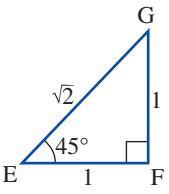
$$\sin B = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 30° = \frac{1}{2} \qquad \sin A = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 30° = \frac{\sqrt{3}}{2} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 60° = \frac{1}{2}$$

$$\tan B = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \qquad \tan A = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 60° = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}$$

- Consider a right-angled isosceles triangle EFG whose equal sides are of 1 unit. The hypotenuse EG is obtained by using Pythagoras' theorem.

$$\begin{aligned}(EG)^2 &= (EF)^2 + (FG)^2\\ &= 1^2 + 1^2\\ &= 2\\ EG &= \sqrt{2}\end{aligned}$$



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- Using triangle EFG and the three trigonometric ratios, the following exact values are obtained:

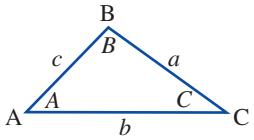
$$\sin E = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$
$$\cos E = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$
$$\tan E = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 45° = \frac{1}{1} \text{ or } 1$$

Summary of exact values

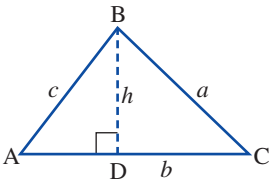
θ	30°	45°	60°
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

The sine rule

- In non–right-angled triangles, it is usual to label the angles A, B and C, and the sides a, b and c, so that side a is the side opposite angle A, side b is the side opposite angle B and side c is the side opposite angle C.



- In a non–right-angled triangle, a perpendicular line, h, can be drawn from the angle B to side b.
  This divides the triangle into two right-angled triangles, ABD and CBD.
- Using triangle ABD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin A = \frac{h}{c}$. Using triangle CBD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin C = \frac{h}{a}$.



$\frac{h}{c} = \sin A$ and $\frac{h}{a} = \sin C$

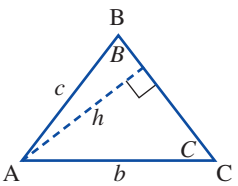
- Transposing each equation to make h the subject, we obtain: $h = c \sin A$ and $h = a \sin C$. Since h is common to both triangles the two equations may be equated and we get $c \sin A = a \sin C$.
  Dividing both sides of the equation by $\sin A$ gives:

$$c = \frac{a \sin C}{\sin A}$$

  Dividing both sides of the equation by $\sin C$ gives:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

- In a similar way, if a perpendicular line is drawn from angle A to side a, the two right-angled triangles would give $h = c \sin B$ and $h = b \sin C$.



$h = c \sin B$ and $h = b \sin C$

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This would give: $\frac{b}{\sin B} = \frac{c}{\sin C}$

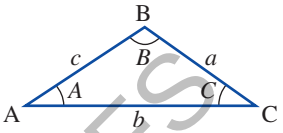
From this, the **sine rule** can be derived.

- In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Notes

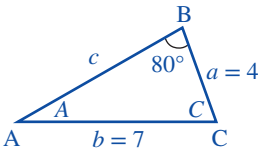
1. When using this rule, depending on the values given, any combination of the two equalities may be used to solve a particular triangle.
2. To solve a triangle means to find all unknown side lengths and angles.



- The sine rule can be used to solve non–right-angled triangles if we are given:
  1. two angles and one side length
  2. two side lengths and an angle opposite one of these side lengths.

WORKED EXAMPLE 1 TI CASIO

In the triangle ABC, $a = 4$ m, $b = 7$ m and $B = 80°$. Find A, C and c.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write down the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{b}{\sin B}$
4 Substitute the known values into the rule.	$\frac{4}{\sin A} = \frac{7}{\sin 80°}$
5 Transpose the equation to make $\sin A$ the subject.	$4 \sin 80° = 7 \sin A$ $\sin A = \frac{4 \sin 80°}{7}$
6 Evaluate.	$A = \sin^{-1}\left(\frac{4 \sin 80°}{7}\right)$ $\approx 34.246\,004\,71°$
7 Round off the answer to degrees and minutes.	$\approx 34°15'$
8 Determine the value of angle C using the fact that the angle sum of any triangle is 180°.	$C \approx 180° - (80° + 34°15')$ $= 65°45'$
9 Write down the sine rule to find c.	To find side length c: $\frac{c}{\sin C} = \frac{b}{\sin B}$

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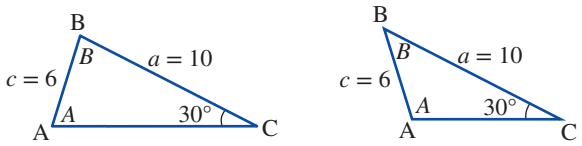
10	Substitute the known values into the rule.	$\frac{c}{\sin\ 65°45'} = \frac{b}{\sin\ 80°}$
11	Transpose the equation to make c the subject.	$c = \frac{7 \sin\ 65°45'}{\sin\ 80°}$
12	Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 6.48 m

The ambiguous case

- If two side lengths and an angle opposite one of these side lengths are given, then two different triangles may be drawn. For example, if $a = 10$, $c = 6$ and $C = 30°$, two possible triangles could be created.



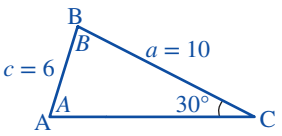
In the first case (above left), angle A is an acute angle, while in the second case (above right), angle A is an obtuse angle.

- When using the sine rule to find an angle, the inverse sine function is used. If we are finding an angle, given the sine value, it is important to remember that an angle between 0° and 90° has the same sine value as its supplement. For example, $\sin\ 40° = 0.6427$, and $\sin\ 140° = 0.6427$.

WORKED EXAMPLE 2 TI CASIO

In the triangle ABC, $a = 10$ m, $c = 6$ m and $C = 30°$. Find two possible values of A, and hence two possible values of B and b.

Case 1

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write down the sine rule to find A.	To find angle A: $\frac{a}{\sin\ A} = \frac{c}{\sin\ C}$
4 Substitute the known values into the rule.	$\frac{10}{\sin A} = \frac{6}{\sin 30°}$ $10 \sin\ 30° = 6 \sin\ A$

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5 Transpose the equation to make sin A the subject.

$$\sin A = \frac{10 \sin 30°}{6}$$

6 Evaluate angle A.

$$A = \sin^{-1}\left(\frac{10 \sin 30°}{6}\right)$$
$$\approx 56.442\,690\,24°$$

7 Round off the answer to degrees and minutes.

$$A = 56°27'$$

8 Determine the value of angle B, using the fact that the angle sum of any triangle is 180°.

$$B \approx 180° - (30° + 56°27')$$
$$= 93°33'$$

9 Write down the sine rule to find b.

To find side length b:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

10 Substitute the known values into the rule.

$$\frac{b}{\sin 93°33'} = \frac{6}{\sin 30°}$$

11 Transpose the equation to make b the subject.

$$b = \frac{6 \sin 93°33'}{\sin 30°}$$

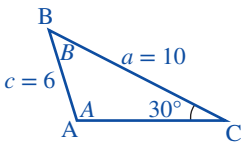
12 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.

$$\approx 11.98 \text{ m}$$

The values we have just obtained are only one set of possible answers for the given dimensions of the triangle ABC.

We are told that $a = 10$ m, $c = 6$ m and $C = 30°$. Since side a is larger than side c, it follows that angle A will be larger than angle C. Angle A must be larger than 30°; therefore it may be an acute angle or an obtuse angle.

Case 2

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Write down the alternative value for angle A. Simply subtract the value obtained for A in Case 1 from 180°.	To find the alternative angle A: If $\sin A = 0.8333$, then A could also be: $A \approx 180° - 56°27'$ $= 123°33'$

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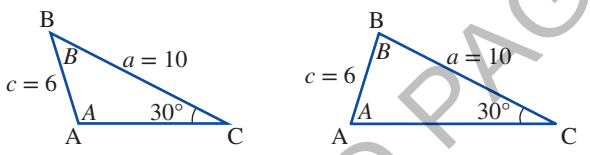
3	Determine the alternative value of angle B, using the fact that the angle sum of any triangle is 180°.	$B \approx 180° - (30° + 123°33')$ $= 26°27'$
4	Write down the sine rule to find the alternative b.	To find side length b: $\frac{b}{\sin B} = \frac{c}{\sin C}$
5	Substitute the known values into the rule.	$\frac{b}{\sin 26°27'} = \frac{6}{\sin 30°}$
6	Transpose the equation to make b the subject.	$b = \frac{6 \sin 26°27'}{\sin 30°}$
7	Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 5.34 m

Hence, for this example there were two possible solutions as shown by the diagrams below.



- The ambiguous case does not apply to every question. Consider Worked example 1, in which we were required to solve the triangle ABC given $a = 4$ m, $b = 7$ m and $B = 80°$. For angle A, we obtained $A = 34°15'$. However, angle A could also have been $A = 145°45'$ (since there are two possible values of A between 0° and 180° whose sine is the same; that is, $\sin 34°15' = 0.5628$ and $\sin 145°45' = 0.5628$).
  We will now see whether or not $A = 145°45'$ is a possible solution.
  To obtain C subtract angles A and B from 180°.

$$\begin{aligned} C &= 180° - (80° + 145°45') \\ &= 180° - 225°45' \\ &= -45°45' \text{ (not possible)} \end{aligned}$$

  Hence, for Worked example 1 only one possible solution exists.
- The ambiguous case exists if C is an acute angle and $a > c > a \sin C$, or any equivalent statement; for example, if B is an acute angle and $a > b > a \sin B$, and so.
  - In Worked example 2, where $a = 10$ m, $c = 6$ m and $C = 30°$, there were two possible solutions because C was an acute angle and $a > c > a \sin C$, since $10 > 6 > 10 \times 0.5$.
  - In Worked example 1, where $a = 4$ m, $b = 7$ m and $B = 80°$, there was only one possible solution because even though B was an acute angle, the condition $a > b > a \sin B$ could not be satisfied.

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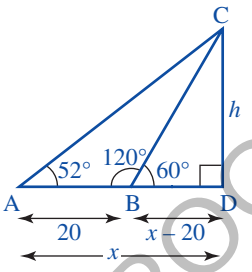
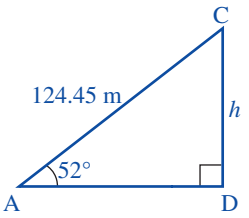
WORKED EXAMPLE 3

To calculate the height of a building, Kevin measures the angle of elevation to the top as 52°. He then walks 20 m closer to the building and measures the angle of elevation as 60°. How high is the building?

THINK	WRITE/DRAW
1 Draw a labelled diagram of the situation and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied for triangle ABC.	The sine rule can be used for triangle ABC since two angles and one side length have been given.
3 Determine the value of angle ACB, using the fact that the angle sum of any triangle is 180°.	$\angle ACB = 180° - (52° + 120°)$ $= 8°$
4 Write down the sine rule to find b (or AC).	To find side length b of triangle ABC: $\frac{b}{\sin B} = \frac{c}{\sin C}$
5 Substitute the known values into the rule.	$\frac{b}{\sin 120°} = \frac{20}{\sin 8°}$
6 Transpose the equation to make b the subject.	$b = \frac{20 \times \sin 120°}{\sin 8°}$
7 Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	≈ 124.45 m
8 Draw a diagram of the situation, that is, triangle ADC, labelling the required information. *Note:* There is no need to solve the rest of the triangle in this case as the values will not assist in finding the height of the building.	
9 Write down what is given for the triangle.	Have: angle and hypotenuse
10 Write down what is needed for the triangle.	Need: opposite side
11 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).	$\sin\theta = \frac{O}{H}$

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12	Substitute the given values into the appropriate ratio.	$\sin 52° = \frac{h}{124.45}$
13	Transpose the equation and solve for h.	$124.45 \sin 52° = h$ $h = 124.45 \sin 52°$
14	Round off the answer to 2 decimal places.	≈ 98.07
15	Answer the question.	The height of the building is 98.07 m.

Exercise 20.2 The sine rule

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INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 4, 5, 7, 10, 11, 14, 16, 18, 20, 22, 23, 24	Questions: 1, 3, 5, 6, 8, 10–12, 14, 16, 18, 20, 22, 24	Questions: 1, 5–17, 19, 21, 23, 24, 25

REFLECTION
In what situations can the sine rule be used?

