CHAPTER 1

First-Order Differential Equations and Models

How many tons of fish can be harvested each year without killing off the population? When you double the dose of your cold medication, do you fall asleep in your math class? Does it take a ball longer to rise than to fall? In this chapter we model natural processes with differential equations in order to answer these and many other questions.

1.1 A Modeling Adventure

Differential equations provide powerful tools for explaining the behavior of dynamically changing processes. We will use them to answer questions about processes that are hard to answer in any other way.

Take a look, for example, at the fish population in one of the Great Lakes. What harvesting rates maintain both the population and the fishing industry at acceptable levels? We will use differential equations to find out how the population changes over time given birth, death, and harvesting rates.

The clue that a differential equation may describe what is going on lies in the words “birth, death, and harvesting rates.” The key word here is “rates.” Rates are
derivatives with respect to time, but what quantity is to be differentiated in this case? Let’s measure the population of living fish at time \( t \) by the total tonnage \( y(t) \), and the time in years. Then the net rate of change of the fish population in tons of fish per year is \( dy(t)/dt \), written as \( y'(t) \) or simply \( y' \). At any time \( t \), we have

\[
y'(t) = \text{Birth rate} - \text{Death rate} - \text{Harvest rate}
\]

where we measure all the rates in tons per year. We suppose that fish immigration and emigration rates from rivers that meet the lake cancel each other out, so they don’t need to appear in (1). Close observation of many species over many years suggests that birth and death rates are each roughly proportional to the size of the population:

- Birth rate at time \( t \): \( by(t) \)
- Death rate at time \( t \): \( (m + cy(t))y(t) \)

where \( b \), \( m \), and \( c \) are nonnegative proportionality constants. The extra twist here is that the natural mortality coefficient \( m \) is augmented by the term \( cy(t) \) which accounts for overcrowding. As a population increases in a fixed habitat, the death rate often increases much faster than can be accounted for by a single constant coefficient \( m \). The overcrowding term is needed to model this accelerated mortality factor.

Now let’s pull all of these bits and pieces together and create a model.

### Making the Mathematical Model

Denoting the harvest rate by \( H \) and using the law given by (1), we have a differential equation for \( y(t) \):

\[
y' = by - (m + cy)y - H
\]

or

\[
y' = ay - cy^2 - H
\]

where \( a = b - m \) is assumed to be positive. An equation like (2) that involves a to-be-determined function of a single variable and its derivatives is called an **ordinary differential equation** (ODE, for short).

Referring to our fishing model, we note that observation of an actual fish population gives us a fairly good idea of the birth and death rates (so we suppose that \( a \) and \( c \) are known), and the harvest rate \( H \) is under our control. That leaves the tonnage \( y(t) \) to be determined from ODE (2). A function \( y(t) \) for which

\[
y'(t) = ay(t) - c(y(t))^2 - H
\]

for all \( t \) in an interval is called a **solution** of ODE (2). The value \( y_0 \) of \( y(t) \) at some time \( t_0 \) can be estimated, and must surely be a critical factor in predicting later values of \( y(t) \). The condition \( y(t_0) = y_0 \) is called an **initial condition**.

Measuring time forward from the time \( t_0 \), we have created a problem whose solution \( y(t) \) is the predicted tonnage of fish at future times:
Mathematical Model for the Fish Population over Time

Given the constants $a$ and $c$, the harvesting rate $H$, and the values $t_0$ and $y_0$, find a function $y(t)$ for which

$$y' = ay - cy^2 - H, \quad y(t_0) = y_0$$

on some $t$-interval containing $t_0$.

The ODE and the initial condition in (3) form an initial value problem (IVP) for $y(t)$. We will see in Chapter 2 that the general IVP (3) has a unique solution on some $t$-interval if the harvesting rate $H$ is a constant, or if $H$ is a continuous function of time. It is nice to know that we are dealing with a problem that has exactly one solution, even though we don’t yet know how to construct that solution. It is like knowing in advance that the pieces of a jigsaw puzzle will indeed fit together.

So how do we describe the solution $y(t)$ of IVP (3)? Do we use words, graphs, or formulas? We will use all three.

A Solution Formula for IVP (3): No Overcrowding

We have put together a general model IVP for the fish tonnage. To describe the solution, it might be a good idea not to tackle the full-blown initial value problem, but to look at particular cases first.

Suppose that there is no overcrowding (so $c = 0$). Start the clock when the value $y_0$ is known. This gives us the following IVP: Find $y(t)$ so that

$$y' = ay - H, \quad y(0) = y_0, \quad t \geq 0$$

We assume that $a$, $H$, and $y_0$ are nonnegative constants. Here’s a way to find a solution formula for IVP (4).

Suppose that $y(t)$ is a solution of IVP (4), that is,

$$y'(t) = ay(t) - H, \quad y(0) = y_0$$

Moving all the terms in the ODE of (5) to the left-hand side and multiplying through by $e^{-at}$, we have that

$$e^{-at}(y' - ay + H) = 0$$

Since $(e^{-at})' = -ae^{-at}$ and $(e^{-at}y(t))' = e^{-at}y'(t) - ae^{-at}y(t)$, ODE (6) becomes

$$\left(e^{-at}y - \frac{H}{a}e^{-at}\right)' = 0$$

But from calculus we know that the only functions with zero derivatives everywhere are the constant functions. So for some constant $C$ we have

$$e^{-at}y(t) - \frac{H}{a}e^{-at} = C$$
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Setting \( t = 0 \) in formula (7), we can solve for \( C \). We get

\[
y_0 - \frac{H}{a} = C
\]

since \( y(0) = y_0 \). So, multiplying each side of formula (7) by \( e^{at} \), using the value for \( C \) given in (8), and rearranging terms, we finally see that the solution of IVP (4) has the form

\[
y(t) = \frac{H}{a} + \left( y_0 - \frac{H}{a} \right) e^{at}, \quad \text{for } t \geq 0
\]  

(9)

To complete the construction process you may want to verify that the function \( y(t) \) given in (9) actually is a solution of IVP (4).

What does formula (9) tell us about the fish population? First, if initial tonnage \( y_0 \) is exactly \( H/a \), then (9) yields \( y(t) = H/a \) for all \( t \geq 0 \). This constant solution \( y(t) = H/a \) is called an equilibrium solution. Second, note that if \( y_0 \) is the slightest bit greater than \( H/a \), then exponential growth sets in. If \( y_0 \) is less than \( H/a \), then the fish population becomes extinct since there is a time \( t^* > 0 \) such that \( y(t^*) = 0 \).

The graph in the \( ty \)-plane of a solution \( y(t) \) of an ODE is called a solution curve. Figure 1.1.1 shows the exponential growth of the population if there isn’t any fishing (\( H = 0 \)). Figure 1.1.2 shows both exponential growth and decline away from equilibrium if there is fishing (\( H = 5/3 \) tons per year). These two figures can be generated directly by using formula (9) and graphing software.

If \( y_0 < H/a \) we soon end up with extinction, but if \( y_0 > H/a \), then the fish population grows without bound (which never happens in real life). So we need a better model. Maybe we need to put the overcrowding term back into play.

**FIGURE 1.1.1** Exponential growth (no harvesting): IVP (4) with \( a = 1, \ H = 0 \).

**FIGURE 1.1.2** Exponential growth and decline with harvesting: IVP (4) with \( a = 1, \ H = 5/3 \).
Overcrowding, No Harvesting

So let’s temporarily drop the harvesting term from the ODE and put the overcrowding term back in to obtain the IVP

\[ y' = ay - cy^2, \quad y(0) = y_0, \quad t \geq 0 \quad (10) \]

where \( a, c, \) and \( y_0 \) are positive constants. Although there is a formula for the solution of IVP (10) (look ahead to Example 1.6.5), the formula isn’t particularly easy to derive, so we need another way to describe the solution of IVP (10). There are computer programs called numerical solvers that compute very good approximations of the solution to an IVP like (10), even when there is no solution formula. Let’s see what we can do with IVP (10) using a numerical solver.

Figure 1.1.3 shows approximate solution curves for IVP (10) with \( a = 1, c = 1/12:\)

\[ y' = y - y^2/12, \quad y(0) = y_0, \quad y_0 = \text{various positive values}, \quad t \geq 0 \quad (11) \]

We have set the computer solve-time interval at \( 0 \leq t \leq 10 \) to predict future tonnage and the tonnage range at \( 0 \leq y \leq 20; \) negative tonnage makes no sense here.

What does Figure 1.1.3 suggest about the evolving fish tonnage as time advances? First of all, there seem to be two equilibrium levels, \( y(t) = 12 \) for all \( t \geq 0 \) and \( y(t) = 0 \) for all \( t \geq 0. \) Are these actual solutions of the ODE in (11)? Yes, because the constant functions \( y(t) = 12 \) and \( y(t) = 0 \) satisfy the ODE, as can be verified by direct substitution. Intriguingly, the upper equilibrium seems to attract all other nonconstant solution curves in the population quadrant \( y \geq 0, \quad t \geq 0. \) Left alone, the fish population tends toward this equilibrium level, no matter what the initial population might be.

Since we will use numerical solvers often, let’s see how they work.

Some Tips on Using a Numerical Solver

A numerical solver plots an approximate value of the solution \( y(t) \) at hundreds of different instants of time and then connects these points on the computer screen with line segments. How well this graph approximates the true solution curve depends on the sophistication of the solver. Numerical analysts have done a remarkable job in coming up with reliable solvers; we have a great deal of confidence in ours.

For now we only need to concern ourselves with the basics of how to communicate with the solver. The first thing to do is to write the IVP in the form

\[ y' = f(t, y), \quad y(t_0) = y_0 \]

because the numerical solver has to know the function \( f(t, y) \) and the initial point \( (t_0, y_0). \) Since \( dy/dt \) is the time rate of change of the solution \( y(t) \) of the IVP, the function \( f(t, y) \) is often called a rate function. Next, the user needs to specify the solve-time interval as running from the initial point \( t_0 \) to the final point \( t_1. \) The IVP is said to be solved forward if \( t_1 > t_0, \) and backward if \( t_1 < t_0. \)

The solver must be told how to display solution curves. We like to select the screen size (i.e., the axis ranges) before telling our solver to find and plot solution curves. There are two reasons for this:
\[ y' = y - \frac{y^2}{12}, \quad 0 \leq y_0 \leq 20 \]

\[ y(\text{tons}) \quad t \text{ (years)} \]

**FIGURE 1.1.3** Overcrowding, no harvesting: equilibrium solutions \( y = 0, 12; \) IVP (11).

\[ y' = y - \frac{y^2}{12} - \frac{5}{3}, \quad 0 \leq y_0 \leq 20 \]

\[ y(\text{tons}) \quad t \text{ (years)} \]

**FIGURE 1.1.4** Overcrowding, harvesting: equilibrium solutions \( y = 2, 10; \) IVP (12).

• Well-designed solvers often shut down automatically when the solution curve gets too far beyond the specified screen area because of a poorly selected solve-time interval. This prevents the computer from working too hard (and perhaps crashing).

• Some solvers have a default setting that automatically scales the screen size to the solution curve over the solve-time interval. If you have a runaway solution curve, you won’t see much on the screen.

Choosing the right screen size to bring out the features you wish to examine is often as much of an art as it is a science. Your skill at setting screen sizes will improve with experience.

Now we are ready to return to the fish population model. Let’s put the fishing industry back in business and see what happens.

**Overcrowding and Harvesting**

Let’s start out by including light harvesting, say \( H = \frac{5}{3} \) tons per year, so IVP (11) becomes

\[ y' = y - \frac{y^2}{12} - \frac{5}{3} = -\frac{1}{12}(y - 2)(y - 10), \quad y(0) = y_0 \geq 0 \quad (12) \]

Let’s use our solver to plot approximate solution curves to IVP (12) for positive values of \( y_0 \) (Figure 1.1.4). There are two equilibrium solutions: \( y = 2 \) and \( y = 10 \), all \( t \). The upper equilibrium line still attracts solution curves, but now not all of them. Those starting out below the lower equilibrium line curve downward toward extinction. This model of a low harvesting rate flashes a yellow caution signal: light harvesting doesn’t appear to be very harmful, at least if the initial tonnage \( y_0 \) is high enough, but even a light harvesting rate could drive a population to extinction if the population level is low.
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\begin{align*}
y' &= y - \frac{y^2}{12} - 4, \quad 0 < y_0 < 20 \\
\end{align*}

\begin{align*}
y' &= y - \frac{y^2}{12} - H(t), \quad 0 < y_0 < 20 \\
H(t) &= \begin{cases} 
4, & 0 \leq t < 5 \\
0, & t \geq 5 
\end{cases} \\
y &= 12 \\
\end{align*}

FIGURE 1.1.5 Extinction; IVP (13) for various \( y_0 \) values.

FIGURE 1.1.6 Ban on fishing over a five-year period restores fish population; IVP (14).

to begin with. Still, this is a scenario where both the fish population and the fishing industry do fairly well.

Now let’s give the fishermen a free hand and suppose that the harvesting rate is much higher. Let’s say the harvesting rate rises to 4 tons per year. We have the heavy harvesting IVP

\begin{align*}
y' &= y - \frac{y^2}{12} - 4, \quad y(0) = y_0, \quad t \geq 0 \\
\end{align*}

This time if we search for equilibrium solutions by setting \( y' = 0 \) and using the quadratic formula to find the roots of \( y - \frac{y^2}{12} - 4 \), we find that there are none. In fact, \( y' \) is always negative and Figure 1.1.5 shows the resulting catastrophe.

**Ban on Fishing**

We can’t let the fish population die out. Let’s see what happens in our model if, after five years of fishing at the rate of 4 tons per year, we ban fishing for five years. Now the harvest rate is given by the function

\begin{align*}
H(t) &= \begin{cases} 
4, & 0 \leq t < 5 \\
0, & t \geq 5 
\end{cases} \\
\end{align*}

and the IVP is

\begin{align*}
y' &= y - \frac{y^2}{12} - H(t), \quad y(0) = y_0, \quad 0 \leq t \leq 10 \\
\end{align*}

Fortunately, it is known that even if the harvesting rate is an on-off function like \( H(t) \), an initial value problem such as (14) still has a unique solution \( y(t) \) for each value of \( y_0 \). We don’t have a formula for \( y(t) \), but our numerical solver gives us a good idea of just how \( y(t) \) behaves.
As you might expect, the fish population is rescued from extinction if \( y_0 \) is large. Figure 1.1.6 shows that after five years of heavy harvesting the surviving population heads toward the level of \( y = 12 \). We have saved the fish, but at the expense of the fishing industry.

Figure 1.1.6 shows a strange feature not seen in any of the other graphs: corners on the solution curves. These appear precisely at \( t = 5 \) when harvesting suddenly stops. So a discontinuity in the harvesting rate shows up in the graphs as a sudden change in the slope of a solution curve. That is not surprising because the slope of a solution \( y(t) \) is the derivative \( y'(t) \), and \( y'(t) \) in ODE (14) involves the on-off harvesting rate.

**Comments**

We created a mathematical model using ODEs for changes in population size, a model that includes internal controls (the overcrowding factor) and external controls (the fishing rate). We found formulas for the solutions of the mathematical model in a simple case, used a numerical solver to graph solutions in more complex cases, and interpreted all of these solutions in terms of what happens to the fish population. The model introduced here has its flaws, as all models do. But the modeling process has allowed us to examine the consequences of various assumptions about the rate of change of the fish population.

There are many good solvers that require little or no programming skills. No specific solver is presumed in this text.

**PROBLEMS**

1. *(Exponential Growth).* Say that the model IVP for a fish population is given by \( y'(t) = ay(t) \), \( y(0) = y_0 \), where \( a \) and \( y_0 \) are positive constants (no overcrowding and no harvesting).

   \(<\text{a}> \) Find a solution formula for \( y(t) \).

   \(<\text{b}> \) What happens to the population as time advances? Is this a realistic model? Explain.

2. *(Control by Overcrowding and by Harvesting).* The IVP \( y' = y - \frac{y^2}{9} - \frac{8}{9}, \ y(0) = y_0 \), where \( y_0 \) is a positive constant, is a special case of IVP (3).

   \(<\text{a}> \) What is the overcrowding coefficient and its units? What is the harvesting rate?

   \(<\text{b}> \) Find the two positive equilibrium levels. \([\text{Hint: Find the roots of } y - \frac{y^2}{9} - \frac{8}{9}.]\]

   \(<\text{c}> \) Graph solution curves of the IVP for various values of \( y_0 \). Use the axis ranges \( 0 \leq t \leq 10, 0 \leq y \leq 15 \). Interpret what you see in terms of the future of the fish population.

3. *(Restocking).* Restocking the fish population with \( R \) tons of fish per year leads to the model ODE \( y' = ay - cy^2 + R \), where \( a \) and \( c \) are positive constants.

   \(<\text{a}> \) Explain each term in the model ODE.

   \(<\text{b}> \) Test the model on the IVP \( y' = y - \frac{y^2}{12} + \frac{7}{3}, \ y(t_0) = y_0 \), for various nonnegative values of \( t_0 \) and \( y_0 \). Carry solution curves forward and backward in time. Use the screen \( 0 \leq t \leq 10, 0 \leq y \leq 25 \). Interpret what you see.
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4. (Periodic Harvesting/Restocking). Look at the IVP $y' = y - y^2 + 0.3 \sin(2\pi t)$, $y(t_0) = y_0$.
   (a) Discuss the meaning of the ODE in terms of a fish population. Graph solution curves for $t_0 = 0$ and values of $y_0$ ranging from 0 to 2. Use the ranges $0 \leq t \leq 10$, $0 \leq y \leq 2$. Repeat for $t_0 = 1, 2, \ldots, 9$ and $y_0 = 0$. Interpret what you see in terms of the fish population.
   (b) Explain why the solution curves starting at $(t_0, y_0)$ and $(t_0 + 1, y_0)$ look alike. In the rectangle $0 \leq t \leq 10$, $-1 \leq y \leq 2$, plot the solution curve through the point $t_0 = 0.5$, $y_0 = 0$. Why is this curve meaningless in terms of the fish population?

5. (Constant Effort Harvesting). The models in this section have a flaw. At low population levels a fixed high harvesting rate can’t be sustained for long because the population dies out. A safer model (for the fish) is $y' = ay - cy^2 - H_0 y$, $y(0) = y_0$, where $a$, $c$, $H_0$, and $y_0$ are positive constants. In this model the lower the population, the lower the harvesting rate.
   (a) Interpret each term in the ODE. Why is this called constant effort harvesting?
   (b) For values of $H_0$ less than $a$, explain why this model produces solution curves similar to those in Figure 1.1.3, but possibly with a different stable equilibrium population.

6. (Heavy Harvesting, Light Harvesting). What happens when a five-year period of heavy harvesting is followed by five years of light harvesting? Combine IVPs (12) and (13) by supposing that $y' = y - y^2/12 - H(t)$, where

$$H(t) = \begin{cases} 
4, & 0 \leq t < 5 \\
5/3, & 5 \leq t \leq 10 
\end{cases}$$

Plot solution curves for $0 \leq t \leq 10$, $0 \leq y \leq 20$, and interpret what you see. Draw the lines $y = 10$ and $y = 2$ on your plot and explain their significance for the population for $t \geq 5$.

7. (Seasonal Harvesting). Say that harvesting is seasonal, “on” for the first few months of each year and “off” for the rest of the year. The ODE is $y' = ay - cy^2 - H(t)$, where $H(t)$ has value $H_0$ during the on-season, and value 0 during the off-season. The harvesting season is the first two months of each year in Figure 1.1.7 and the first eight months in Figure 1.1.8; in both figures $a = 1$, $c = 1/2$, $H_0 = 4$. Duplicate the graphs in Figures 1.1.7 and 1.1.8. Discuss what you see in terms of population behavior. [Hint: Try $H(t) = H_0 \text{sqw}(t, d, 1)$, where $d = 100(2/12) = 50/3$ for a two-month season and $d = 100(8/12)$ for an eight-month season. See Appendix B.1 for more information about the on-off function sqw.]
1.2 Visualizing Solution Curves

In the previous section we used a numerical solver to plot some solution curves for a few differential equations, and we accepted the results without question. This is a risky practice because every computer has its limitations, and that goes for software, too. It’s always good to have more than one way to look at things so that results can be checked in as many ways as possible. In this section we show how to visualize the behavior of a solution curve based on the differential equation itself, rather than relying solely on computer output for the solution. First, though, we take a brief detour.

There are good reasons that most of the differential equations we have looked at so far have been written in the normal form

\[ y' = f(t, y) \]  

(1)

where \( f(t, y) \) is a function defined on some portion (or all) of the \( ty \)-plane. For example, most numerical solvers only accept ODEs that have been written in normal form. In addition, the general theory of ODEs applies only to differential equations in this form.

Let’s define what we mean by a solution of ODE (1). A function \( y(t) \) defined on a \( t \)-interval \( I \) is a solution of ODE (1) if \( f(t, y(t)) \) is defined and \( y'(t) = f(t, y(t)) \) for all \( t \) in \( I \). With this definition and the normal form in hand, we can begin a general discussion of the relation of the solutions \( y(t) \) of ODE (1) to its solution curves.

Solution Curves

Examples in the last section show that ODEs can have many solutions; we were fortunate in finding a formula for one of the simpler ODEs. We will see in Section 2.1 that under certain mild conditions, ODEs have solutions even though we can’t always find explicit formulas for them. Armed with this knowledge, we can use a numerical solver to plot approximate solution curves. We will often use theory and geometric methods to examine properties of solutions without the benefit of solution formulas.