FLUENCY

1 WE1 In the triangle ABC, $a = 10$, $b = 12$ and $B = 58°$. Find A, C and c.
2 In the triangle ABC, $c = 17.35$, $a = 26.82$ and $A = 101°47'$. Find C, B and b.
3 In the triangle ABC, $a = 5$, $A = 30°$ and $B = 80°$. Find C, b and c.
4 In the triangle ABC, $c = 27$, $C = 42°$ and $A = 105°$. Find B, a and b.
5 In the triangle ABC, $a = 7$, $c = 5$ and $A = 68°$. Find the perimeter of the triangle.
6 Find all unknown sides and angles for the triangle ABC, given $A = 57°$, $B = 72°$ and $a = 48.2$.
7 Find all unknown sides and angles for the triangle ABC, given $a = 105$, $B = 105°$ and $C = 15°$.
8 Find all unknown sides and angles for the triangle ABC, given $a = 32$, $b = 51$ and $A = 28°$.
9 Find the perimeter of the triangle ABC if $a = 7.8$, $b = 6.2$ and $A = 50°$.
10 MC *Note:* There may be more than one correct answer.
In a triangle ABC, $B = 40°$, $b = 2.6$ and $c = 3$. The value of C is approximately:
A 47° B 48° C 132° D 133°
11 WE2 In the triangle ABC, $a = 10$, $c = 8$ and $C = 50°$. Find two possible values of A, and hence two possible values of b.
12 In the triangle ABC, $a = 20$, $b = 12$ and $B = 35°$. Find two possible values for the perimeter of the triangle.
13 Find all unknown sides and angles for the triangle ABC, given $A = 27°$, $B = 43°$ and $c = 6.4$.
14 Find all unknown sides and angles for the triangle ABC, given $A = 100°$, $b = 2.1$ and $C = 42°$.
15 Find all unknown sides and angles for the triangle ABC, given $A = 25°$, $b = 17$ and $a = 13$.

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SkillSHEET
Labelling right-angled triangles
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SkillSHEET
Calculating sin, cos or tan of an angle
doc-5399
SkillSHEET
Finding side lengths in right-angled triangles
doc-5400
SkillSHEET
Calculating the angle from a sin, cos or tan ratio
doc-5401
SkillSHEET
Finding angles in right-angled triangles
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UNDERSTANDING

16 WE3 To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. How high is the building?

REASONING

17 A river has parallel banks that run directly east–west. From the south bank, Kylie takes a bearing to a tree on the north side. The bearing is 047°T. She then walks 10 m due east, and takes a second bearing to the tree. This is 305°T. Find:

- **a** her distance from the second measuring point to the tree
- **b** the width of the river, to the nearest metre.

18 A ship sails on a bearing of S20°W for 14 km; then it changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.

- **a** How far is it from the starting point?
- **b** On what bearing did it sail the 20 km leg?

19 A cross-country runner runs at 8 km/h on a bearing of 150°T for 45 mins; then she changes direction to a bearing of 053°T and runs for 80 mins at a different speed until she is due east of the starting point.

- **a** How far was the second part of the run?
- **b** What was her speed for this section?
- **c** How far does she need to run to get back to the starting point?

20 From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. How far is the fire from each tower?

21 MC A boat sails on a bearing of N15°E for 10 km and then on a bearing of S85°E until it is due east of the starting point. The distance from the starting point to the nearest kilometre is:

A 10 km B 38 km C 113 km D 114 km

22 MC A hill slopes at an angle of 30° to the horizontal. A tree that is 8 m tall and leaning downhill is growing at an angle of 10° to the vertical and is part-way up the slope. The vertical height of the top of the tree above the slope is:

A 7.37 m B 8.68 m C 10.84 m D 15.04 m

23 A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal and then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.

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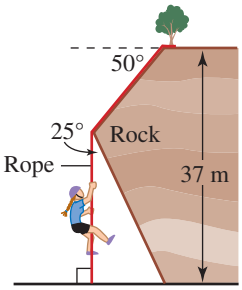
Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Will the rope be long enough to allow her to reach the ground?



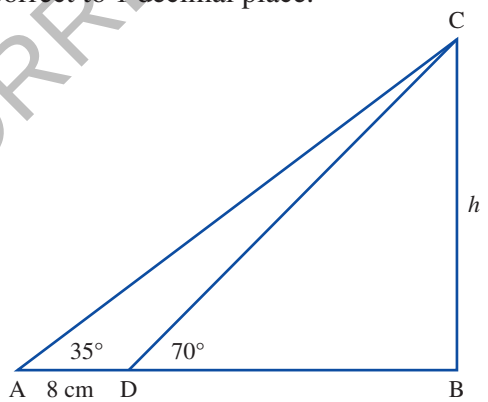
PROBLEM SOLVING

24 A yacht sets sail from a marina and sails on a bearing of 065°T for 3.5 km. It then turns and sails on a bearing of 127°T for another 5 km.

a How far is the yacht from the marina?

b On what bearing to the nearest minute should the yacht travel if it was to sail directly back to the marina?

25 Find the value of h, correct to 1 decimal place.



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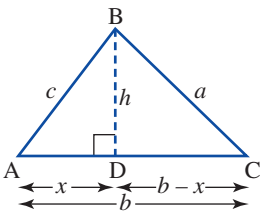
20.3 The cosine rule

- In any non–right-angled triangle ABC, a perpendicular line can be drawn from angle B to side b. Let D be the point where the perpendicular line meets side b, and let the length of the perpendicular line be h. Let the length AD $= x$ units. The perpendicular line creates two right-angled triangles, ADB and CDB.



- Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \quad [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \quad [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

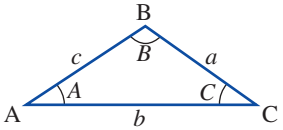
$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

From triangle ABD, $x = c \cos A$; therefore $a^2 = b^2 + c^2 - 2bx$ becomes

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

- This is called the **cosine rule** and is a generalisation of Pythagoras' theorem.
- In a similar way, if the perpendicular line was drawn from angle A to side a or from angle C to side c, the two right-angled triangles would give $c^2 = a^2 + b^2 - 2ab \cos C$ and $b^2 = a^2 + c^2 - 2ac \cos B$ respectively. From this, the cosine rule can be stated:
  In any triangle ABC:

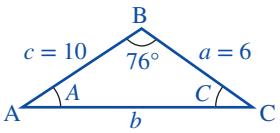
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



- The cosine rule can be used to solve non–right-angled triangles if we are given:
  1. three sides of the triangle
  2. two sides of the triangle and the included angle (the angle between the given sides).

WORKED EXAMPLE 4

Find the third side of triangle ABC given $a = 6$, $c = 10$ and $B = 76°$.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	Yes, the cosine rule can be used since two side lengths and the included angle have been given.
3 Write down the appropriate cosine rule to find side b.	To find side b: $b^2 = a^2 + c^2 - 2ac \cos B$
4 Substitute the given values into the rule.	$= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 76°$
5 Evaluate.	$\approx 106.969\,372\,5$ $b \approx \sqrt{106.969\,372\,5}$
6 Round off the answer to 2 decimal places.	≈ 10.34

UNCORRECTED PAGE PROOFS

- *Note:* Once the third side has been found, the sine rule could be used to find other angles if necessary.
- If three sides of a triangle are known, an angle could be found by transposing the cosine rule to make cos *A*, cos *B* or cos *C* the subject.

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \quad \Rightarrow \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

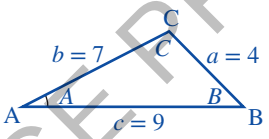
WORKED EXAMPLE 5 TI CASIO

Find the smallest angle in the triangle with sides 4 cm, 7 cm and 9 cm.

THINK

1. Draw a labelled diagram of the triangle, call it ABC and fill in the given information.
   Note: The smallest angle will correspond to the smallest side.
2. Check that one of the criteria for the cosine rule has been satisfied.
3. Write down the appropriate cosine rule to find angle *A*.
4. Substitute the given values into the rearranged rule.
5. Evaluate.
6. Transpose the equation to make *A* the subject by taking the inverse cos of both sides.
7. Round off the answer to degrees and minutes.

WRITE/DRAW



Let $a = 4$
$b = 7$
$c = 9$

The cosine rule can be used since three side lengths have been given.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{7^2 + 9^2 - 4^2}{2 \times 7 \times 9}$$
$$= \frac{114}{126}$$
$$A = \cos^{-1}\left(\frac{114}{126}\right)$$
$$\approx 25.208\,765\,3°$$
$$\approx 25°13'$$

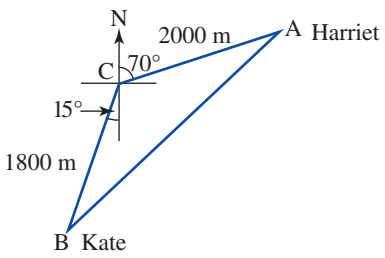
WORKED EXAMPLE 6

Two rowers, Harriet and Kate, set out from the same point. Harriet rows N70°E for 2000 m and Kate rows S15°W for 1800 m. How far apart are the two rowers?

THINK

1. Draw a labelled diagram of the triangle, call it ABC and fill in the given information.

WRITE/DRAW



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2	Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since two side lengths and the included angle have been given.
3	Write down the appropriate cosine rule to find side c.	To find side c: $c^2 = a^2 + b^2 - 2ab\cos C$
4	Substitute the given values into the rule.	$= 2000^2 + 1800^2 - 2 \times 2000 \times 1800 \cos 125°$
5	Evaluate.	$\approx 11\,369\,750.342$ $c \approx \sqrt{11\,369\,750.342}$
6	Round off the answer to 2 decimal places.	≈ 3371.91
7	Answer the question.	The rowers are 3371.91 m apart.

assess on

Exercise 20.3 The cosine rule

REFLECTION
In what situations would you use the sine rule rather than the cosine rule?

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3–5, 7, 9, 11, 13, 15	Questions: 1, 2, 4, 6–9, 11–13, 15, 16, 18	Questions: 1, 3, 4, 6, 7, 9–14, 16–19

FLUENCY

1 WE4 Find the third side of triangle ABC given $a = 3.4$, $b = 7.8$ and $C = 80°$.
2 In triangle ABC, $b = 64.5$, $c = 38.1$ and $A = 58°34'$. Find a.
3 In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Find b, and hence find A and C.
4 WE5 Find the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm. (*Hint:* The smallest angle is opposite the smallest side.)
5 In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Find the largest angle.
6 In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Find the size of all the angles.
7 WE6 Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. How far apart are the two rowers?
8 Maria cycles 12 km in a direction N68°W and then 7 km in a direction of N34°E.
   a How far is she from her starting point?
   b What is the bearing of the starting point from her finishing point?

UNDERSTANDING

9 A garden bed is in the shape of a triangle, with sides of length 3 m, 4.5 m and 5.2 m.
   a Calculate the smallest angle.
   b Hence, find the area of the garden. (*Hint:* Draw a diagram, with the longest length as the base of the triangle.)

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10 A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle, to the nearest degree, must the shot be made if it is to score a goal?

11 An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. How high can the balloon fly?

12 A plane flies in a direction of N70°E for 80 km and then on a bearing of S10°W for 150 km.

a How far is the plane from its starting point?

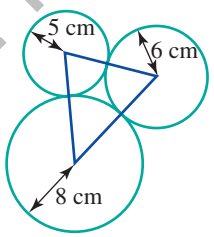
b What direction is the plane from its starting point?

13 Ship A is 16.2 km from port on a bearing of 053°T and ship B is 31.6 km from the same port on a bearing of 117°T. Calculate the distance between the two ships.

14 A plane takes off at 10.00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10.05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. How far apart are the planes at 10.25 am?

REASONING

15 Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Find the largest angle in the triangle.

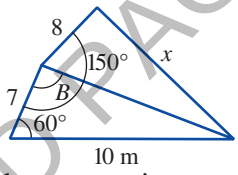


16 For the given shape at near right, determine:

a the length of the diagonal

b the magnitude (size) of angle B

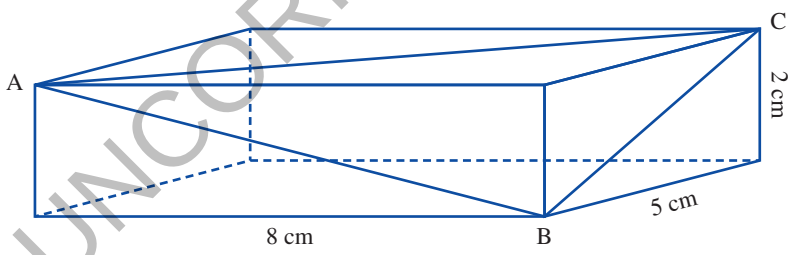
c the length of x.