Look at the normal ODE \( y' = f(t, y) \), where \( f(t, y) \) is defined over a closed rectangle \( R \), that is, a rectangle that contains its boundary lines. If \( (t_0, y_0) \) is any point in \( R \) and if \( f \) and \( \partial f / \partial y \) are continuous on \( R \), then (as we show in Section 2.1) there is exactly one solution \( y(t) \) to the initial value problem (IVP)

\[ y' = f(t, y), \quad y(t_0) = y_0 \]  

(2)

The point \((t_0, y_0)\) in \( R \) is called the initial point for the IVP. The solution curve for \( y(t) \) extends to the boundary of \( R \) both for \( t > t_0 \) and for \( t < t_0 \). In other words, the solution curve does not suddenly stop inside \( R \). The unique solvability of IVP (2) for each point \((t_0, y_0)\) in \( R \) implies that no two solution curves of the ODE \( y' = f(t, y) \) can intersect inside \( R \). We can think of \( R \) as being completely covered by solution curves, each point of \( R \) on exactly one solution curve.

Knowing that IVP (2) has a unique solution, we can use a numerical solver to compute and plot approximate solution curves. In Chapter 2 we will see that numerical
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Solvers use a step-by-step process to generate approximate points on a solution curve, and then graph approximate solutions by connecting these points with line segments.

Let’s turn now to some geometric interpretations.

**Geometry of Solution Curves**

There is a way to view the solvability of IVP (2) that appeals to geometric intuition and lends itself to a graphical approach to finding solution curves. The ODE says that at each point \((\tilde{t}, \tilde{y})\) in \(R\) the number \(f(\tilde{t}, \tilde{y})\) is the slope of the tangent line to the solution curve through that point (see margin figure).

On the other hand, suppose that the graph of a function \(y(t)\) lies in \(R\). Then \(y(t)\) defines a solution curve of \(y' = f(t, y)\) if at each point \((\tilde{t}, \tilde{y})\) on its graph the slope of the tangent line has the value \(f(\tilde{t}, \tilde{y})\). This is the geometric way of saying that \(y'(t) = f(t, y(t))\), that is, \(y(t)\) is a solution for the ODE.

This change of viewpoint gives us an imaginative way to see solution curves for the ODE. By drawing short line segments with slopes \(f(\tilde{t}, \tilde{y})\) and centered at a grid of points \((\tilde{t}, \tilde{y})\) in \(R\), we obtain a diagram, called a direction field. A direction field suggests curves in \(R\) with the property that at each point on each curve the tangent line to the curve at that point lies along the direction field line segment at the point. This process reveals solution curves in much the same way as iron filings sprinkled on paper held over the poles of a magnet reveal magnetic field lines.

Given the rate function \(f(t, y)\), we can draw the line segments of a direction field by hand, but it’s a lot easier to have a numerical solver do the work. Most solvers can do this, and many let the user choose the density of the grid points and the length of the segments.

Let’s illustrate these ideas with an example.

**Example 1.2.1 A Direction Field and a Solution Curve**

Figure 1.2.1 shows a direction field for the ODE

\[ y' = y - t^2 \]

You can almost see the solution curves. They rise wherever \(y > t^2\) because \(y'\) is positive; the curves fall where \(y < t^2\). The field line segments in the figure are all the same length, even if it doesn’t appear so at first glance. The reason for this is that the computer screen length of a vertical unit is not the same as the screen length of a horizontal unit. The ratio of the former to the latter is the aspect ratio of the display.

Now let’s plot the solution curve through the initial point \(t_0 = 0, y_0 = 1\). Figure 1.2.2 shows this curve (solid) extended forward and backward in time from \(t_0 = 0\) until the curve leaves the rectangle defined by the computer screen. Notice how nicely the solution curve fits the direction field.

The **nullclines** (the curves of zero inclination) of \(y' = f(t, y)\) are defined by \(f(t, y) = 0\). For example, in the ODE \(y' = y - t^2\) we see that \(f(t, y) = y - t^2\), so the nullcline is the curve defined by \(y = t^2\) (the dashed curve in Figure 1.2.2). The nullcline divides the \(ty\)-plane into the region above the parabola \((y > t^2)\), where solution curves rise, and the region below \((y < t^2)\), where they fall. Solution curves cross
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The nullcline with zero slope because the direction line segment centered at a point on
the nullcline is horizontal.

Nullclines are usually not solution curves. But if \( y_0 \) is a root of a function \( f(y) \),
then the constant function \( y = y_0 \) is both a nullcline and a solution curve for the ODE

\[
y' = f(y)
\]

For example, the horizontal lines \( y = 1, \ y = 2, \) and \( y = 3 \) in the \( ty \)-plane are at the
same time nullclines and solution curves for the ODE

\[
y' = (y - 1)(y - 2)(y - 3)
\]

Such curves are equilibrium solution curves and the solutions that generate them are
called equilibrium solutions.

Finding Equilibrium Solutions

To find the equilibrium solutions of the ODE \( y' = f(y) \), first set \( f(y) = 0 \) and solve
for \( y \). If \( y_0 \) is a number such that \( f(y_0) = 0 \) then the constant function \( y = y_0 \), for all \( t \),
is an equilibrium solution.

**EXAMPLE 1.2.2**

Equilibrium Solutions

Let’s look at the ODE \( y' = 12y - y^2 - 20 \) which models a fish population that is
harvested at a constant rate. Now \( f(y) = 12y - y^2 - 20 \), and since the equation \( 12y - y^2 - 20 = 0 \) has the two roots \( y_0 = 2 \) and \( y_0 = 10 \) we have two equilibrium solutions:

\( y = 2, \) for all \( t \) and \( y = 10, \) for all \( t \).

Here is another example where the direction field guides your eye along the solution curves.
1.2/ Visualizing Solution Curves

1.2.1 Time Dependent Harvesting

Let’s look at an ODE model of a fish population that is harvested and restocked sinusoidally:

\[ y' = y - y^2 - 0.2 \sin t \]  

We used our solver to produce the direction field and solution curves in Figure 1.2.3 for this ODE. We can almost see the solution curves being traced out, guided along by the direction field. It appears that if the initial population is not too small, then the resulting population curve eventually looks like a sinusoid of period \( 2\pi \), which is the period of the harvesting/restocking function \( H(t) \). We haven’t proven that this is so, but the visual evidence is fairly strong.

Compression, Expansion, and Zooming

The solution curves in Figure 1.2.3 were not as easy to graph as you might think. What initial point on the y-axis would you choose so that the solution curve passes through the “target point” \((7.5, -1)\)? Choosing the exact initial point is very difficult since many solution curves intersect the y-axis very near our desired solution curve. If our choice of initial point is even slightly off we will end up with a solution curve which misses our target by a wide margin.

Why is it so hard to hit the target point? It appears from the graph that any two solution curves of ODE (3) either compress together or spread apart without bound as \( t \rightarrow +\infty \). To plot the solution curve through \((7.5, -1)\), choose the initial point to be \((7.5, -1)\) and solve backward in time until the solution curve hits the y-axis.

The solution curves in the upper half of Figure 1.2.3 all seem to flow together and appear to touch inside the rectangle \( R \) defined by the screen. But we know this can’t be

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**FIGURE 1.2.3** Direction field and some solution curves for ODE (3).

**FIGURE 1.2.4** Zoom on a portion of Figure 1.2.3 centered at the point \((6.50, 1.07)\).
so for the following reason: compare ODE (3) with ODE (2) to find the rate function
\[ f(t, y) = y - y^2 - 0.2 \sin t \]

The functions \( f(t, y) \) and \( \frac{\partial f}{\partial y} = 1 - 2y \) are both continuous on the entire \( ty \)-plane, and theory tells us that distinct solution curves of ODE (3) can never touch. So our comfortable world of theory collides with the practical problem of displaying data on a computer screen. Since the screen only contains a finite number of pixels, points that are less than a pixel apart are not distinguishable from one another, which explains the apparent contradiction. In Figure 1.2.4 we separate the apparently touching curves by zooming in on them. When we look at the direction field and the solution curves through a microscope this way, the field line segments seem to be parallel and the solution curves are very nearly parallel lines.

Comments

We have seen how to recognize solution curves of \( y' = f(t, y) \) geometrically by looking at the direction field determined by the rate function \( f \). We saw, too, that the nullclines [where \( f(t, y) \) is zero] divide the \( ty \)-plane into regions where solution curves rise and regions where they fall. All of this shows the intimate relationship between the rate function and the way solution curves behave.

We have also indicated conditions (the continuity of \( f \) and \( \frac{\partial f}{\partial y} \) in a rectangle \( R \)) under which we can be assured that the IVP \( y' = f(t, y), \ y(t_0) = y_0 \) has a unique solution curve in \( R \), even when there is no known solution formula.

PROBLEMS

1. (Ban on Fishing May Come Too Late). Say that a fish population is modeled by the IVP \( y' = y - y^2/12 - H(t), \ y(0) = y_0 \), where \( H(t) = 4 \) for \( 0 \leq t \leq 5 \), \( H(t) = 0 \) for \( 5 \leq t \leq 10 \). Estimate the smallest value of \( y_0 \) such that the fish population recovers after fishing is banned. Give reasons for your estimate. [Hint: See Figure 1.1.6.]

2. (Equilibrium Solutions). First, find all equilibrium solutions for each ODE. Then describe in words how the nonequilibrium solution curves behave in various regions in the \( ty \)-plane. Then plot a direction field and solution curves in the given rectangle. [Hint: Find the sign of \( y' \) above and below each equilibrium solution.]
   (a) \( y' = y^2 - 11y + 10 \), \( |t| \leq 1 \), \(-5 \leq y \leq 15 \)  
   (b) \( y' = |y| - y^2 \), \( |t| \leq 5 \), \( |y| \leq 2 \)  
   (c) \( y' = \sin(2\pi y/(1 + y^2)) \), \( |t| \leq 5 \), \( |y| \leq 2 \)

3. Use a numerical solver to plot solution curves of the ODEs. Use the four initial conditions \( y(0) = -3, -1, 1, 3 \) and the ranges \( 0 \leq t \leq 4 \), \( -4 \leq y \leq 4 \). Shade (or describe in words) the regions where solution curves rise. [Hint: Plot the nullclines to find the boundaries of these regions. Before using your numerical solver, write the ODE in normal form.]
   (a) \( y' + y = 1 \)  
   (b) \( y' + y = t \)  
   (c) \( y' + y = t + 1 \)  
   (d) \( y' = \sin(3t) - y \)
1.3 The Search for Solution Formulas

Solutions of an ODE can be pursued on two levels: we can search for a formula that describes a solution, or we can search for solution curves with the use of a numerical solver. The process of finding solutions of differential equations at either level is often referred to as “solving a differential equation.” In this section we will concentrate on finding solution formulas.

How can we find solution formulas for ODEs? Good guessing is one way. Guessing a solution has a long and honorable history as a way of finding solutions, but some differential equations defy anyone’s ingenuity to guess a solution formula. We will see soon enough that our bag of tricks (including guessing) for finding solution formulas is rather small. That’s where a numerical solver comes in handy. It finds and plots approximate numerical solutions even though there is no solution formula in sight.

But an important class of ODEs does have solution formulas, and we turn now to a description of that class.

First-Order ODEs: Linear or Nonlinear?

The order of an ODE is the order of the highest derivative of the to-be-determined function that appears in the equations. For example, \( y' = y + \sin t \) is a first-order
ODE. All of the ODEs mentioned in the previous sections are first-order ODEs. Later in Sections 1.5 and 1.6 we will see some second-order ODEs like

\[ y'' = -9.8 + 0.15y' \]

and in Section 1.7 the second-order ODE

\[ y'' = \frac{-k}{(y + R)^2} \]

where \( k \) and \( R \) are positive constants.

Let’s look more closely at a class of first-order ODEs which are encountered frequently in modeling natural processes.

A first-order ODE is **linear** if it can be written in the form

\[ y' + p(t)y = q(t) \tag{1} \]

where \( p(t) \) and \( q(t) \) are functions that do not depend on \( y \), but may depend on \( t \). Linear ODEs written as in (1) are in normal linear form. The ODE \( t^2y' = e^t y + \sin 3t \) is a first-order ODE and it is linear because by dividing by \( t^2 \) it can be written in the normal linear form

\[ y' - \frac{e^t}{t^2} y = \frac{\sin 3t}{t^2} \]

First-order ODEs which cannot be written in the form (1) are **nonlinear**.

In Section 1.1 we looked at the first-order ODE

\[ y' - ay = -H \tag{2} \]

where \( a \) and \( H \) are constants. ODE (2) is a first-order linear ODE which is written in normal linear form. We saw that all solutions of ODE (2) are described by the formula

\[ y = \frac{H}{a} + \left( y_0 - \frac{H}{a} \right) e^{at} \]

where \( y_0 \) is any constant.

In Section 1.2 we used a numerical solver to come up with approximate solution curves of the first-order nonlinear ODE

\[ y' = y - y^2 - 0.2 \sin t \tag{3} \]

Figure 1.2.3 displays the results. ODE (3) is nonlinear because the term \( y^2 \) prevents it from being written in the normal linear form (1). Whether a first-order ODE is linear or nonlinear, antidifferentiation is often used to construct a formula for the solutions.

**Looking for Solution Formulas**

The Fundamental Theorem of Calculus turns out to be a key tool for finding solution formulas for ODEs. The basic concept in that theorem is antidifferentiation: an antiderivative of a function \( f(t) \) is a function \( F(t) \) such that \( F'(t) = f(t) \).

Let’s start by finding all solutions of the ODE \( y' = f(t) \).
**THEOREM 1.3.1**

**Antiderivative Theorem.** Suppose that $F(t)$ is an antiderivative of a continuous function $f(t)$ on a $t$-interval. Then all solutions of the ODE

$$y' = f(t)$$

are given on that $t$-interval by

$$y = F(t) + C,$$

where $C$ is any constant (4)

To see this, suppose that $y(t)$ is any solution of $y' = f(t)$, that $F(t)$ is any antiderivative of $f(t)$, and that $t_0$ is any point in the $t$-interval. Then integrating from $t_0$ to $t$ and using the Fundamental Theorem of Calculus, we see that

$$y(t) - y(t_0) = \int_{t_0}^{t} y'(s) \, ds = \int_{t_0}^{t} f(s) \, ds = F(t) - F(t_0)$$

for all $t$ in the interval, so $y(t)$ has the form $F(t) + C$ [the constant $C$ in formula (4) is $y(t_0) - F(t_0)$ in this case]. On the other hand, $y = F(t) + C$ is a solution of the ODE $y' = f(t)$ for any value of the constant $C$, so we have captured all the solutions of the ODE.

The Antiderivative Theorem is the source of many methods for finding solution formulas for ODEs. You might say that it is the “mother of all methods.” Let’s use it to find all solutions of several ODEs.

**EXAMPLE 1.3.1**

**Just Antidifferentiate!**

Since $(\sin t)' = \cos t$, we see from Theorem 1.3.1 that the ODE

$$y' = \cos t$$

has all of its solutions given by the formula

$$y(t) = \sin t + C, \quad C \text{ any constant}$$

Here is an example of how the Antiderivative Theorem is used to find a solution formula for a normal first-order linear ODE.

**EXAMPLE 1.3.2**

**Preparing a Linear ODE for Antidifferentiation**

Suppose that $y(t)$ is a solution of the linear ODE in normal linear form

$$y' - 2y = 2$$

(5)

Multiply each side of the ODE by the function $e^{-2t}$ to obtain the identity

$$e^{-2t}y' - 2e^{-2t}y = 2e^{-2t}$$

(6)
Since $e^{-2t}$ is never zero, every solution of ODE (6) is also a solution of ODE (5), and the other way around. The product rule for derivatives shows that

$$\left[e^{-2t}y(t)\right]' = e^{-2t}y'(t) + (e^{-2t})'y(t) = e^{-2t}y'(t) - 2e^{-2t}y(t)$$

end so identity (6) can be written as

$$\left[e^{-2t}y(t)\right]' = 2e^{-2t}$$

(7)

The trick of multiplying ODE (6) by $e^{-2t}$ was not pulled out of a hat. Its purpose was to produce ODE (7). Now apply Theorem 1.3.1 to ODE (7) to obtain

$$e^{-2t}y(t) = -e^{-2t} + C$$

where $C$ is any constant. Multiply through by $e^{2t}$:

$$y = -1 + e^{2t}C, \quad \text{all } t$$

We now have a formula for all solutions of ODE (5).

Multiplying both side of ODE (5) by $e^{-2t}$ turned out to be an excellent strategy. You may wonder how we hit upon the magic factor $e^{-2t}$. Read on to see how to choose such “integrating factors” for linear ODEs such as (1).

### The Integrating Factor Approach for Linear ODEs

Let’s look at the first-order normal linear ODE

$$y' + p(t)y = q(t)$$

(8)

where the coefficient $p(t)$ and driving term (or input) $q(t)$ are continuous on a $t$-interval $I$. If $q(t) = 0$ for all $t$ in $I$, then ODE (8) is said to be **undriven**. Otherwise, ODE (8) is **driven** by the input $q(t)$. We now find a formula for all solutions of ODE (8).

Here’s the approach that will give us a formula for all solutions. Suppose that $P(t)$ is any antiderivative of the coefficient $p(t)$ in ODE (8) on the interval $I$, that is, $P'(t) = p(t)$ on $I$. The exponential function $e^{P(t)}$ is called an **integrating factor** for ODE (8). Using the Chain Rule we have that $(e^{P(t)})' = e^{P(t)}P'(t) = e^{P(t)}p(t)$ and so we have the identity

$$\left[e^{P(t)}y(t)\right]' = e^{P(t)}y'(t) + e^{P(t)}P'(t)y(t)$$

$$= e^{P(t)}y'(t) + e^{P(t)}p(t)y(t)$$

$$= e^{P(t)}[y'(t) + p(t)y(t)]$$

(9)

We will use identity (9) to help us solve ODE (8).

Suppose that $y(t)$ is any solution of ODE (8) on the interval $I$. Multiplying ODE (8) by the integrating factor $e^{P(t)}$, we have the identity

$$e^{P(t)}[y'(t) + p(t)y(t)] = e^{P(t)}q(t)$$
which, because of the identity (9), can be written as

\[ [e^{P(t)}y(t)]' = e^{P(t)}q(t) \]  

(10)

Suppose that \( R(t) \) is any antiderivative of \( e^{P(t)}q(t) \). Since \( R'(t) = e^{P(t)}q(t) \), and applying Theorem 1.3.1 to ODE (10), we obtain

\[ e^{P(t)}y(t) = R(t) + C \]

where \( C \) is some constant. After multiplying each side of this equality by \( e^{-P(t)} \), we have a formula for the solution \( y(t) \) of ODE (8) that we started with:

\[ y = Ce^{-P(t)} + e^{-P(t)}R(t) \]  

(11)

where \( C \) is any constant. Because \( P(t) \) and \( R(t) \) are defined on the \( t \)-interval \( I \) where \( p(t) \) and \( q(t) \) are continuous, we see that the solution is defined on \( I \). So we have shown that every solution \( y(t) \) of ODE (8) has the form (11) for some value of the constant \( C \). But what values of \( C \) in formula (8) actually produce a solution of ODE (8)?

The answer: \textit{any} value of \( C \)!

To see this, start with \textit{any} constant \( C \) to define a function \( y \) via formula (11) and then reverse the above steps to show that this function \( y(t) \) solves the ODE (8).

Summarizing, here are the steps we followed in finding the formula (11):

\[ \text{Solving a First-Order Linear ODE} \]

1. Write the ODE in the normal linear form \( y' + p(t)y = q(t) \) and identify the coefficient \( p(t) \) and the driving term \( q(t) \).
2. Find an antiderivative \( P(t) \) for \( p(t) \); any one will do.
3. Multiply the ODE by the integrating factor \( e^{P(t)} \) and write the new ODE as
   \[ (e^{P(t)}y)' = e^{P(t)}q(t) \]
4. Find an antiderivative \( R(t) \) for \( e^{P(t)}q(t) \); any one will do.
5. Apply the Antiderivative Theorem 1.3.1 to the new ODE, multiply, and rearrange terms to find the general solution formula (11).

Since formula (11) captures all solutions of ODE (8), it is called the \textit{general solution} of the ODE. One implication of our constructive solution process is that after \textit{any} choice of the antiderivatives \( P(t) \) and \( R(t) \) as described above, all solutions of ODE (8) always have the form (11). It is not obvious from formula (11) itself that this is so, but the construction process does not lie.

Since the constant \( C \) in the general solution formula (11) is arbitrary, we see that ODE (8) has infinitely many solutions, one for each value of the constant \( C \). Now different values for the constant \( C \) in formula (11) give solution curves that never touch in \( I \). To show this, insert distinct constants \( C_1 \) and \( C_2 \) into formula (11) to obtain two solutions, \( y_1(t) \) and \( y_2(t) \). Subtracting, we have

\[ y_1(t) - y_2(t) = (C_1 - C_2)e^{-P(t)} \]
But exponential functions never have the value 0, and \( C_1 \) and \( C_2 \) are distinct. So \( y_1(t) \) and \( y_2(t) \) are never equal for any \( t \) in \( I \), and the two solution curves don’t touch.

When following the procedure above, we have to be fairly good at finding antiderivatives. In case you’re a bit rusty at this, here is a short table of several of the antiderivatives you’ll need in this text.