17 From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. How fast does the yacht sail?

PROBLEM SOLVING

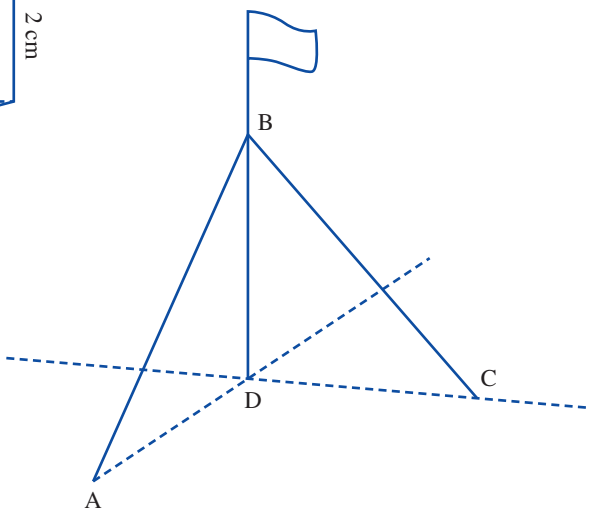
18 Find the measure of angles CAB, ABC and BCA. Give your answers correct to 2 decimal places.



19 A vertical flag pole DB is supported by two wires AB and BC. AB is 5.2 metres long, BC is 4.7 metres long and B is 3.7 metres above ground level. Angle ADC is a right angle.

a Find the distance from A to C.

b Calculate the angle between AB and BC.



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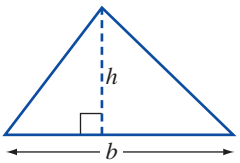
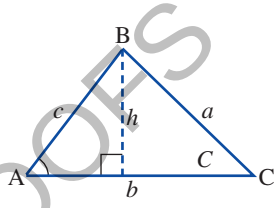
20.4 Area of triangles

- The area of any triangle is given by the rule area $= \frac{1}{2}bh$, where b is the base and h is the perpendicular height of the triangle.



- However, often the perpendicular height is not given and needs to be calculated first. In the triangle ABC, b is the base and h is the perpendicular height of the triangle.
- Using the trigonometric ratio for sine:



$$\sin A = \frac{h}{c}$$

Transposing the equation to make h the subject, we obtain:

$$h = c \sin A$$

- Therefore, the area of triangle ABC becomes:

$$\text{area} = \frac{1}{2}bc \sin A$$

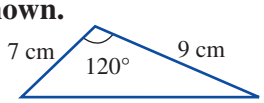
- Depending on how the triangle is labelled, the formula could read:

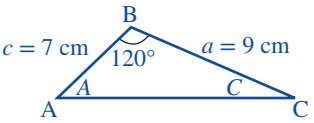
$$\text{area} = \frac{1}{2}ab \sin C \quad \text{area} = \frac{1}{2}ac \sin B \quad \text{area} = \frac{1}{2}bc \sin A$$

- The area formula may be used on any triangle provided that two sides of the triangle and the included angle (that is, the angle between the two given sides) are known.

WORKED EXAMPLE 7

Find the area of the triangle shown.



THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 9$ cm, $c = 7$ cm, $B = 120°$.
2 Check that the criterion for the area rule has been satisfied.	The area rule can be used since two side lengths and the included angle have been given.
3 Write down the appropriate rule for the area.	Area $= \frac{1}{2}ac \sin B$
4 Substitute the known values into the rule.	$= \frac{1}{2} \times 9 \times 7 \times \sin 120°$

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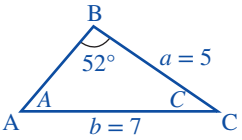
5	Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	$\approx 27.28 \text{ cm}^2$

WORKED EXAMPLE 8 TI CASIO

A triangle has known dimensions of $a = 5$ cm, $b = 7$ cm and $B = 52°$. Find A and C and hence the area.

	THINK	WRITE/DRAW
1	Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 5, b = 7, B = 52°$.
2	Check whether the criterion for the area rule has been satisfied.	The area rule cannot be used since the included angle has not been given.
3	Write down the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{b}{\sin B}$
4	Substitute the known values into the rule.	$\frac{5}{\sin A} = \frac{7}{\sin 52°}$
5	Transpose the equation to make $\sin A$ the subject.	$5 \sin 52° = 7 \sin A$ $\sin A = \frac{5 \sin 52°}{7}$
6	Evaluate.	$A = \sin^{-1}\left(\frac{5 \sin 52°}{7}\right)$ $\approx 34.254\,151\,87°$
7	Round off the answer to degrees and minutes.	$\approx 34°15'$
8	Determine the value of the included angle, C, using the fact that the angle sum of any triangle is 180°.	$C \approx 180° - (52° + 34°\ 15')$ $= 93°45'$
9	Write down the appropriate rule for the area.	Area $= \frac{1}{2} ab \sin C$
10	Substitute the known values into the rule.	$\approx \frac{1}{2} \times 5 \times 7 \times \sin 93°45'$
11	Evaluate. Round off the answer to 2 decimal places and include the appropriate unit.	$\approx 17.46 \text{ cm}^2$

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Heron's formula

- If the lengths of all the sides of the triangle are known but none of the angles are known, the cosine rule can be used to find an angle and then the area can be calculated. Alternatively, **Heron's formula** could be used to find the area.
- Heron's formula gives the area of a triangle as:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

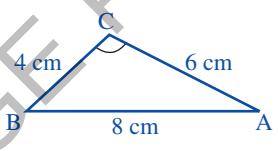
where s is the *semi-perimeter* of the triangle; that is,

$$s = \frac{1}{2}(a+b+c)$$

Note: The proof of this formula is beyond the scope of this course.

WORKED EXAMPLE 9

Find the area of the triangle with sides of 4 cm, 6 cm and 8 cm.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 4, b = 6, c = 8$.
2 Determine which area rule will be used.	Since three side lengths have been given, use Heron's formula.
3 Write down the rule for Heron's formula.	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
4 Write down the rule for s, the semi-perimeter of the triangle.	$s = \frac{1}{2}(a+b+c)$
5 Substitute the given values into the rule for the semi-perimeter.	$= \frac{1}{2}(4+6+8)$ $= 9$
6 Substitute all of the known values into Heron's formula.	$\text{Area} = \sqrt{9(9-4)(9-6)(9-8)}$
7 Evaluate.	$= \sqrt{9 \times 5 \times 3 \times 1}$ $= \sqrt{135}$ $\approx 11.618\,950\,04$
8 Round off the answer to 2 decimal places and include the appropriate unit.	$\approx 11.62 \text{ cm}^2$

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assess on

Exercise 20.4 Area of triangles

INDIVIDUAL PATHWAYS

REFLECTION
List three formulas for finding the area of a triangle.

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 12, 14, 16, 18, 20, 23	Questions: 1–6, 8, 9, 11, 13, 15, 17, 19, 21, 23	Questions: 1, 4, 5, 8–10, 12–14, 16–24

FLUENCY

1 WE7 Find the area of the triangle ABC with $a = 7$, $b = 4$ and $C = 68°$.

2 Find the area of the triangle ABC with $a = 7.3$, $c = 10.8$ and $B = 104°40'$.

3 Find the area of the triangle ABC with $b = 23.1$, $c = 18.6$ and $A = 82°17'$.

4 A triangle has $a = 10$ cm, $c = 14$ cm and $C = 48°$. Find A and B and hence the area.

5 WE8 A triangle has $a = 17$ m, $c = 22$ m and $C = 56°$. Find A and B and hence the area.

6 A triangle has $b = 32$ mm, $c = 15$ mm and $B = 38°$. Find A and C and hence the area.

7 MC In a triangle, $a = 15$ m, $b = 20$ m and $B = 50°$. The area of the triangle is:

A 86.2 m^2 B 114.9 m^2 C 149.4 m^2 D 172.4 m^2

8 WE9 Find the area of the triangle with sides of 5 cm, 6 cm and 8 cm.

9 Find the area of the triangle with sides of 40 mm, 30 mm and 5.7 cm.

10 Find the area of the triangle with sides of 16 mm, 3 cm and 2.7 cm.

11 MC A triangle has sides of length 10 cm, 14 cm and 20 cm. The area of the triangle is:

A 41 cm^2 B 65 cm^2 C 106 cm^2 D 137 cm^2

12 A piece of metal is in the shape of a triangle with sides of length 114 mm, 72 mm and 87 mm. Find its area using Heron's formula.

13 A triangle has the largest angle of 115°. The longest side is 62 cm and another side is 35 cm. Find the area of the triangle.

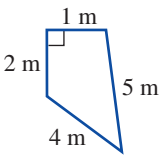
14 A triangle has two sides of 25 cm and 30 cm. The angle between the two sides is 30°. Find:

a its area b the length of its third side

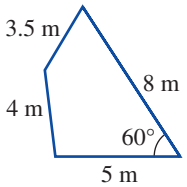
c its area using Heron's formula.

UNDERSTANDING

15 The surface of a fish pond has the shape shown in the diagram at right. How many goldfish can the pond support if each fish requires 0.3 m^2 surface area of water?



16 Find the area of this quadrilateral.



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REASONING

17 A parallelogram has diagonals of length 10 cm and 17 cm. An angle between them is 125°. Find:

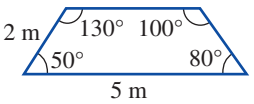
a the area of the parallelogram

b the dimensions of the parallelogram.

18 A lawn is to be made in the shape of a triangle, with sides of length 11 m, 15 m and 17.2 m. How much grass seed, to the nearest kilogram, needs to be purchased if it is sown at the rate of 1 kg per 5 m^2?

19 A bushfire burns out an area of level grassland shown in the diagram. (*Note*: This is a sketch of the area and is not drawn to scale.) What is the area, in hectares, of the land that is burned?

20 An earth embankment is 27 m long and has a vertical cross-section shown in the diagram. Find the volume of earth needed to build the embankment.



21 MC A parallelogram has sides of 14 cm and 18 cm and an angle between them of 72°. The area of the parallelogram is:

A 118.4 cm^2 **B** 172.4 cm^2 **C** 239.7 cm^2 **D** 252 cm^2

22 MC An advertising hoarding is in the shape of an isosceles triangle, with sides of length 15 m, 15 m and 18 m. It is to be painted with two coats of purple paint. If the paint covers 12 m^2 per litre, the amount of paint needed, to the nearest litre, would be:

A 9 L **B** 18 L

C 24 L **D** 36 L

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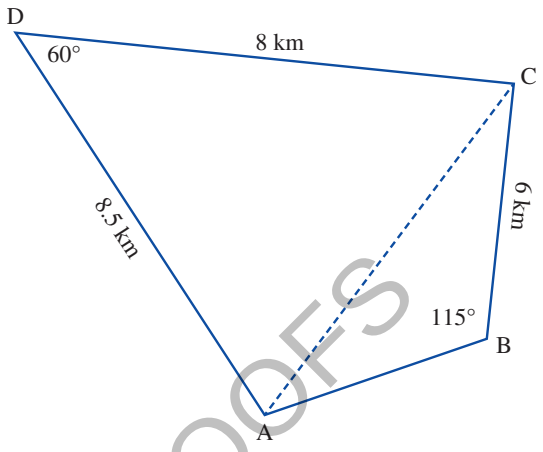
PROBLEM SOLVING

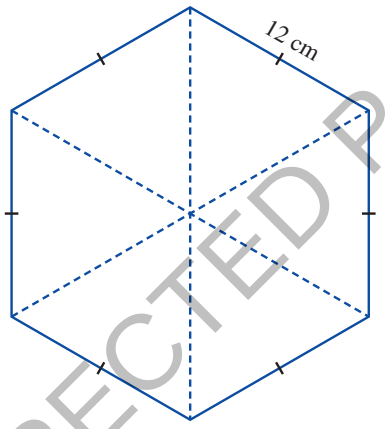
23 A surveyor measured the boundaries of a property as shown. The side AB could not be measured because it crossed through a marsh. The owner of the property wanted to know the total area and the length of the side AB.



- **a** Find the area of the triangle ACD.
- **b** Find the distance AC.
- **c** Find the angle CAB.
- **d** Find the angle ACB.
- **e** Find the length AB.
- **f** Find the area of the triangle ABC.
- **g** What is the area of the property?

Give all lengths correct to 2 decimal places and angles to the nearest degree.