### TABLE 1.3.1 Some Useful Antiderivatives

- For any constant \( a \neq 0 \):
  
  \[
  \int e^{at} \, dt = \frac{1}{a} e^{at}, \quad \int t e^{at} \, dt = \frac{e^{at}}{a^2} (at - 1), \quad \int t^2 e^{at} \, dt = \frac{e^{at}}{a^3} (a^2 t^2 - 2at + 2)
  \]

  \[
  \int t \sin at \, dt = -\frac{1}{a} t \cos at + \frac{1}{a^2} \sin at, \quad \int t \cos at \, dt = \frac{1}{a} t \sin at + \frac{1}{a^2} \cos at
  \]

- For any constants \( a \) and \( b \), not both zero:
  
  \[
  \int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2 + b^2} (a \cos bt + b \sin bt), \quad \int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2 + b^2} (a \sin bt - b \cos bt)
  \]

### EXAMPLE 1.3.3 Use an Integrating Factor and Find the General Solution

Let’s look at the ODE from Example 1.2.1:

\[
y' - y = -t^2 \tag{12}
\]

This ODE is already in normal linear form with \( p(t) = -1 \), and \( q(t) = -t^2 \), and these functions are continuous on the whole real line. Using the notation of the procedure just described we see that \( P(t) = -t \), and the integrating factor is \( e^{-t} \). Multiplying ODE (12) through by \( e^{-t} \), we have

\[
(e^{-t} y)' = -t^2 e^{-t} \tag{13}
\]

So using Table 1.3.1 we see that \( R(t) = e^{-t}(t^2 + 2t + 2) \) is an antiderivative of \(-t^2 e^{-t}\). Applying Theorem 1.3.1 to ODE (13), we have

\[
e^{-t} y = e^{-t}(t^2 + 2t + 2) + C \tag{14}
\]

where \( C \) is any constant. Multiplying (14) through by \( e^t \), we see that all solutions of ODE (12) are given by the general solution formula

\[
y = C e^t + t^2 + 2t + 2
\]

where \( C \) is any constant.

Now let’s use the general solution approach to solve an initial value problem.
Solving an Initial Value Problem

The constant \( C \) in formula (11) plays an important role in solving an IVP.

**EXAMPLE 1.3.4 Find the General Solution, Solve an IVP**

Here is a first-order ODE in normal linear form with \( p(t) = 2, q(t) = 3e^t \):

\[
y' + 2y = 3e^t \tag{15}
\]

Since \( 2t \) is an antiderivative of \( 2 \), \( e^{2t} \) is an integrating factor. Multiply each side of ODE (15) by \( e^{2t} \) and apply Theorem 1.3.1 to obtain

\[
e^{2t}y' + 2e^{2t}y = 3e^{3t}
\]

Then, since \( e^{2t}y' + 2e^{2t}y = (e^{2t}y)' \) by the formula for differentiating a product, we have

\[
(e^{2t}y)' = 3e^{3t}
\]

\[
e^{2t}y = C + e^{3t}
\]

\[
y = Ce^{-2t} + e^t, \quad \text{all } t \tag{16}
\]

where \( C \) is any constant. Formula (16) gives the general solution of ODE (15).

An initial condition will determine \( C \). For example, to find the solution of the IVP

\[
y' + 2y = 3e^t, \quad y(0) = -3 \tag{17}
\]

we set \( y = -3 \) and \( t = 0 \) in formula (16):

\[
-3 = e^{-2\cdot0}C + e^0 = C + 1
\]

and so \( C = -4 \). Replacing \( C \) in solution formula (16) by \(-4\) gives the solution of IVP (17):

\[
y = -4e^{-2t} + e^t, \quad -\infty < t < \infty
\]

If we had used the initial condition \( y(1) = 0 \) in IVP (17) instead of \( y(0) = -3 \), then we would have found that \( C = -e^3 \) and so the solution formula would be \( y = -e^3e^{-2t} + e^t \).

The formula \( y = Ce^{-2t} + e^t \) for all solutions of \( y' + 2y = 3e^t \) is very helpful if we want to describe the long-term behavior of solutions. Since \( e^{-2t} \to 0 \) as \( t \to +\infty \), we see from (16) that all solutions look more and more like the solution \( y = e^t \) as \( t \) increases.

The constructive process used in Example 1.3.4 suggests that an IVP involving a linear differential equation has exactly one solution. And that is exactly right!
THEOREM 1.3.2

Existence and Uniqueness. Suppose that \( p(t) \) and \( q(t) \) are continuous on a \( t \)-interval \( I \), and that \( t_0 \) is any point in \( I \). If \( y_0 \) is any number, then the IVP

\[
y' + p(t)y = q(t), \quad y(t_0) = y_0
\]

has a solution \( y(t) \), which is defined on all of \( I \) (existence), and there is no other solution (uniqueness).

Let’s show this as follows: According to solution formula (11), the general solution of the ODE in (18) is

\[
y(t) = Ce^{-P(t)} + e^{-P(t)} R(t), \quad t \text{ in } I
\]

where \( P \) and \( R \) are the respective antiderivatives of \( p(t) \) and \( e^{P(t)} q(t) \), and \( C \) is an arbitrary constant. To satisfy the condition \( y(t_0) = y_0 \), let’s substitute \( t_0 \) and \( y_0 \) into the solution formula and obtain

\[
y_0 = Ce^{-P(t_0)} + e^{-P(t_0)} R(t_0)
\]

This algebraic equation for \( C \) has the unique solution \( C = y_0 e^{P(t_0)} - R(t_0) \). This means that IVP (18) does indeed have a unique solution.

The Existence and Uniqueness Theorem provides the foundation for all of the applications of first-order linear IVPs because it guarantees that the IVP we are working with has exactly one solution.

Comments

Our search for solution formulas has paid off handsomely for first-order linear ODEs. The only tool we needed was the Fundamental Theorem of Calculus in the form of the Antiderivative Theorem (Theorem 1.3.1). So we see that first-order linear ODEs have explicit solution formulas. But there is a down side: We may not be able to find the required antiderivatives, so the solution formula (11) may not be especially helpful. You always have the option, though, of using a numerical solver to plot approximate solution curves of an ODE even if a solution formula is unrevealing.

PROBLEMS

1. Find the order of each ODE. Identify each ODE as linear or nonlinear in \( y \), and write each linear ODE in normal linear form. [Hint: For part (d), find \( dy/dt \).]

(a) \( y' = \sin t - t^3 y \)
(b) \( e^y' + 3y = t^2 \)
(c) \( (t^2 + y^2)^{1/2} = y' + t \)
(d) \( dt/dy = 1/(t^2 - ty) \)
(e) \( y'' - t^3 y = \sin t + (y')^2 \)
(f) \( y' = (1 + r^2) y'' - \cos t \)
(g) \( e^{y'} + (\sin t)y' + 3y = 5e^t \)
(h) \( (y'')^2 = y^5 \)
2. (Finding Solutions). Simple exponential functions like \( y = e^{rt} \) are often solutions of ODEs. Find all values of the constant \( r \) so that \( y = e^{rt} \) is a solution. [Hint: Insert \( e^{rt} \) into the ODE and find values of \( r \) that yield a solution. For example, \( y = e^{rt} \) solves \( y' - y = 0 \) if \( re^{rt} - e^{rt} = (r-1)e^{rt} = 0 \). Since \( e^{rt} \neq 0 \), we must have \( r = 1 \). So \( y = e^{rt} \) solves the given ODE.]

(a) \( y' + 3y = 0 \) \hspace{1cm} (b) \( y'' + 5y' + 6y = 0 \)
(c) \( y''' - 3y'' + 2y' = 0 \) \hspace{1cm} (d) \( y'' + 2y' + 2y = e^{-t} \)

3. (Finding Solutions). Sometimes multiples of a power of \( t \) solve an ODE. Find all values of the constant \( r \) so that \( y = rt^r \) is a solution.

(a) \( t^2y'' + 6ty' + 5y = 0 \) \hspace{1cm} (b) \( t^2y'' + 6ty' + 5y = 2t^3 \) \hspace{1cm} (c) \( ty' = y^2 \)

4. (Finding Solutions). Choosing the right value for \( r \) may give a solution \( y = r' \) of an ODE. Find all values of the constant \( r \) so that \( y = r' \) is a solution.

(a) \( t^2y'' + 4ty' + y = 0 \) \hspace{1cm} (b) \( t^2y'' + 7ty'' + 3t^2y' - 6ty' + 6y = 0 \)

5. (Using the Antiderivative Theorem). Use Theorem 1.3.1 to find all solutions in each case. [Hint: Write \( y'' = (y')' \) and \( y''' = (y'')' \).

(a) \( y'' = 5 + \cos t \) \hspace{1cm} (b) \( y'' = t^2 + t + e^{-t} \) \hspace{1cm} (c) \( y'' = e^{-t}\cos 2t \)
(d) \( y''' = 0 \) \hspace{1cm} (e) \( y'' = \sin t \) \hspace{1cm} (f) \( y'' = 0, \ y(0) = 0, \ y'(0) = 1 \) \hspace{1cm} (g) \( y'' = e^{-t} \) [Hint: Note that \( (e^t y' )' = e^t y'' + y' \)].

6. (Finding Solution Formulas). Use Theorem 1.3.1 to find all the solutions. [Hint: In parts (a)–(c) multiply each side of the ODE by \( e^{\int} \).

(a) \( y' + y = 1 \) \hspace{1cm} (b) \( y' + y = t \) \hspace{1cm} (c) \( y' + y = t + 1 \)
(d) \( 2yy' = 1 \) [Hint: \( (y' y)' = 2yy' \)] \hspace{1cm} (e) \( 2yy' = t \)

7. (Integrating Factors). Find the general solution of each ODE by using an integrating factor.

(a) \( y' - 2y = t \) \hspace{1cm} (b) \( y' - y = e^{2t} - 1 \) \hspace{1cm} (c) \( y' = \sin t - y \sin t \)
(d) \( 2y' + 3y = e^{-t} \) \hspace{1cm} (e) \( t(2y - 1) + 2y' = 0 \) \hspace{1cm} (f) \( y' + y = te^{-t} + 1 \)

8. (Solving an IVP, Long-Term Behavior). First find the general solution of the ODE. Then use the initial condition and find the solution of the IVP. Finally, discuss that solution’s behavior as \( t \to +\infty \).

(a) \( y' + y = e^{-t}, \ y(0) = 1 \) \hspace{1cm} (b) \( y' + 2y = 3, \ y(0) = -1 \)
(c) \( y' + 2ty = 2t, \ y(0) = 1 \) \hspace{1cm} (d) \( y' + (\cos t)y = \cos t, \ y(\pi) = 2 \)

9. Use a numerical solver to plot the solution curve of each IVP of Problem 8 on the given \( t \)-interval and compare with the graph of the “true” solution.

(a) \([-2, 6]\) \hspace{1cm} (b) \([-1.5]\) \hspace{1cm} (c) \([-5.5]\) \hspace{1cm} (d) \([-8.8]\)

10. (Using an Integrating Factor). Find the general solution of each ODE over the indicated \( t \)-interval by using an integrating factor. Discuss the behavior of the solution as \( t \to 0^+ \). Discuss the behavior of the ODEs in parts (a), (b), and (d) as \( t \to +\infty \).

(a) \( ty' + 2y = t^2, \ t > 0 \) \hspace{1cm} (b) \( (3t - y) + 2ty^2 = 0, \ t > 0 \)
(c) \( y' = (\tan t)y + \sin 2t, \ |t| < \pi/2 \) \hspace{1cm} (d) \( y' = y/t + e^t, \ t > 0, \ \text{integer} \ n \)

11. (Solving an IVP). Find the general solution of the ODE over the indicated \( t \)-interval. Then use the initial condition to find the solution of the IVP. Finally, discuss the behavior of the solution as \( t \) tends to the given value.

(a) \( ty' + 2y = \sin t, \ t > 0; \ y(\pi) = 1/\pi; \ t \to +\infty \)
(b) \( (\sin t)y' + (\cos t)y = 0, \ 0 < t < \pi; \ y(3\pi/4) = 2; \ t \to 0^+ \)
(c) \( y' + (\cot t)y = 2\cos t, \ 0 < t < \pi; \ y(\pi/2) = 3; \ t \to 0^+ \)
(d) \( y'' + (2/t)y = (\cos t)/t^2, \ t > 0; \ y(\pi) = 0; \ t \to 0^+, \ t \to +\infty \)
12. Use a numerical solver to plot the solution curve of each IVP of Problem 11 on the given interval and compare with the “true” solution.
   (a) [0, 50]  (b) [0, 3]  (c) [0, 3]  (d) [0, 4]

13. (Do-It-Yourself Linear ODE). Make up several linear ODEs for which a solution formula is not helpful in deciding how solutions behave. Use your numerical solver to find out what happens to solutions as $t$ increases and as $t$ decreases. For example, try the ODE $y' + 2ty = 1/(1 + t^2)$.

### 1.4 Modeling with Linear ODEs

First-order linear ODEs model natural processes that range from the changing concentration of a pollutant in a water tank to the rising temperature of an egg as it is being hard-boiled. We will model one of these processes and leave others to the problem set. Next we will take another look at the solution formula constructed in the last section. That formula reveals a simple structure for all solutions of a first-order linear ODE, and we will use this structure to understand what is going on in our models.

#### The Balance Law and a Compartment Model

First-order linear ODEs often arise in applications as the result of an underlying basic principle: If $y(t)$ denotes the size of a population or the amount of a substance in a **compartment** at time $t$, then the rate of change $y'(t)$ can be calculated as the “rate in” minus the “rate out” for the compartment. We formalize this principle as the **Balance Law**.

**Balance Law.** Net rate of change = Rate in − Rate out.

Let’s apply the Balance Law to a mixture process.

#### Example 1.4.1 A Model IVP for Accumulation of a Pollutant

A tank contains 100 gallons of contaminated water in which $y_0$ pounds of pollutant are dissolved. Contaminated water starts to run into the tank at the rate of 10 gallons per minute. The concentration of pollutant in this incoming stream at time $t$ is $c(t)$ pounds per gallon. The solution in the tank is thoroughly mixed, and contaminated water flows out at the rate of 10 gallons per minute. Find the IVP for the amount of pollutant $y(t)$ in the tank.

Think of the tank as the compartment that the pollutant occupies, and apply the Balance Law to find an expression for $y'(t)$. The initial time is 0, so $y_0$ is the amount of pollutant in the water at $t = 0$. Pollutant is added to the tank at the inflow rate

\[
10 \text{ gal/min} \cdot \left( \frac{c(t) \text{ lb}}{\text{gal}} \right) = 10c(t) \text{ lb/min}
\]
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Pollutant leaves the tank at the rate
\[
\left(10 \text{ gal/min}\right) \cdot \left(\frac{y(t)}{100 \text{ gal}}\right) = 0.1 y(t) \frac{\text{lb}}{\text{min}}
\]
since contaminated water drains at 10 gal/min, and the pollutant concentration in the tank and in the exit stream at time \(t\) is \(y(t)/100\) lb/gal. The Balance Law says that
\[
y'(t) = \text{Rate in} - \text{Rate out} = 10c(t) - 0.1 y(t)
\]
The corresponding IVP is
\[
y'(t) + 0.1 y(t) = 10c(t), \quad y(0) = y_0, \quad t \geq 0
\]
which is our model.

Mathematical formulas are a dime a dozen, but understanding their meaning and how they can be used takes thought and theory. One way to do this is to look at and interpret each part of a formula. Before solving the specific IVP (1) let’s take some time out to examine the solution formula of a first-order linear ODE more closely.

Output Depends on Initial Data and Input

The output of a process depends on the initial data and the input. In the context of a first-order linear ODE we can describe the dependence in a quite precise way.

**THEOREM 1.4.1**

**Response to Data and Input.** Suppose that \(p(t)\) and \(q(t)\) are continuous on an interval \(I\) containing \(t_0\) and that \(P(t)\) is any antiderivative of \(p(t)\). Then the IVP
\[
y' + p(t)y = q(t), \quad y(t_0) = y_0
\]
has a unique solution, which can both be written as a formula and interpreted in terms of initial data \(y_0\) and input \(q(t)\):
\[
y(t) = e^{P(t_0)}y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{P(s)} q(s) \, ds, \quad t \in I
\]

Total Response = Response to initial data + Response to input

To show this, note that the general solution formula for \(y' + p(t)y = q(t)\) is
\[
y = C e^{-P(t)} + e^{-P(t)} R(t)
\]
where \(C\) is any constant and \(P(t)\) and \(R(t)\) are any antiderivatives of \(p(t)\) and \(e^{P(t)}q(t)\), respectively. Let’s take a specific antiderivative of \(e^{P(t)}q(t)\):
\[
R(t) = \int_{t_0}^{t} e^{P(s)} q(s) \, ds
\]
Now set $y = y_0$ and $t = t_0$:

$$y_0 = Ce^{-P(t_0)} + e^{-P(t_0)} \int_{t_0}^{t} e^{P(s)} q(s) \, ds = Ce^{-P(t_0)}$$

So $C = e^{P(t_0)} y_0$ and solution formula (3) follows.

We will use the response interpretation often, particularly in the applications, because we can see from formula (3) exactly how changes in the initial value $y_0$ and in the driving function $q(t)$ will affect the output $y(t)$.

Since the formula (3) defines a solution $y(t)$ for IVP (2) for any choice of $y_0$ and $q(t)$, we see that

$$\text{Response to initial data} \quad y(t) = e^{P(t_0)} y_0 e^{P(t)} \quad (6)$$

solves the IVP

$$y' + p(t)y = 0, \quad y(t_0) = y_0 \quad (7)$$

[just put $q(t) = 0$ in formula (3)], and that

$$\text{Response to input} \quad y(t) = e^{-P(t)} \int_{t_0}^{t} e^{P(s)} q(s) \, ds \quad (8)$$

solves the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = 0 \quad (9)$$

[just put $y_0 = 0$ in formula (3)]. Adding the right sides of (6) and (8) gives the total response.

**Response Analysis for the Pollutant Model**

Now that we know something about the structure of solutions for first-order linear ODEs, we will return to the pollutant accumulation model to analyze and to interpret its solutions.

**EXAMPLE 1.4.2 How Much Pollutant Is in the Tank?**

Now let’s solve IVP (1). After multiplying through by the integrating factor $e^{0.1t}$, the ODE in (1) becomes

$$e^{0.1t}[y'(t) + 0.1y(t)] = [e^{0.1t} y(t)]' = e^{0.1t} 10c(t)$$

So we see from Theorem 1.4.1 that

$$y(t) = y_0 e^{-0.1t} + e^{-0.1t} \int_{0}^{t} e^{0.1s} [10c(s)] \, ds, \quad t \geq 0 \quad (10)$$

**Total Response = Response to initial amount + Response to inflow**

The first term on the right-hand side of formula (10) shows the declining amount of pollutant in the tank if pure water were to run in and contaminated water run out at the
same rate. The second term shows the amount of pollutant in the tank at time \( t \) due solely to the incoming stream of contaminated water. In the long run we expect the second term to dominate.

It is the linearity of the ODE that allows the decoupling of the effect of the initial amount of pollutant from the effect of the incoming stream. We can illustrate all this by graphing the terms of formula (10) separately. To be specific, let’s take \( y_0 = 15 \) lb and assume an input concentration \( c(t) \) that varies sinusoidally about a mean of 0.2 lb of pollutant per gallon. For example, suppose that

\[
c(t) = 0.2 + 0.1 \sin t
\]  

(11)

We could have used an antiderivative in Table 1.3.1 to work out the integral in formula (10), but instead we used our numerical solver to solve IVP (1) directly. The results are displayed in Figure 1.4.1. We plotted the “Response to initial amount” and the “Response to inflow” curves by applying our solver to the respective IVPs (7) and (9). Over time, the amount of pollutant in the tank at the start becomes almost irrelevant, and it is pollutant in the inflow stream that determines the amount of pollutant in the outflow.

It looks as if the long-term response of the ODE to an oscillating input is an oscillatory output with the same frequency as the input. More (a lot more) on this curious behavior in Section 2.2.
Cleaning Up the Water Supply: Step Functions

Step functions can be used to switch midstream from one model ODE to another. The basic step function is defined like this:

\[ \text{step}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \]  

(12)

We can work with this function as we would with any other. For example,

\[ 3 \text{step}(t - 5) = \begin{cases} 0, & t < 5 \\ 3, & t \geq 5 \end{cases} \]  

and  

\[ -4 \text{step}(15 - t) = \begin{cases} -4, & t \leq 15 \\ 0, & t > 15 \end{cases} \]

Step functions and other piecewise linear functions are commonly used to model on-off behavior in natural processes. We call them the engineering functions. Here’s an example where a step function comes in handy.

**Example 1.4.3 Cleaning Up the Water Supply**

In the tank model of Example 1.4.1 let’s suppose that the inflow pollutant concentration \( c(t) \) is 0.2 lb/gal, but that after 20 minutes all the pollutants are filtered out of the inflow stream. We can express \( c(t) \) in terms of the step function in the following way:

\[ c(t) = 0.2 \text{step}(20 - t), \quad t \geq 0 \]  

(13)

So the model IVP

\[ y' + 0.1y = 10c(t), \quad y(0) = y_0, \quad t \geq 0 \]  

(14)
has, using formula (10), the solution

\[ y = y_0 e^{-0.1t} + e^{-0.1t} \int_0^t e^{0.1s}[2 \text{step}(20 - s)] ds, \quad t \geq 0 \quad (15) \]

Notice that the integral in formula (15) changes its form at \( t = 20 \):

\[
\int_0^t e^{0.1s}[10 \cdot 0.2] ds = \begin{cases} 
\int_0^t e^{0.1s}[10 \cdot 0.2] ds = 20(e^{0.1t} - 1), & 0 \leq t \leq 20 \\
\int_0^t e^{0.1s}[10 \cdot 0.2] ds = 20(e^2 - 1), & t > 20 
\end{cases}
\]

We can put this integral into formula (15) and obtain the solution formula

\[ y = y_0 e^{-0.1t} + \begin{cases} 
20(1 - e^{-0.1t}), & 0 \leq t \leq 20 \\
20(e^2 - 1)e^{-0.1t}, & t > 20 
\end{cases} \]

We can graph solutions using this formula, or alternatively we can use a numerical solver to solve IVP (14) directly. We prefer the latter approach.