24 A regular hexagon has sides of length 12 centimetres. It is divided into six smaller equilateral triangles. Find the area of the hexagon, giving your answer correct to 2 decimal places.



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CHALLENGE 20.1

Cosine, sine and area rule applied to right-angled triangles

The cosine rule states that $c^2 = a^2 + b^2 - 2ab \cos C$. If $C° = 90°$ and $\cos 90° = 0$, then:

$c^2 = a^2 + b^2 - 2ab(0)$

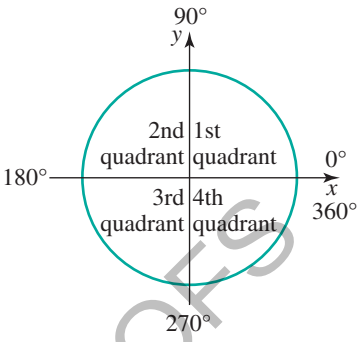
$c^2 = a^2 + b^2$.

Hence, when the consine rule is applied to a right-angled triangle, you are left with Pythagoras' theorem for right-angled triangles.

Investigate what happens when the sine and area rules are applied to right-angled triangles.

20.5 The unit circle

- A **unit circle** is a circle with a radius of 1 unit.
- The unit circle is divided into 4 quadrants.
- All angles in quadrant 1 are between 0° and 90°. All angles in quadrant 2 are between 90° and 180°, in quadrant 3 between 180° and 270°, and in quadrant 4 between 270° and 360°.
- The quadrants are numbered in the anticlockwise direction, and positive angles are measured anticlockwise from 0°. Negative angles are measured clockwise from 0°.



WORKED EXAMPLE 10

State the quadrant of the unit circle in which each of the following angles is found.

a **145°** **b** **282°**

THINK **WRITE**

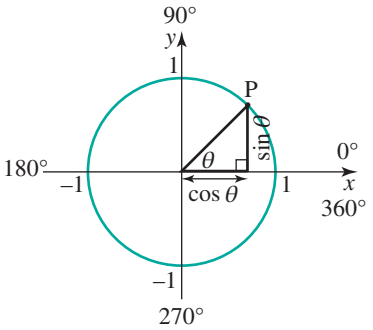
a The given angle is between 90° and 180°. State the appropriate quadrant.

a 145° is in quadrant 2.

b The given angle is between 270° and 360°. State the appropriate quadrant.

b 282° is in quadrant 4.

- If a right-angled triangle containing angle θ is constructed in quadrant 1 of the unit circle, then the value of sin θ can be found by measuring the length of the opposite side and the value of cos θ by measuring the length of the adjacent side.



- The point of intersection of the radius (which is one of the arms of angle θ) with the unit circle is P. From the diagram above, observe that cos θ represents the x-coordinate of point P and sin θ represents its y-coordinate. This observation provides us with the technique for finding sine and cosine of any angle in the unit circle, as shown in the diagram at the top of this page.
- To find the value of sine, cosine or tangent of any angle θ from the unit circle, follow these steps:
  1. Draw a unit circle.
  2. Construct the required angle so that its vertex is at the origin and the angle itself is measured from 0° (as marked on the x-axis) in an anticlockwise direction. Label the point of intersection of the radius and the unit circle, P.

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3. Use a ruler to find the coordinates of point P.
4. Interpret the results: $x = \cos\theta$, $y = \sin\theta$ and $\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$, where x and y are coordinates of P.

The 4 quadrants of the unit circle

For the following activity and exercise, we will need to be able to read values for sine and cosine from a unit circle.

Constructing your own unit circle

Step 1. Using graph paper, carefully draw a circle centred at the origin and with a radius of 5 cm. Label the x- and y-axes.

Step 2. On your graph, mark in 0°, 90°, 180°, 270° and 360°.

Step 3. Since we need a unit circle, 5 cm will represent 1 unit; that is, 5 cm = 1 unit. Carefully mark a scale on each axis, where each centimetre represents 0.2 units. (Draw as carefully as possible, since you will need to read values from your axes in Exercise 20.5.)

Use a unit circle to investigate the following.

1. If P has coordinates (x, y) and is located on the unit circle, what is the highest value that the x-coordinate can take? Hence, what is the largest value that the cosine of an angle can take?
2. What is the lowest value that the x-coordinate can take? Hence, what is the smallest value that the cosine of an angle can take?
3. What is the highest value that the y-coordinate can take? Hence, what is the largest value that the sine of an angle can take?
4. What is the lowest value that the y-coordinate can take? Hence, what is the smallest value that the sine of an angle can take?
5. Note that P could be in any of the four quadrants (depending on the size of the angle). Hence, its coordinates could take either positive or negative values, or zero.
   Copy and complete the table below to summarise whether sine, cosine and tangent are positive or negative for angles in each of the four quadrants.

	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	positive (+)		negative (−)	
$\cos\theta$	positive (+)	negative (−)		
$\tan\theta$	positive (+)			negative (−)

6. Copy and complete the following sentences.
   Sine is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.
   Cosine is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.
7. Use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to work out whether the tangent of an angle in each quadrant is positive or negative. Copy and complete the following sentence: Tangent is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.

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WORKED EXAMPLE 11 TI CASIO

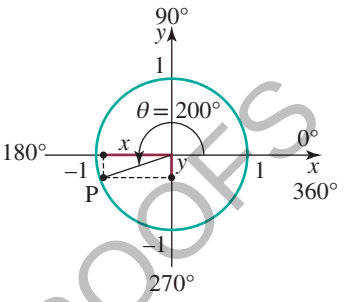
Find the approximate value of each of the following using the unit circle.

a **sin 200°** **b** **cos 200°** **c** **tan 200°**

THINK

Draw a unit circle and construct an angle of 200°. Label the point corresponding to the angle of 200° on the circle P. Highlight the lengths, representing the x- and y-coordinates of point P.

WRITE/DRAW



a The sine of the angle is given by the y-coordinate of P. Find the y-coordinate of P by measuring the distance along the y-axis. State the value of sin 200°. (*Note:* The sine value will be negative as the y-coordinate is negative.)

a $\sin 200° = -0.3$

b The cosine of the angle is given by the x-coordinate of P. Find the x-coordinate of P by measuring the distance along the x-axis. State the value of cos 200°. (*Note:* Cosine is also negative in quadrant 3, as the x-coordinate is negative.)

b $\cos 200° = -0.9$

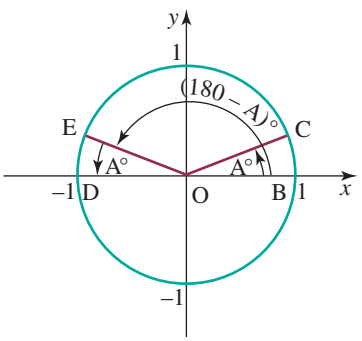
c $\tan 200° = \dfrac{\sin 200°}{\cos 200°}$

c $\dfrac{-0.3}{-0.9} = \dfrac{1}{3} = 0.3333$

- The approximate results obtained in Worked example 11 can be verified with the aid of a calculator:

  $\sin 200° = -0.342\,020\,143$, $\cos 200° = -0.939\,692\,62$ and $\tan 200° = 0.3640$.

  Rounding these values to 1 decimal place would give -0.3, -0.9 and 0.4 respectively, which match the values obtained from the unit circle.
- Consider the special relationship between the sine, cosine and tangent of supplementary angles, say $A°$ and $(180 - A)°$.
- In the diagram, triangle OBC is reflected in the y-axis to form triangle ODE. Clearly, $\angle BOC = \angle DOE = A°$. It follows that the obtuse angle BOE equals $(180 - A)°$.



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From the diagram, $|BC| = |ED|$ and $|OB| = |OD|$.
$ED = \sin(180 - A)$ and is positive.
$OD = \cos(180 - A)$ and is negative.
Thus, $\sin(180 - A) = \sin A$ and $\cos(180 - A) = -\cos A$
It follows that $\tan(180 - A) = \dfrac{\sin(180 - A)}{\cos(180 - A)} = \dfrac{\sin A}{-\cos A} = -\tan A$.

Exercise 20.5 The unit circle

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER	REFLECTION
Questions: 1, 4–15, 18	Questions: 1–6, 8, 10, 12, 15, 17, 18	Questions: 1–19	What is the length of the diameter of the unit circle?

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FLUENCY

1 WE10 State which quadrant of the unit circle each of the following angles is in.

a 60° **b** 130° **c** 310° **d** 260°
e 100° **f** 185° **g** 275° **h** 295°

2 MC If $\theta = 43°$, the triangle drawn to show this would be in:

A quadrant 1 **B** quadrant 2 **C** quadrant 3 **D** quadrant 4

3 MC If $\theta = 295°$, the triangle drawn to show this would be in:

A quadrant 1 **B** quadrant 2 **C** quadrant 3 **D** quadrant 4

4 WE11 Find the value of each of the following using the unit circle.

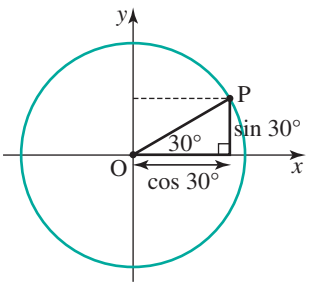
a sin 20° **b** cos 20° **c** cos 100° **d** sin 100°
e sin 320° **f** cos 320° **g** sin 215° **h** cos 215°

5 Use the unit circle to find each of the following.

a sin 90° **b** cos 90° **c** sin 180° **d** cos 180°
e sin 270° **f** cos 270° **g** sin 360° **h** cos 360°

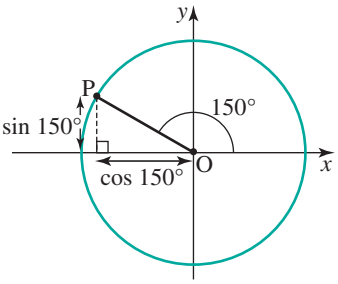
UNDERSTANDING

6 On the unit circle, use a protractor to measure an angle of 30° from the positive x-axis. Mark the point P on the circle. Use this point to construct a triangle in quadrant 1 as shown.



a Find cos 30°. (Remember that the length of the adjacent side of the triangle is cos 30°.)
b Find sin 30°. (This is the length of the opposite side of the triangle.)
c Check your answers in **a** and **b** by finding these values with a calculator.

7 Using your graph of the unit circle, measure 150° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 2 as shown.



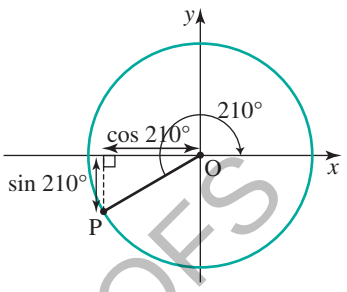
a What angle does the radius OP make with the negative x-axis?
b Remembering that $x = \cos\theta$, use your circle to find the value of cos 150°.

c How does cos 150° compare to cos 30°?

d Remembering that $y = \sin\theta$, use your circle to find the value of sin 150°.

e How does sin 150° compare with sin 30°?

8 On the unit circle, measure 210° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 3 as shown.



a What angle does the radius OP make with the negative x-axis?

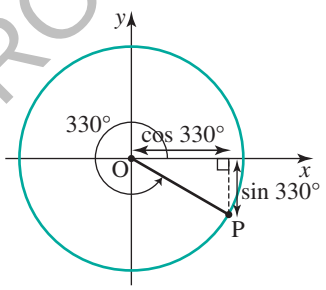
b Use your circle to find the value of cos 210°.

c How does cos 210° compare to cos 30°?

d Use your circle to find the value of sin 210°.

e How does sin 210° compare with sin 30°?

9 On the unit circle, measure 330° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 4 as shown.



a What angle does the radius OP make with the positive x-axis?

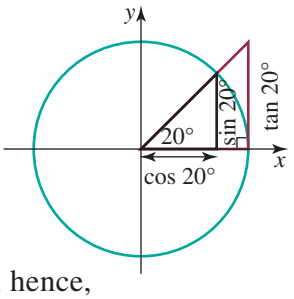
b Use your circle to find the value of cos 330°.

c How does cos 330° compare to cos 30°?

d Use your circle to find the value of sin 330°.

e How does sin 330° compare with sin 30°?

10 On the unit circle, draw an appropriate triangle for the angle of 20° in quadrant 1.



a Find sin 20°.

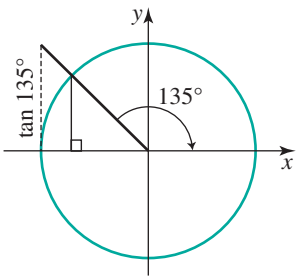
b Find cos 20°.

c Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
Accurately measure the length of the tangent between the x-axis and the point where it meets the hypotenuse and, hence, state the value of tan 20°.

d What is the value of $\frac{\sin 20°}{\cos 20°}$?

e How does tan 20° compare with $\frac{\sin 20°}{\cos 20°}$?

11 On the unit circle, draw an appropriate triangle for the angle of 135° in quadrant 2.



a Find sin 135°, using sin 45°.

b Find cos 135°, using cos 45°.

c Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.

Accurately measure the length of the tangent to where it meets the hypotenuse to find the value of tan 135°.

d What is the value of $\frac{\sin 135°}{\cos 135°}$?

e How does tan 135° compare with $\frac{\sin 135°}{\cos 135°}$?

f How does tan 135° compare with tan 45°?

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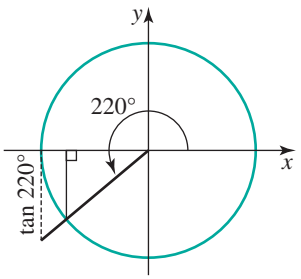
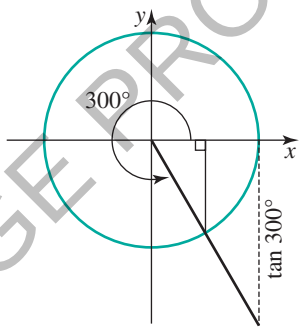
12 On the unit circle, draw an appropriate triangle for the angle of 220° in quadrant 3.



- **a** Find sin 220°.
- **b** Find cos 220°.
- **c** Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
  Find tan 220° by accurately measuring the length of the tangent to where it meets the hypotenuse.
- **d** What is the value of $\frac{\sin 220°}{\cos 220°}$?
- **e** How does tan 220° compare with $\frac{\sin 220°}{\cos 220°}$?
- **f** How does tan 220° compare with tan 40°? (Use a calculator.)

13 On the unit circle, draw an appropriate triangle for the angle of 300° in quadrant 4.



- **a** Find sin 300°.
- **b** Find cos 300°.
- **c** Draw a tangent line and extend the hypotenuse of the triangle to meet the tangent as shown.
  Find tan 300° by accurately measuring the length of the tangent to where it meets the hypotenuse.
- **d** What is the value of $\frac{\sin 300°}{\cos 300°}$?
- **e** How does tan 300° compare with $\frac{\sin 300°}{\cos 300°}$?
- **f** How does tan 300° compare with tan 60°? (Use a calculator.)

14 MC In a unit circle, the length of the radius is equal to:

A $\sin\theta$ **B** $\cos\theta$ **C** $\tan\theta$ **D** 1

REASONING

15 If $\sin x° = p$, $0 \leq x \leq 90°$, write each of the following in terms of p.

a $\cos x°$ **b** $\sin(180 - x)°$ **c** $\cos(180 - x)°$

16 Simplify $\sin(180 - x)° - \sin x°$.

17 The point P $(\cos\theta, \sin\theta)$ lies in the first quadrant on the circumference of a unit circle. Find the new coordinates of P under the following transformations.

- **a** Dilated by a factor of 2 horizontally and then 3 vertically
- **b** Rotated 180° anticlockwise about the origin
- **c** Reflected in both axes

PROBLEM SOLVING

18 Simplify:

a $\frac{\sin(180° - \theta)\cos(270° + \theta)}{\tan(-\theta)}$, given that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b $\frac{\sin^2(90° - \theta) - \cos^2(270° - \theta)}{\cos(90° - \theta) + \cos(-\theta)}$.

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19 Simplify:

a $\dfrac{\sin(90° - x)\cos(90° - x)}{\cos^2(x)}$

b $\sin^2(180° - x)\ \text{cosec}(270° + x) + \cos^2(360° - x)\sec(180° - x)$

where $\sec(x) = \dfrac{1}{\cos(x)}$ and $\text{cosec}(x) = \dfrac{1}{\sin(x)}$.

20.6 Trigonometric functions

Sine, cosine and tangent graphs

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Trigonometric functions
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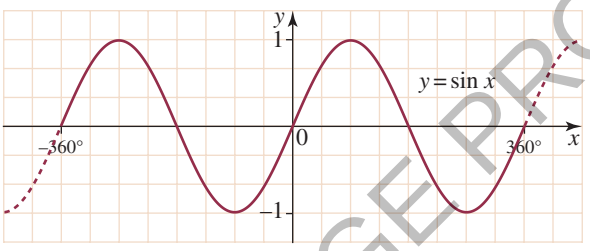
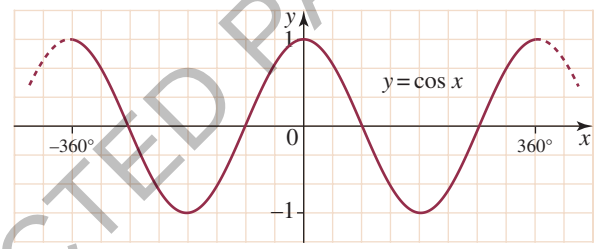
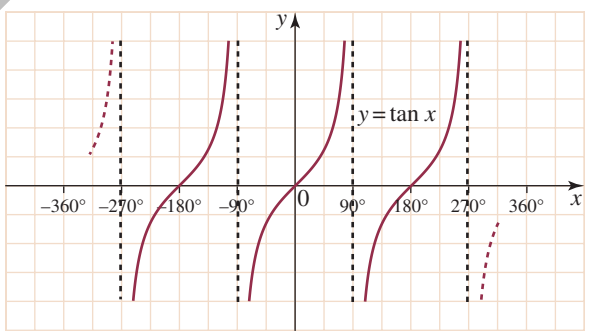
- The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ are shown below.







- Trigonometric graphs repeat themselves continuously in cycles, and hence they are called **periodic functions**.
- The period of the graph is the distance between repeating peaks or troughs. In the graphs shown above, the period between the repeating peaks for $y = \sin x$ and $y = \cos x$ is 360°. The graph of $y = \tan x$ shows a periodic function, but it is not of the same form as $y = \sin x$ and $y = \cos x$. The period of the basic tan graph is 180°, and asymptotes occur at $x = 90°$ and intervals of 180°.

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- The **amplitude** of a periodic graph is half the distance between the maximum and minimum values of the function. In the graphs of $y = \sin x$ and $y = \cos x$ shown on the previous page, the distance is half of two units (the distance between -1 and 1), hence the amplitude is 1 unit. Amplitude can also be described as the amount by which the graph goes above and below its mean value. In the above examples, the mean value lies along the x-axis. The amplitude of $y = \tan x$ is undefined.

WORKED EXAMPLE 12

TI | CASIO

Sketch the graphs of a $y = 2 \sin x$ and b $y = \cos 2x$ for $0° \leq x \leq 360°$.

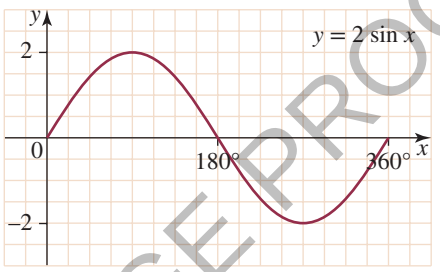
THINK

a
1. The graph must be drawn from 0° to 360°.
2. Compared to the graph of $y = \sin x$ each value of $\sin x$ has been multiplied by 2, therefore the amplitude of the graph must be 2.
3. Label the graph $y = 2 \sin x$.

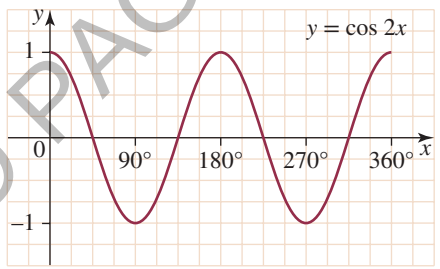
b
1. The graph must be drawn from 0° to 360°.
2. Compared to the graph of $y = \cos x$, each value of x has been multiplied by 2, therefore the period of the graph must become 180°.
3. Label the graph $y = \cos 2x$.

WRITE/DRAW

a



b



- For the graph of $y = a \sin nx$, or $y = a \cos nx$, the amplitude is a and the period becomes $\frac{360°}{n}$.
- If the graph has a negative value of a, the amplitude is the positive value of a, i.e. the amplitude is always $|a|$.
- For the graph of $y = a \tan nx$, the amplitude is undefined and the period is $\frac{180°}{n}$.

WORKED EXAMPLE 13

For each of the following graphs, state:
i the amplitude
ii the period.

a $y = 2 \sin 3x$

b $y = \cos \frac{x}{3}$

c $y = \tan 2x$

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THINK	WRITE
a The value of a is 2. The period is $\frac{360°}{n}$.	**a** **i** Amplitude $= 2$ **ii** Period $= \frac{360}{3}$ $= 120°$
b The value of a is 1. The period is $\frac{360°}{n}$.	**b** **i** Amplitude $= 1$ **ii** Period $= \frac{360}{\frac{1}{3}}$ $= 1080°$
c The tangent curve has infinite amplitude. The period is $\frac{180°}{2}$.	**c** **i** Amplitude infinite **ii** Period $= \frac{180°}{2}$ $= 90°$

Exercise 20.6 Trigonometric functions

REFLECTION
For the graph of $y = a \tan nx$, what would be the period and amplitude?

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–19, 22, 24, 27	Questions: 1–21, 23–27	Questions: 1–20, 22, 24, 26–28

FLUENCY

1 Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\sin x$													

2 On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from −1 to 1. Carefully plot the graph of $y = \sin x$ using the values from the table in question **1**.

3 How long does it take for the graph of $y = \sin x$ to complete one full cycle?

4 From your graph of $y = \sin x$, find the value of y for each of the following.

a $x = 42°$ **b** $x = 130°$ **c** $x = 160°$ **d** $x = 200°$
e $x = 180°$ **f** $x = 70°$ **g** $x = 350°$ **h** $x = 290°$

5 From your graph of $y = \sin x$, find the value of x for each of the following.

a $y = 0.9$ **b** $y = -0.9$ **c** $y = 0.7$
d $y = -0.5$ **e** $y = -0.8$ **f** $y = 0.4$

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6 Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos x$													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\cos x$													

7 On graph paper, rule x- and y-axes and carefully mark a scale along each axis.
Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°.
Use 2 cm = 1 unit along the y-axis to show y-values from −1 to 1.
Carefully plot the graph of $y = \cos x$ using the values from the table in question **6**.

8 If you were to continue the graph of $y = \cos x$, what shape would you expect it to take?

9 Is the graph of $y = \cos x$ the same as the graph of $y = \sin x$? How does it differ? What features are the same?

10 Using the graph of $y = \cos x$, find a value of y for each of the following.

a 48° **b** 170° **c** 180° **d** 340°
e 240° **f** 140° **g** 40° **h** 165°

11 Using the graph of $y = \cos x$, find a value of x for each of the following.

a $y = -0.5$ **b** $y = 0.8$ **c** $y = 0.7$
d $y = -0.6$ **e** $y = 0.9$ **f** $y = -0.9$

12 Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\tan x$													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\tan x$													

13 On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from −2 to 2. Carefully plot the graph of $y = \tan x$ using the values from the table in question **12**.

14 If you were to continue the graph of $y = \tan x$, what shape would you expect it to take?

15 Is the graph of $y = \tan x$ the same as the graphs of $y = \sin x$ and $y = \cos x$? How does it differ? What features are the same?

16 Using the graph of $y = \tan x$, find a value of y for each of the following.

a 60° **b** 135° **c** 310° **d** 220°
e 500° **f** 590° **g** 710° **h** 585°

17 Using the graph of $y = \tan x$, find a value of x for each of the following.

a $y = 1$ **b** $y = 1.5$ **c** $y = -0.4$
d $y = -2$ **e** $y = 0.2$ **f** $y = -1$

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18 WE12 Sketch the following graphs

a $y = \cos x$, for $x \in [-180°, 180°]$ **b** $y = \sin x$, for $x \in [0°, 720°]$

c $y = \sin 2x$, for $x \in [0°, 360°]$ **d** $y = 2\cos x$, for $x \in [-360°, 0°]$

19 WE13 For each of the graphs in question **18**, state

i the period **ii** the amplitude.