Figure 1.4.2 shows the changing pollutant concentration in the tank if \( y_0 = 15 \) lb. The corners on two of the solution curves are caused by the discontinuities at \( t = 20 \) in the step function used to model the inflow stream.

The set of solutions of a first-order linear ODE has a structure that is often used to find solutions without extensive use of integrating factors. Let’s see how this approach works.

**Another Approach to Finding a Solution Formula**

There is a way to find a formula for all solutions of a first-order linear ODE other than the method of integrating factors presented in Section 1.3. In practice, engineers and scientists use this alternative approach in solving linear ODEs. Here is the result we need:

**THEOREM 1.4.2**

**Structure of Solutions.** Suppose that \( p(t) \) and \( q(t) \) are continuous on an interval \( I \). Then the general solution \( y(t) \) of the driven linear ODE

\[ y' + p(t)y = q(t) \quad (16) \]

is given by

\[ y(t) = y_u(t) + y_d(t), \quad \text{all } t \text{ in } I \quad (17) \]

where \( y_u(t) \) is the general solution of the undriven linear ODE

\[ y' + p(t)y = 0 \quad (18) \]

and \( y_d(t) \) is any particular solution of the driven ODE (16) itself (any one will do).
To see why this is true, take any particular solution \( y_d(t) \) of the driven ODE (16); it can be any one. Now we show that \( y(t) \) is a solution of ODE (16) if and only if the function \( w = y - y_d \) is a solution of the undriven ODE (18). This fact results from the computation

\[
w' + pw = (y - y_d)' + p(y - y_d) = (y' + py) - (y'_d + py_d) = q - q = 0
\]

So this means that \( y = y_u + y_d \), where \( y_u \) is the general solution of the undriven ODE (16). We saw earlier that \( y_u = Ce^{-P(t)} \), where \( C \) is any constant and \( P(t) \) is any single antiderivative of the coefficient \( p(t) \).

Here’s an example that uses Theorem 1.4.2 to find the general solution.

**EXAMPLE 1.4.4 Good Guessing Works**

The undriven form of the ODE

\[
y' + y = 17 \sin 4t
\]

is \( y' + y = 0 \), which has \( y_u = Ce^{-t} \) as its general solution. So the general solution of the driven ODE (19) has the form

\[
y = Ce^{-t} + y_d(t)
\]

where \( C \) is any constant and \( y_d \) is any one particular solution.

We could use the procedure of the last section to find a particular solution, but there is another way. Because the input function is \( 17 \sin 4t \), a good guess for a particular solution would be

\[
y_d = A \sin 4t + B \cos 4t
\]

where \( A \) and \( B \) are constants to be determined. To see this, put this form of \( y_d \) into the left side of ODE (19) to obtain

\[
y'_d + y_d = 4A \cos 4t - 4B \sin 4t + A \sin 4t + B \cos 4t = (A - 4B) \sin 4t + (4A + B) \cos 4t
\]

So for \( y_d \) in (20) to solve ODE (19) we must have

\[
(A - 4B) \sin 4t + (4A + B) \cos 4t = 17 \sin 4t
\]

The only way for this equality to hold for all \( t \) is for \( A \) and \( B \) to satisfy the equations

\[
A - 4B = 17, \quad 4A + B = 0
\]

Solving for \( A \) by multiplying the second equation by 4 and then adding the two equations, we see that \( A = 1 \). So \( B = -4 \), and inserting these values into our trial solution (20), we have constructed a particular solution,

\[
y_d = \sin 4t - 4 \cos 4t
\]
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The general solution is then
\[ y = Ce^{-t} + \sin 4t - 4\cos 4t, \quad C \text{ any constant} \quad (21) \]
so our guessing approach has paid off.

The structure formula (17) sometimes reveals interesting properties of solutions.

**EXAMPLE 1.4.5** Attraction to a Unique Periodic Solution
In Example 1.4.4 we showed that all solutions of the ODE \( y' + y = 17 \sin 4t \) have the form \( y = Ce^{-t} + y_d(t) \), where \( y_d(t) = \sin 4t - 4\cos 4t \) and \( C \) is an arbitrary constant. Since \( Ce^{-t} \to 0 \) as \( t \to +\infty \), we see that all solutions are attracted to the single solution \( y_d(t) \), so the ODE has a unique periodic solution. This periodic solution \( y_d(t) \) has period \( \pi/2 \), exactly the same period as the input \( 17 \sin 4t \). Figure 1.4.3 shows the strong attraction of all solutions toward \( y_d \).

**Comments**
In this section we have shown how to interpret the formulas for the general solution of a linear ODE and for the solution of a corresponding IVP. That is a crucial aspect when you use an IVP to model a natural process, and the application of the Balance Law to the pollutant concentration process bears this out.
PROBLEMS

1. (Pollution). Contaminated waste water is pumped at the rate of 1 gal/min into a tank containing 1000 gal of clean water, and the well-stirred mixture leaves the tank at the same rate.
   (a) Find the amount $y(t)$ of waste in the tank at time $t$. What happens as $t \to +\infty$?
   (b) How long does it take the concentration of waste to reach 20% of its maximum level?
   (c) After an hour, the waste inflow is stopped and clean water is pumped in at 1 gal/min. Use a step function and create a model IVP for this situation. Use a numerical solver to plot $y(t)$ over the time span $0 \leq t \leq 1500$. What happens as $t \to +\infty$? Explain what you see.

2. (Pollution). Water with pollutant concentration $c_0$ lb/gal starts to run at 1 gal/min into a vat holding 10 gal of water mixed with 10 lb of pollutant. The mixture runs out at 1 gal/min.
   (a) Find the amount $y(t)$ of pollutant in the vat at time $t$. Find $\lim_{t \to +\infty} y(t)$.
   (b) Plot $y(t)$ over the interval $0 \leq t \leq 40$ for values of $c_0$ in the range $0.1 \leq c_0 \leq 2$. Include $c_0 = 1.0$ and values of $c_0$ above and below 1.0. Explain why as $t \to +\infty$, $y(t) \to 10c_0$.
   (c) Suppose that $c_0 = 1$ for the first 10 min, and then an efficient filter removes all of the pollutant from the inflow stream. Find the model IVP, using a step function to represent the input. As in Figure 1.4.2, use a numerical solver to plot the total response, the response to the initial data, and the response to the input. If possible, put all three curves on the same graph; otherwise use three graphs with the same $t$ and $y$ scales. What happens as $t \to +\infty$?
   (d) Repeat part (c), but with an inefficient filter that only removes 50% of the pollutant.
   (e) Repeat part (c), but with an efficient filter that is in place for just 10 minutes at the start of each hour. Plot for $0 \leq t \leq 150$, $-5 \leq y \leq 15$. What happens to each response as $t$ increases? [Hint: Use the function $sqw(t, 100/6, 60)$.]

3. (Salt Solution). A solution containing 2 lb of salt per gallon starts to flow into a tank of 50 gal of pure water at a rate of 3 gal/min. After 3 minutes the mixture starts to flow out at 3 gal/min.
   (a) How much salt is in the tank at $t = 2$ min? At $t = 25$ min? [Hint: Solve two IVPs: for $t_0 = 0$ and for $t_0 = 3$.]
   (b) How much salt is in the tank as $t \to +\infty$? Can you guess without any calculation?

4. (Good Guessing and Undetermined Coefficients). Find a particular solution for the linear ODE by first guessing the form of the solution, and then determining the coefficients. Then find the general solution.
   (a) $y' + y = t^2$ [Hint: Try $y_p = At^2 + Bt + C$.]
   (b) $y' + ty = t^2 - t + 1$
   (c) $y' + 2y = e^{-2t}$ [Hint: Try $y_p = Ate^{-2t}$.]
   (d) $y' + 2y = 3e^{-t}$
   (e) $y' + y = 5 \cos 2t$
   (f) $y' + y = e^{-t} \cos t$

5. (Long-Term Behavior). Describe the long-term behavior (i.e., as $t \to +\infty$) of all solutions of the corresponding ODE in parts (a)-(f) of Problem 4 by using the general solution formula. Justify your conclusions.

6. (More Guessing and Undetermined Coefficients). For all values of the constants $a$, $b$, $c$, find a particular solution of each ODE.
   (a) $y' + ay = b \cos t + c \sin t$
   (b) $y' + ay = bt + c$, $a \neq 0$

7. (Long-Term Behavior). Find all solutions of the undriven ODE $y' + 2y = 0$, and describe what happens as $t \to +\infty$. Use your numerical solver to plot several solutions of the driven ODE $y' + 2y = q(t)$ over a long enough time span that you can conjecture what happens to its general solution as $t \to +\infty$. Why is the general solution formula of little use here?
   (a) $q(t) = 1$
   (b) $q(t) = (1 + 2t)^{-1}$
   (c) $q(t) = 1 + t^2$
   (d) $q(t) = 1 + 2t^2$ [Hint: Guess a solution $y_d = A + Bt$]
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8. (Long-Term Behavior). The examples in this section may give the impression that as \( t \to +\infty \) the general solution \( y(t) \) of the undriven ODE \( y' + p(t)y = 0 \) tends to 0, so the general solution \( y(t) \) of the driven ODE \( y' + p(t)y = q(t) \) would tend to a particular solution \( y_d(t) \) as \( t \to +\infty \). Show that this is not always the case by creating your own linear ODE \( y' + p(t)y = q(t) \) where there is some solution of the undriven ODE \( q = 0 \) that tends to \( +\infty \) as \( t \to +\infty \), and there is some solution \( y_d(t) \) of the driven ODE such that \( y_d(t) \to 0 \) as \( t \to +\infty \).

9. Use a numerical solver in plotting solutions of the given IVPs.
   (a) Plot solutions of the IVPs \( y' + y = 0 \), \( y(-5) = 0, \pm 2, \pm 4, -5 \leq t \leq 10 \).
   (b) Plot the solution \( y = y_d(t) \) of the IVP \( y' + y = t \cos(t^2) \), \( y(-5) = 0, -5 \leq t \leq 10 \).
   (c) Plot the solutions of the IVPs \( y' + y = t \cos(t^2) \), \( y(-5) = 0, \pm 2, \pm 4, -5 \leq t \leq 10 \). Use (a) and (b) to explain the behavior of the solutions. What happens as \( t \to +\infty \)?

10. Plot solutions of the IVPs \( y' = -(\cos t)y + \sin t \), \( y(-10) = -6, -2, 0, 5, |t| < 10 \), and explain why a general solution formula isn’t any help here.