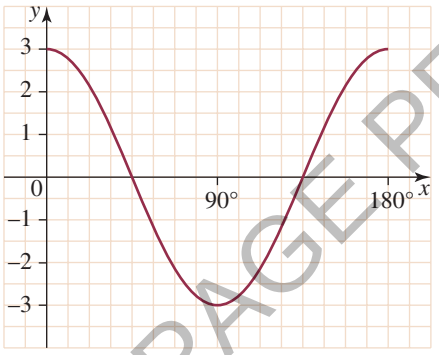
20 For each of the following, state:

i the period **ii** the amplitude.

a $y = 3\cos 2x$ **b** $y = 4\sin 3x$ **c** $y = 2\cos\frac{x}{2}$

d $y = \frac{1}{2}\sin\frac{x}{4}$ **e** $y = -\sin x$ **f** $y = -\cos 2x$

21 MC Parts **a** to **c** refer to the graph below.



a The amplitude of the graph is:

A 180° **B** 90° **C** 3 **D** −3 **E** 6

b The period of the graph is:

A 180° **B** 360° **C** 90° **D** 3 **E** −3

c The equation of the graph could be:

A $y = \cos x$ **B** $y = \sin x$ **C** $y = 3\cos\frac{x}{3}$

D $y = 3\cos 2x$ **E** $y = 3\sin 2x$

UNDERSTANDING

22 Sketch each of the following graphs, stating the period and amplitude of each.

a $y = 2\cos\frac{x}{3}$, for $x \in [0°, 360°]$ **b** $y = -3\sin 2x$, for $x \in [0°, 360°]$

c $y = 3\sin\frac{x}{2}$, for $x \in [-180°, 180°]$ **d** $y = -\cos 3x$, for $x \in [0°, 360°]$

e $y = 5\cos 2x$, for $x \in [0°, 180°]$ **f** $y = -\sin 4x$, for $x \in [0°, 180°]$

23 Use technology to sketch the graphs of each of the following for $0° \leq x \leq 360°$.

a $y = \cos x + 1$ **b** $y = \sin 2x - 2$

c $y = \cos(x - 60°)$ **d** $y = 2\sin 4x + 3$

REASONING

24 a Sketch the graph of $y = \cos 2x$ for $x \in [0°, 360°]$.

i What is the minimum value of y for this graph?

ii What is the maximum value of y for this graph?

b Using the answers obtained in part **a** write down the maximum and minimum values of $y = \cos 2x + 2$.

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c What would be the maximum and minimum values of the graph of $y = 2 \sin x + 3$? Explain how you obtained these values.

25 a Complete the table below by filling in the exact values of $y = \tan x$

x	0°	30°	60°	90°	120°	150°	180°
$y = \tan x$							

b Sketch the graph of $y = \tan x$ for $[0°, 180°]$.

c What happens at $x = 90°$?

d For the graph of $y = \tan x$, $x = 90°$ is called an asymptote. Write down when the next asymptote would occur.

e State the period and amplitude of $y = \tan x$.

26 a Sketch the graph of $y = \tan 2x$ for $[0°, 180°]$.

b When do the asymptotes occur?

c State the period and amplitude of $y = \tan 2x$.

PROBLEM SOLVING

27 The height of the tide above the mean sea level on the first day of the month is given by the rule

$$h = 3 \sin(30t°)$$

where t is the time in hours since midnight.

a Sketch the graph of h versus t.

b What was the height of the high tide?

c Calculate the height of the tide at 8 am.

28 The temperature inside a house t hours after 3 am is given by the rule

$$T = 22 - 2 \cos(15t°) \text{ for } 0 \leq t \leq 24$$

where t is the temperature in degrees Celsius.

a What is the temperature inside the house at 6 am?

b Sketch the graph of T versus t.

c What is the warmest and coolest that it gets inside the house over the 24-hour period?

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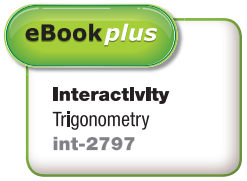
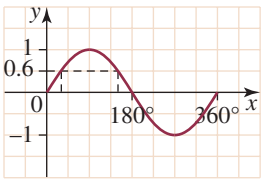
20.7 Solving trigonometric equations

Solving trigonometric equations graphically



- Because of the periodic nature of circular functions, there are infinitely many solutions to trigonometric equations.
- Equations are usually solved within a particular domain (x-values), to restrict the number of solutions.
- The sine graph below shows the solutions between 0° and 360° for the equation $\sin x° = 0.6$.



In the example above, it can clearly be seen that there are two solutions to this equation, which are approximately $x = 37°$ and $x = 143°$.

- The smaller the period, the greater the number of solutions within a particular domain.
- It is difficult to obtain accurate answers from a graph. More accurate answers can be obtained using technology.

Solving trigonometric equations algebraically

Exact answers can be found for some trigonometric equations using the table at the top of page 845.

WORKED EXAMPLE 14 TI CASIO

Solve the following equations.

a $\sin x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b** $\cos 2x = -\frac{1}{\sqrt{2}}, x \in [0°, 360°]$

THINK	WRITE
a 1 The inverse operation of sine is $\sin^{-1}$.	**a** $x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
2 The first solution in the given domain from the table at the top of page 845 is $x = 60°$.	
3 Since sine is positive in the first and second quadrants, another solution must be $x = 180° - 60° = 120°$.	There are two solutions in the given domain, $x = 60°$ and $x = 120°$.
b 1 The inverse operation of cosine is $\cos^{-1}$.	**b** $2x = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$
2 From the table of values, $\cos^{-1}\frac{1}{\sqrt{2}} = 45°$.	
3 Cosine is negative in the second and third quadrants, which gives the first two solutions to the equation as: $180° - 45°$ and $180° + 45°$.	$2x = 135°, 225°$
4 Solve for x by dividing by 2.	$x = 67.5°, 112.5°$
5 Since the domain in this case is $[0°, 360°]$ and the period has been halved, there must be 4 solutions altogether. The other 2 solutions can be found by adding the period onto each solution.	The period $= \frac{360°}{2} = 180°$ $x = 67.5° + 180°, 112.5° + 180°$ $x = 67.5°, 112.5°, 247.5°, 292.5°$

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assess on

Exercise 20.7 Solving trigonometric equations

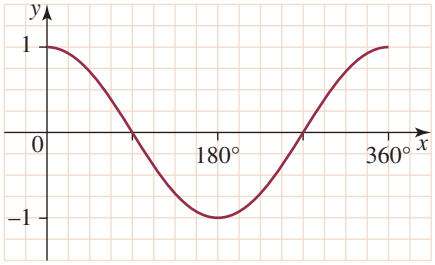
INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6	Questions: 1–7	Questions: 1–7

REFLECTION
Explain why sine and cosine functions can be used to model situations that occur in nature such as tide heights and sound waves.

FLUENCY

1 Use the graph at right to find approximate answers to the following equations for the domain $0 \leq x \leq 360°$. Check your answers using a calculator.



i $\cos x = 0.9$

ii $\cos x = 0.3$

iii $\cos x = -0.2$

iv $\cos x = -0.6$

2 Solve the following equations for the domain $0° \leq x \leq 360°$.

a $\sin x = \frac{1}{2}$ **b** $\sin x = \frac{\sqrt{3}}{2}$ **c** $\cos x = -\frac{1}{2}$ **d** $\cos x = -\frac{1}{\sqrt{2}}$

e $\sin x = 1$ **f** $\cos x = -1$ **g** $\sin x = -\frac{1}{2}$ **h** $\sin x = -\frac{1}{\sqrt{2}}$

i $\cos x = \frac{\sqrt{3}}{2}$ **j** $\cos x = -\frac{\sqrt{3}}{2}$ **k** $\sin x = 1$ **l** $\cos x = 0$

UNDERSTANDING

3 WE14 Solve the following equations for the given values of x.

a $\sin 2x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b** $\cos 2x = -\frac{\sqrt{3}}{2}, x \in [0°, 360°]$

c $\tan 2x = \frac{1}{\sqrt{3}}, x \in [0°, 360°]$ **d** $\sin 3x = \frac{-1}{\sqrt{2}}, x \in [-180°, 180°]$

e $\sin 4x = -\frac{1}{2}, x \in [0°, 180°]$ **f** $\sin 3x = -\frac{1}{\sqrt{2}}, x \in [-180°, 180°]$

g $\tan 3x = -1, x \in [0°, 90°]$ **h** $\cos 3x = 0, x \in [0°, 360°]$

REASONING

4 Solve the following equations for $x \in [0°, 360°]$.

a $2 \sin x - 1 = 0$ **b** $2 \cos x = \sqrt{3}$

c $\sqrt{2} \cos x - 1 = 0$ **d** $\tan x + 1 = 0$

5 Sam measured the depth of water at the end of the Intergate jetty at various times on Friday 13 August 2010. The table below provides her results.

Time	6 am	7	8	9	10	11	12 pm	1	2	3	4	5	6	7	8	9
Depth	1.5	1.8	2.3	2.6	2.5	2.2	1.8	1.2	0.8	0.5	0.6	1.0	1.3	1.8	2.2	2.5

a Plot the data.

b Determine:

i the period **ii** the amplitude.

c Sam fishes from the jetty when the depth is a maximum. Specify these times for the next 3 days.

d Sam's mother can moor her yacht when the depth is above 1.5 m. During what periods can she moor the yacht on Sunday 16 January?

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PROBLEM SOLVING

6 Solve:

a $\sqrt{3}\sin(x°) = \cos(x°)$ for $0° \leq x \leq 360°$

b $2\sin(x°) + \cos(x°) = 0$ for $0° \leq x \leq 360°$.

7 Solve $2\sin^2(x°) + 3\sin(x°) - 2 = 0$ for $0° \leq x \leq 360°$.

CHALLENGE 20.2

The grad (^{g}) is another measurement used when measuring the size of angles. A grad is equivalent to $\frac{1}{400}$ of a full circle. Write each of the following as grads (1 grad is written as 1^g).

1 90° **2** 180° **3** 270° **4** 360°

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20.8 Review

eBookplus ONLINE ONLY

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for students to demonstrate their knowledge of this topic.

The Review contains:

- **Fluency** questions — allowing students to demonstrate the skills they have developed to efficiently answer questions using the most appropriate methods
- **Problem Solving** questions — allowing students to demonstrate their ability to make smart choices, to model and investigate problems, and to communicate solutions effectively.

A summary of the key points covered and a concept map summary of this topic are available as digital documents.

Review questions

Download the Review questions document from the links found in your eBookPLUS.

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eBookplus

Interactivities

Word search
int-2883

Crossword
int-2884

Sudoku
int-3895

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

ambiguous case
amplitude
area
asymptote
cosine
cosine graphs
cosine rule
exact value
Heron's formula
period
periodic function
quadrants
sine
sine graphs
sine rule
trigonometric equation
trigonometric function
unit circle

Link to assessON for questions to test your readiness **FOR** learning, your progress **AS** you learn and your levels **OF** achievement.

assessON

assessON provides sets of questions for every topic in your course, as well as giving instant feedback and worked solutions to help improve your mathematical skills.

www.assesson.com.au

The story of mathematics is an exclusive Jacaranda video series that explores the history of mathematics and how it helped shape the world we live in today.

Agnesi (eles-2023) tells the story of Maria Agnesi, an Italian mathematician who made some remarkable contributions to the field of mathematics, including publishing the influential textbook, *Analytical Institutions*.

INVESTIGATION

RICH TASK

What's an arbelos?

As an introduction to this task, you are required to complete the following construction. The questions that follow require the application of measurement formulas, and an understanding of semicircles related to this construction.

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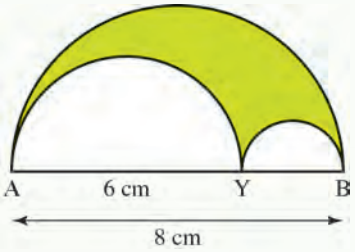
1 Constructing an arbelos

- In the space below, rule a horizontal line AB 8 cm long.
- Find the midpoint of the line and construct a semicircle on top of the line with AB as the diameter.
- Mark Y as a point on AB such that AY = 6 cm.
- Find the midpoint of AY and draw a small semicircle inside the larger semicircle with AY as the diameter.
- Find the midpoint of YB and construct a semicircle (also inside the larger semicircle) with a diameter YB.



The shape enclosed by the three semicircles is known as an arbelos. The word, in Greek, means *shoemaker's knife* as it resembles the blade of a knife used by cobblers. The point Y is not fixed and can be located anywhere along the diameter of the larger semicircle, which can also vary in size.

2 Perimeter of an arbelos

The perimeter of an arbelos is the sum of the arc length of the three semicircles. Perform the following calculations, leaving each answer in terms of π.

a Calculate the arc length of the semicircle with diameter AB.

b Calculate the arc length of the semicircle with diameter AY.

c Calculate the arc length of the semicircle on diameter YB.

d Compare the largest arc length with the two smaller arc lengths. What do you conclude?

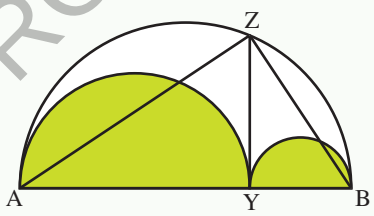
3 We can generalise the arc length of an arbelos. The point Y can be located anywhere on the line AB, which can also vary in length. Let the diameter AB be d cm, AY be d_1 cm and YB be d_2 cm. Prove that your conclusion from question **2 d** holds true for any value of d, where $d_1 + d_2 = d$.

4 Area of an arbelos

The area of an arbelos may be treated as the area of a composite shape.

a Using your original measurements, calculate the area of the arbelos you drew in question **1**. Leave your answer in terms of π.

The area of the arbelos can also be calculated using another method. We can draw the common tangent to the two smaller semicircles at their point of contact and extend this tangent to the larger semicircle. It is said that the area of the arbelos is the same as the area of the circle constructed on this common tangent as diameter.



YZ is the common tangent.

Triangles AYZ, BYZ and AZB are all right-angled triangles. We can use Pythagoras' theorem, together with a set of simultaneous equations, to determine the length of the tangent YZ.

b Complete the following.

In ΔAYZ, $AZ^2 = AY^2 + YZ^2$

$= 6^2 + YZ^2$

In ΔBYZ, $BZ^2 = BY^2 + YZ^2$

$= \ldots\ldots\ldots\ldots + YZ^2$

Adding these two equations,

$AZ^2 + BZ^2 = \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots$

But, in ΔAZB, $AZ^2 + BZ^2 = AB^2$

$\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

So,

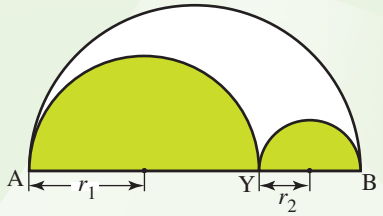
YZ = ... (Leave your answer in surd form.)

c Now calculate the area of the circle with diameter YZ. Is your answer the same as that calculated in question **4 a**?

The area of an arbelos can be generalised.

Let the radii of the two smaller semicircles be r_1 and r_2.

5 Develop a formula for the area of the arbelos in terms of r_1 and r_2. Demonstrate the use of your formula by checking your answer to question **4 a**.



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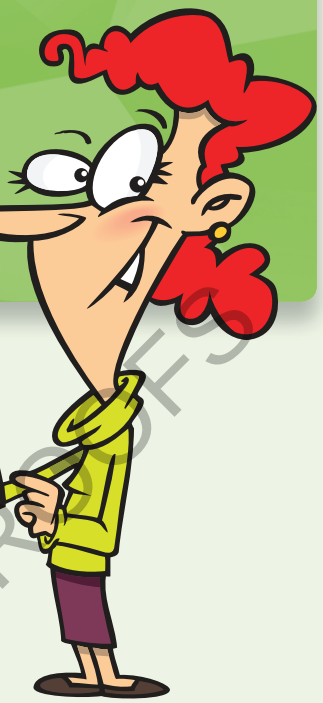
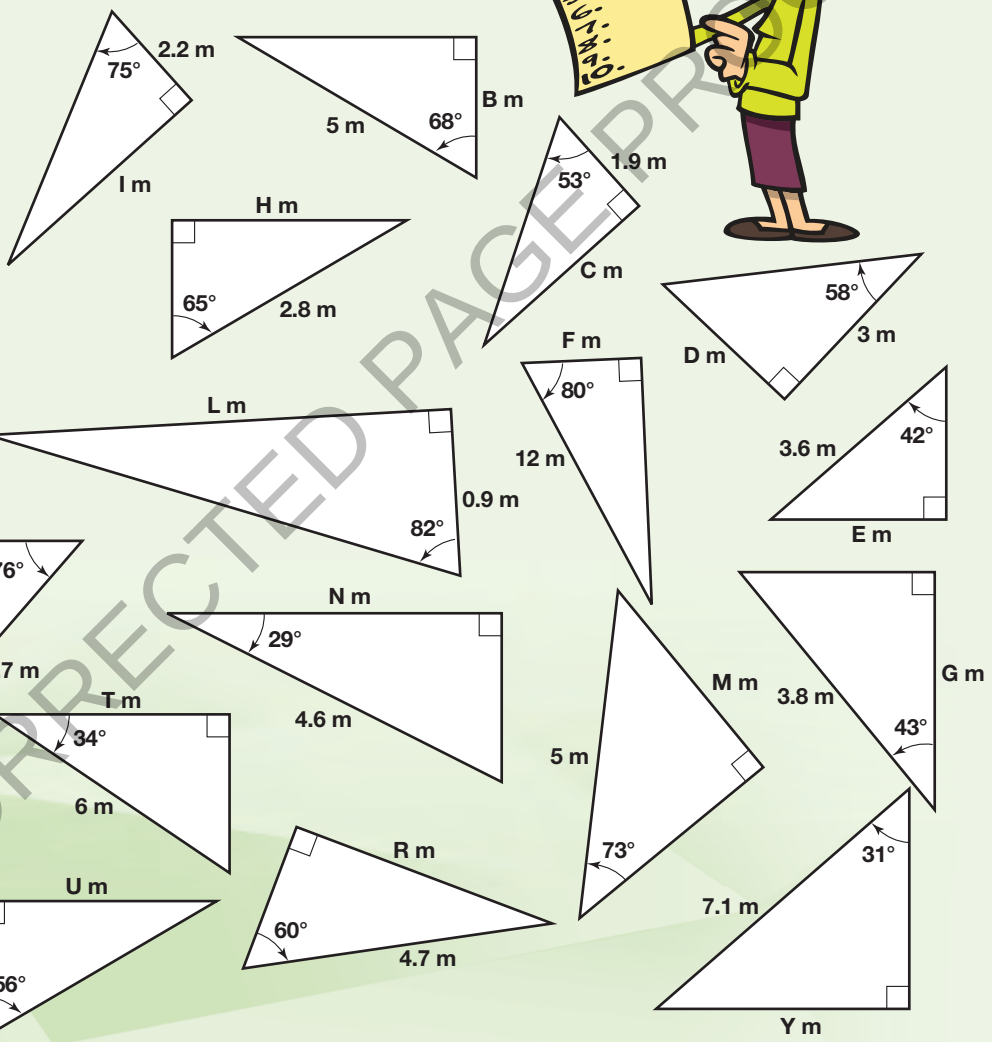
CODE PUZZLE

What is the periodic table?



Find the lengths of the lettered sides accurate to 2 decimal places to discover the puzzle's answer code.



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3.13	4.02	2.10	4.07	4.80	2.41	4.07	2.41	4.80	4.97	3.13	1.87	6.40	2.41

2.10	2.08	3.13	6.40	6.40	4.97	2.54	2.41	2.41	6.40	2.41	4.78	2.41	4.02	4.97	4.29

3.13	4.07	4.07	3.13	4.02	2.78	2.41	4.80	1.87	3.66	4.97	2.54	2.41	8.21	4.07

3.13	4.97	2.10	4.78	8.21	2.52	4.02	4.45	4.78	1.87	2.41	4.07	4.29

eBookplus Activities

20.1 Overview
Video
- The story of mathematics (eles-2023)

20.2 The sine rule
Digital docs
- SkillSHEET (doc-5398): Labelling right-angled triangles
- SkillSHEET (doc-5399): Calculating sin, cos or tan of an angle
- SkillSHEET (doc-5400): Finding side lengths in right-angled triangles
- SkillSHEET (doc-5401): Calculating the angle from a sin, cos or tan ratio
- SkillSHEET (doc-5402): Finding angles in right-angled triangles
- WorkSHEET 20.1 (doc-14631): The sine rule

20.4 Area of triangles
Digital doc
- WorkSHEET 20.2 (doc-14632): Cosine rule and area of triangles

20.6 Trigonometric functions
Interactivity
- Trigonometric functions (int-2796)

20.7 Solving trigonometric equations
Interactivity
- Trigonometry (int-2797)

20.8 Review
Interactivities
- Word search (int-2883)
- Crossword (int-2884)
- Sudoku (int-3895)

Digital docs
- Topic summary (doc-14633)
- Concept map (doc-14634)

To access eBookPLUS activities, log on to www.jacplus.com.au

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Answers

TOPIC 20 Trigonometry II

Exercise 20.2 — The sine rule

1 44°58′, 77°2′, 13.79
2 39°18′, 38°55′, 17.21
3 70°, 9.85, 9.4
4 33°, 38.98, 21.98
5 19.12
6 $C = 51°$, $b = 54.66$, $c = 44.66$
7 $A = 60°$, $b = 117.11$, $c = 31.38$
8 $B = 48°26′$, $C = 103°34′$, $c = 66.26$; or $B = 131°34′$, $C = 20°26′$, $c = 23.8$
9 24.17
10 B, C
11 $A = 73°15′$, $b = 8.73$; or $A = 106°45′$, $b = 4.12$
12 51.9 or 44.86
13 $C = 110°$, $a = 3.09$, $b = 4.64$
14 $B = 38°$, $a = 3.36$, $c = 2.28$
15 $B = 33°33′$, $C = 121°27′$, $c = 26.24$; or $B = 146°27′$, $C = 8°33′$, $c = 4.57$
16 43.62 m
17 a 6.97 m b 4 m
18 a 13.11 km b N20°47′W
19 a 8.63 km b 6.48 km/h c 9.90 km
20 22.09 km from A and 27.46 km from B
21 C
22 B
23 Yes, she needs 43 m altogether.
24 a 7.3 km b 282°3′
25 $h = 7.5$ cm

Exercise 20.3 — The cosine rule

1 7.95
2 55.22
3 23.08, 41°53′, 23°7′
4 28°57′
5 88°15′
6 $A = 61°15′, B = 40°, C = 78°45′$
7 2218 m
8 a 12.57 km b S35°1′E
9 a 35°6′ b 6.73 m^2
10 23°
11 89.12 m
12 a 130 km b S22°12′E
13 28.5 km
14 74.3 km
15 70°49′
16 a 8.89 m b 77° c $x = 10.07$ m
17 1.14 km/h
18 ∠CAB = 34.65°, ∠ABC = 84.83° and ∠BCA = 60.52°
19 a 4.6637 m b 55.93°

Exercise 20.4 — Area of triangles

1 12.98
2 38.14
3 212.88
4 $A = 32°4′, B = 99°56′$, area = 68.95 cm^2
5 $A = 39°50′, B = 84°10′$, area = 186.03 m^2
6 $A = 125°14′, B = 16°46′$, area = 196.03 mm^2
7 C
8 14.98 cm^2
9 570.03 mm^2
10 2.15 cm^2
11 B
12 3131.41 mm^2
13 610.38 cm^2
14 a 187.5 cm^2 b 15.03 cm^2 c 187.47 cm^2
15 17 goldfish
16 22.02 m^2
17 a Area = 69.63 cm^2
b Dimensions are 12.08 cm and 6.96 cm.
18 17 kg
19 52.2 hectares
20 175 m^3
21 C
22 B
23 a 29.44 km^2 b 8.26 km c 41°
d 24° e 3.72 km f 10.11 km^2
g 39.55 km^2
24 374.12 cm^2

Challenge 20.1

$a \sin B = b$
Area $= \frac{1}{2}ac$

Exercise 20.5 — The unit circle

1 a 1st b 2nd c 4th
d 3rd e 2nd f 3rd
g 4th h 4th
2 A
3 D
4 a 0.35 b 0.95 c −0.17
d 0.99 e −0.64 f 0.77
g −0.57 h −0.82
5 a 1 b 0 c 0
d −1 e −1 f 0
g 0 h 1
6 a 0.87 b 0.50
7 a 30° b −0.87
c $\cos 150° = -\cos 30°$ d 0.5
e $\sin 150° = \sin 30°$
8 a 30° b −0.87
c $\cos 210° = -\cos 30°$ d −0.50
e $\sin 210° = -\sin 30°$
9 a 30° b 0.87
c $\cos 330° = -\cos 30°$ d −0.50
e $\sin 330° = -\sin 30°$
10 a 0.34 b 0.94 c 0.36 d 0.36
e They are equal.
11 a 0.71 b −0.71 c −1 d −1
e They are equal. f $\tan 135° = -\tan 45°$
12 a −0.64 b −0.77 c 0.84 d 0.83
e They are approx. equal. f $\tan 220° = \tan 40°$
13 a −0.87 b 0.5 c −1.73 d −1.74
e They are approx. equal. f $\tan 300° = -\tan 60°$
14 D
15 a $\sqrt{1-p^2}$ b p c $-\sqrt{1-p^2}$
16 0
17 a $(2\cos\theta, 3\sin\theta)$ b $(-\cos\theta, -\sin\theta)$
c $(-\cos\theta, -\sin\theta)$
18 a $-\frac{1}{2}\sin(2\theta)$ b $\cos(\theta) - \sin(\theta)$
19 a $\tan(x)$ b $-\sec(x)$

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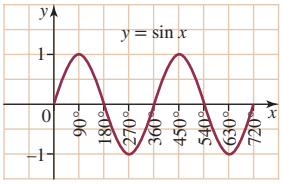
Exercise 20.6 — Trigonometric functions

1

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$	0	0.5	0.87	1	0.87	0.5	0	−0.5°	−0.87	−1	−0.87	−0.5	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\sin x$	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	

2



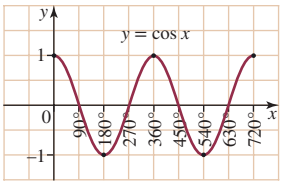
3 360°

4 **a** 0.7
b 0.8
c 0.35
d −0.35
e 0
f 0.9
g −0.2
h −0.9

5 **a** 64°, 116°, 424°, 476°
b 244°, 296°, 604°, 656°
c 44°, 136°, 404°, 496°
d 210°, 330°, 570°, 690°
e 233°, 307°, 593°, 667°
f 24°, 156°, 384°, 516°

6 See the table at the bottom of the page.

7



8 The graph would continue with the cycle.

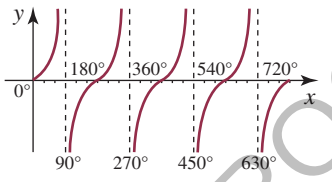
9 It is a very similar graph with the same shape; however, the sine graph starts at (0, 0), whereas the cosine graph starts at (0, 1).

10 **a** 0.7 **b** −0.98 **c** −1 **d** 0.9
e −0.5 **f** −0.8 **g** 0.8 **h** −0.96

11 **a** 120°, 240°, 480°, 600° **b** 37°, 323°, 397°, 683°
c 46°, 314°, 406°, 674° **d** 127°, 233°, 487°, 593°
e 26°, 334°, 386°, 694° **f** 154°, 206°, 514°, 566°

12 See the table at the bottom of the page.

13 $y = \tan x$



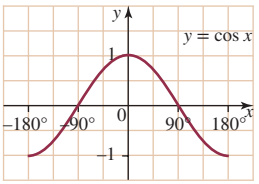
14 The graph would continue repeating every 180° as above.

15 Quite different. $y = \tan x$ has undefined values (asymptotes) and repeats every 180° rather than 360°. It also gives all y-values, rather than just values between −1 and 1.

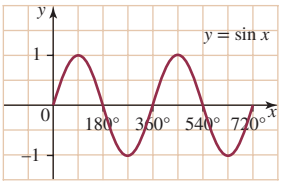
16 **a** 1.7 **b** −1
c −1.2 **d** 0.8
e −0.8 **f** 1.2
g −0.2 **h** 1

17 **a** 45°, 225°, 405°, 585° **b** 56°, 236°, 416°, 596°
c 158°, 338°, 518°, 698° **d** 117°, 297°, 477°, 657°
e 11°, 191°, 371°, 551° **f** 135°, 315°, 495°, 675°

18 **a**



b



6

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\cos x$	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1	

12

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\tan x$	0	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\tan x$	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0	

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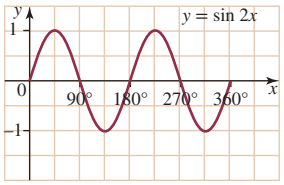
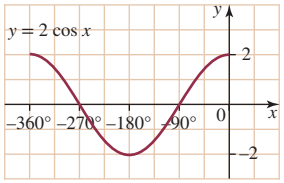
c



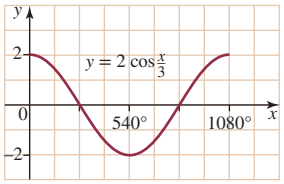
d



19 a i 360° ii 1
b i 360° ii 1
c i 180° ii 1
d i 360° ii 2
20 a i 180° ii 3
b i 120° ii 4
c i 720° ii 2
d i 1440° ii $\frac{1}{2}$
e i 360° ii 1
f i 180° ii 1

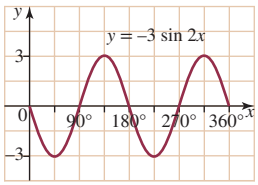
21 a C
b A
c D

22 a



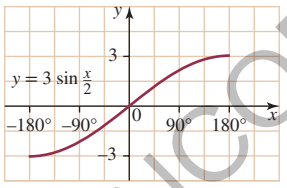
Period = 1080°
Amplitude = 2

b



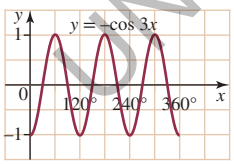
Period = 180°
Amplitude = 3

c



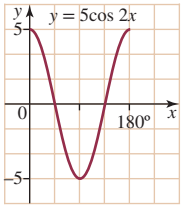
Period = 720°
Amplitude = 3

d



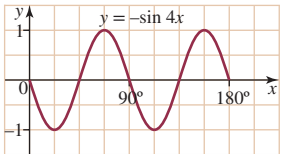
Period = 120°
Amplitude = 1

e



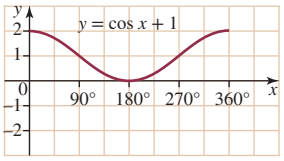
Period = 180°
Amplitude = 5

f

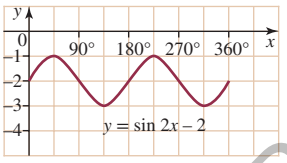


Period = 90°
Amplitude = 1

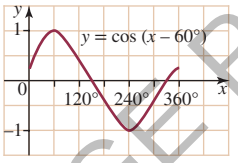
23 a



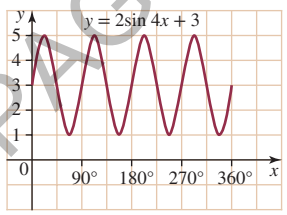
b



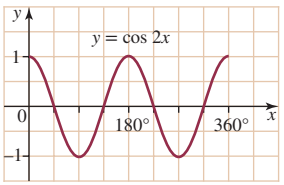
c



d



24 a

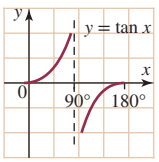


i –1 **ii** 1
b **i** 3 **ii** 1
c Max value of $\sin x = 1$, hence max value of $y = 2 \times 1 + 3 = 5$
Min value of $\sin x = -1$, hence min value of $y = 2 \times -1 + 3 = 1$

25 a

x	0	30°	60°	90°	120°	150°	180°
y	0	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	undef	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0

b



c At $x = 90°$, y is undefined.
d $x = 270°$
e The period = 180°, amplitude is undefined.

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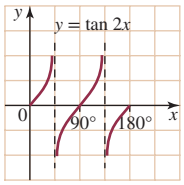
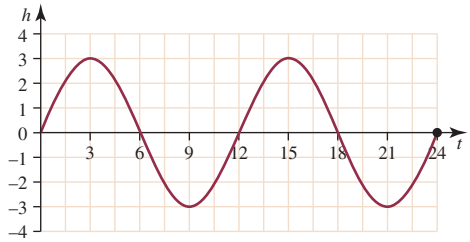
26 a



b $x = 45°$ and $x = 135°$

c Period = 90° and amplitude is undefined.

27 a

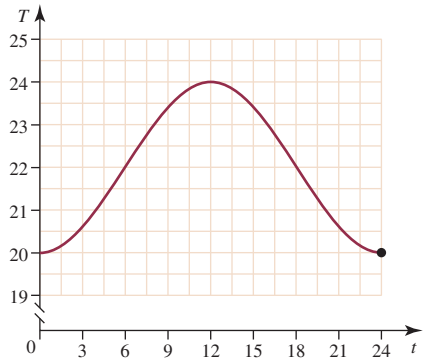


b 3 metres

c −2.6 metres

28 a 22 °C

b



c Coolest 20 °C
Warmest 24 °C

Exercise 20.7 — Solving trigonometric equations

1 Calculator answers

i 25.84°, 334.16°

ii 72.54°, 287.46°

iii 101.54°, 258.46°

iv 126.87°, 233.13°

2 a 30°, 150°
b 60°, 120°
c 120°, 240°
d 135°, 225°
e 90°
f 180°
g 210°, 330°
h 225°, 315°
i 30°, 330°
j 150°, 210°
k 90°
l 90°, 270°

3 a 30°, 60°, 210°, 240°

b 75°, 105°, 255°, 285°

c 15°, 105°, 195°, 285°

d −165°, −135°, −45°, −15°, 75°, 105°

e 52.5°, 82.5°, 142.5°, 172.5°

f −165°, −135°, −45°, −15°, 75°, 105°

g 45°

h 30°, 90°, 150°, 210°, 270°, 330°

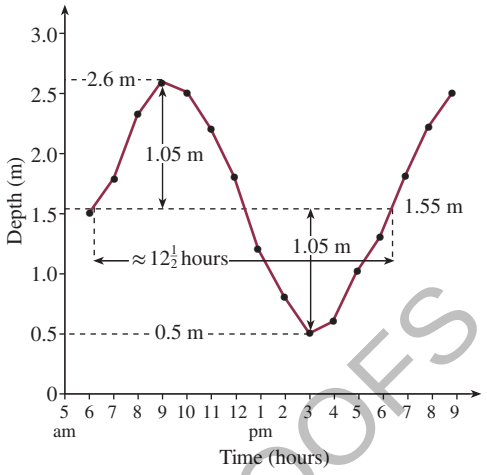
4 a 30°, 150°

b 30°, 330°

c 45°, 315°

d 135°, 315°

5 a



b i $12\frac{1}{2}$ h **ii** 1.05 m

c 10.00 am, 10.30 pm, 11.00 am, 11.30 pm, noon

d Until 1.45 am Sunday, 8 am to 2.15 pm and after 8.30 pm

6 a $x = 30°, 210°$ **b** $x = 153.43°, 333.43°$

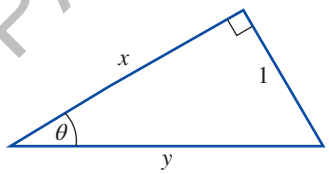
7 $x = 30°, 150°$

Challenge 20.2

1 100^g **2** 200^g **3** 300^g **4** 400^g

Investigation — Rich task

Take one triangle containing the angle θ and label its sides.



This gives

$$\tan(\theta) = \frac{1}{x}$$

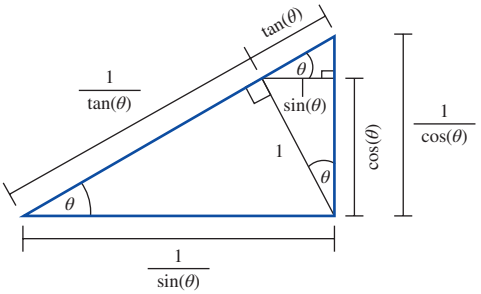
$$\Rightarrow x = \frac{1}{\tan(\theta)}$$

and

$$\sin(\theta) = \frac{1}{y}$$

$$\Rightarrow y = \frac{1}{\sin(\theta)}.$$

Continue until you get the following lengths.



$$\text{Area}_1 = \frac{1}{2}\sin(\theta)\cos(\theta)$$

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$$\text{Area}_2 = \frac{90 - \theta}{360}\pi$$

$$\text{Area}_3 = \frac{1}{2}\left(\frac{1}{\cos(\theta)} - \cos(\theta)\right)\sin(\theta)$$

$$\text{Area}_4 = \frac{1}{2} \times \frac{1}{\tan(\theta)} - \frac{90 - \theta}{360}\pi$$

$$\text{Area}_5 = \frac{3\pi}{4}$$

Code puzzle

An ordered table of all the elements arranged by their atomic numbers

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