11. Use a numerical solver or a solution formula to plot solution curves.
   (a) \( y' = (1 - y)(\sin t) \); \( y(-\pi/2) = -1, 0, 1; -\pi/2 \leq t \leq 2\pi \)
   (b) \( y' + y = e^{t+1}; y(0) = -1, 0, 1; -1 \leq t \leq 3 \)
   (c) \( ty' + 2y = t^2 \); \( y(2) = 0, 1, 2; 0 < t < 4 \). Repeat for \( y(-2) = 0, 1, 2; -4 < t < 0 \). What happens to each solution curve as \( t \to 0 \)?
   (d) \( y' = y\tan t + t\sin 2t; y(0) = -1, 0, 1; -\pi/2 < t < \pi/2 \)

12. (Adjusting the Input). The output \( y(t) \) of the IVP \( y'(t) + p(t)y(t) = q(t) \), \( y(0) = y_0 \), must be kept at the level \( y_0 \) for \( t \geq 0 \). Determine an input \( q(t) \) that will accomplish the task.

13. (Unbounded Solutions). Suppose that \( q(t) \) is continuous for all \( t \), and that \( y' + y = q(t) \) has a particular solution \( y_d(t) \) with the property that \( y_d(t) \to +\infty \) as \( t \to +\infty \). Explain why all solutions have this property.

14. (Newton’s Law of Cooling). According to Newton’s Law of Cooling (or Warming) the rate of change of the temperature of a body is proportional to the difference between the body’s temperature and the surrounding medium’s temperature.

   (a) Write a model ODE for the body’s temperature, given the medium’s temperature \( m(t) \). Is the proportionality constant positive or negative?
   (b) (A Sick Horse). A veterinarian wants to find the temperature of a sick horse. The readings on the thermometer follow Newton’s Law. At the time of insertion the thermometer reads 82°F. After 3 min the reading is 90°F, and 3 min later 94°F. A sudden convulsion destroys the thermometer before a final reading can be obtained. What is the horse’s temperature?
   (c) (Cooling an Egg). A hard-boiled egg is removed from a pot of hot water and set on the table to cool. Initially, the egg’s temperature is 180°F. After an hour its temperature is 140°F. If the room’s temperature is 65°F, when will the egg’s temperature be 120°F, 90°F, 65°F?
   (d) (Cold Body Cools Hot Medium). A cold egg is placed in a pot of hot water. As the egg warms up, the water cools down. Create a model ODE for the temperature of the egg and another for the temperature of the water. What happens to the two temperatures as \( t \to +\infty \)? Explain what you do.

15. (Continuously Compounded Interest). Some banks compound your savings continuously; for example, if the interest rate is 9%, then the ODE for the money in your account is \( A' = 0.09A \).
   (a) What interest rate payable annually is equivalent to 9% interest compounded continuously?
   (b) What is the interest rate if continuously compounded funds double in eight years?
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(c) How long will it take $A$ dollars invested in a continuously compounded savings account to double if the interest rate is 5%; 9%; 12%?

16. (Rule of 72). Two common rules of thumb for bankers are the “Rule of 72” and the “Rule of 42.” These rules say that the number of years for money invested at $r\%$ interest to increase by 100% or 50% is given by $72/r$ or $42/r$, respectively. Assuming continuous compounding, show that these rules overestimate the time required.

17. (Compound Interest). At the end of every month Wilbert deposits a fixed amount in a savings account. He wants to buy a car that will cost $12,400. Currently, he has $5800 saved. If he earns 7% interest compounded continuously on his savings and has an income of $2600 per month, to what amount must Wilbert limit his monthly expenses if he is to have enough saved to buy the car in one year?

1.5 Introduction to Modeling and Systems

Modeling is a process of recasting a natural process from its natural environment into a form, called a model, that can be analyzed via techniques we understand and trust. A model is a device that helps the modeler to predict or explain the behavior of a phenomenon, experiment, or event. For example, say that a rocket is to be put into orbit. Physical intuition alone can’t give us more than a rough idea of what guidance strategy to use once the rocket is launched. Since accuracy will be critical, a mathematical model of the problem is constructed using applicable natural laws. Then we can use the equations, constraints, and control elements in the model to give a reasonably precise description of the orbital elements of the rocket in its course.

Using models to explain outcomes of an observable situation is illustrated in Figure 1.5.1. Broad portions of the natural environment are given a mathematical form by a general model in which all possible outcomes are described by a few basic principles. A specific problem in the environment is translated into a specific mathematical problem in the general model. The mathematical problem is solved (often by a computer simulation) and the results are interpreted in the problem’s natural environment.

Models often end up as a collection of ordinary differential equations in several to-be-determined functions, or for short, differential systems. When we use the word “system” by itself we will most always mean “differential system.” The order of a system is the order of the highest derivative that occurs anywhere in the system. The first-order system

\[
\begin{align*}
\frac{dx}{dt} &= f(t, x, y) \\
\frac{dy}{dt} &= g(t, x, y)
\end{align*}
\]

in the to-be-determined functions $x(t)$ and $y(t)$ is in normal form because the derivatives appear alone on the left-hand side of the ODEs. Numerical solvers prefer to have systems entered in normal form.

System (1) is a linear differential system if the given rate functions $f(t, x, y)$ and $g(t, x, y)$ have the form

\[
\begin{align*}
f(t, x, y) &= a(t)x + b(t)y + h(t) \\
g(t, x, y) &= c(t)x + d(t)y + k(t)
\end{align*}
\]
where the coefficients $a(t), b(t), c(t), d(t)$ and the driving terms $h(t), k(t)$ are functions only of $t$. System (1) is nonlinear if the rate functions do not have the form (2).

Here’s an example of a nonlinear system: If $\theta(t)$ denotes the angle of a moving pendulum bob subject to gravity and a damping force, and if $v(t) = \theta'$, then the pair $\theta(t), v(t)$ solves the first-order system

$$\theta' = v, \quad v' = -0.1v - 4\sin \theta$$

System (3) is nonlinear because of the term $-4\sin \theta$.

In this section we’ll look at two models:

- Radioactive decay and its use in dating old objects.
- Vertical motion: Does the ball take longer to rise or to fall?

The study of radioactivity and moving bodies revolutionized science. That is why we chose to model the basic principles of these two natural processes.

### Modeling Radioactive Decay

Some elements are unstable, decaying into other elements by emission of alpha particles, beta particles, or photons. Such elements are said to be radioactive. For example, a radium atom might decay into a radon atom, giving up an alpha particle in the process. The decay of a single radioactive nucleus is a random event, and the exact time of decay can’t be predicted with certainty. Nevertheless, something definite can be said about the decay process of a large number of radioactive nuclei.

For a collection of radioactive nuclei in a sample, we would like to know the number of radioactive nuclei present at any given time. There is plenty of experimental evidence to suggest that the following decay law is true.

**Radioactive Decay Law.** In a sample containing a large number of radioactive nuclei, the rate of decrease in the number of radioactive nuclei at a given time is proportional to the number of nuclei present at the time.
Denoting the number of radioactive nuclei in the sample at time $t$ by $N(t)$, the law translates to the mathematical equation

$$N'(t) = -kN(t)$$

(4)

where $k$ is a positive coefficient of proportionality. Observation suggests that in most decay processes $k$ is independent of $t$ and $N$. A decay process of this type is said to be of first order with rate constant $k$.

This simple-looking law has all sorts of logical difficulties because we are using a "continuous" mathematical structure to describe discrete events. It’s a remarkable fact that this law leads to a mathematical model that allows us to make accurate predictions.¹

Let’s apply the Radioactive Decay Law to create a mathematical model that predicts how many radioactive nuclei are present in a given sample at any time $t$.

**EXAMPLE 1.5.1 Exponential Decay**

Suppose that there are $N_0$ radioactive nuclei in a sample of an element at time $t_0$. How many radioactive nuclei $N(t)$ are present at a later time? The Radioactive Decay Law describing how $N(t)$ evolves is given by the forward IVP

$$N' = -kN, \quad N(t_0) = N_0, \quad t \geq t_0$$

(5)

Using the method of integrating factors (Section 1.3), we see that IVP (5) has precisely one solution for any given value for $N_0$:

$$N(t) = N_0e^{-k(t-t_0)}, \quad t \geq t_0$$

(6)

Once $k$ and $N_0$ are known, formula (6) can be used to predict the values of $N(t)$.

The rate constant $k$ of a radioactive element is usually calculated from the element’s half-life, which is the time $\tau$ required for half of the nuclei to decay. Curiously, $\tau$ is independent of both the time when the clock starts and the initial amount. For example, we see from formula (6) that at any times $t \geq t_0$ and $\tau > 0$,

$$\frac{N(t + \tau)}{N(t)} = \frac{N_0e^{-k(t+\tau-t_0)}}{N_0e^{-k(t-t_0)}} = e^{-k\tau}$$

So if we require that $N(t + \tau) = \frac{1}{2}N(t)$, then $e^{-k\tau} = \frac{1}{2}$, and (after taking logarithms)

$$\tau = \frac{\ln 2}{k} \quad \text{and} \quad k = \frac{\ln 2}{\tau}$$

(7)

and neither $\tau$ nor $k$ depends on $t_0$ or on $N_0$. The half-lives of many radioactive elements have been determined experimentally. For example, the half-life $\tau$ of radium² is about 1600 years, so the rate constant $k$ for radium is about 0.0004332 (years)⁻¹.


² The Polish scientist Marie Curie (1867–1934) received the Nobel Prize for Physics in 1903 and 1911 for her pioneering experiments with radium and other radioactive substances.
Formula (6) can be used to make predictions about the value of \( N(t) \). These predictions can be checked against experimental determinations of \( N(t) \). Given the logical gaps mentioned above, it might seem surprising that formula (6) provides a remarkably accurate description of radioactive decay processes. But it is a fact of contemporary experimental science that this is so, at least for time spans that are neither too long nor too short. The leap of faith made in ignoring the flaws of the law and the model is justified by the results.

Here is one way that radioactive decay is used to date events that occurred long ago, even before recorded history. Living cells absorb carbon from carbon dioxide in the air. The carbon in some of this carbon dioxide is radioactive carbon-14 (denoted by \( ^{14}\text{C} \)), rather than the common \( ^{12}\text{C} \). Carbon-14 is produced by the collisions of cosmic rays with nitrogen in the atmosphere. The \( ^{14}\text{C} \) nuclei decay back to nitrogen atoms by emitting beta particles. All living things, or things that were once alive, contain some radioactive carbon nuclei. In the late 1940s, Willard Libby\(^3\) showed how a careful measurement of the \( ^{14}\text{C} \) decay rate in a fragment of dead tissue can be used to determine the number of years since its death. This process is called radiocarbon dating. See Problem 10 for the way this technique was used to date the age of prehistoric wall paintings in a cave near Lascaux, France.

Now let’s look at a completely different kind of natural process and model.

The Galilean Approach to Vertical Motion

Let’s suppose that a ball is moving along a vertical line near the ground. Neglecting air resistance, Galileo\(^4\) showed experimentally that the ball moves with constant acceleration \( g \), where \( g \) is the earth’s gravitational constant, and that this motion is independent of the size, shape, or mass of the ball. We will use this information to construct a model for the motion. It is known that the value of \( g \) is

\[
g \approx 9.8 \text{ m/sec}^2 = 980 \text{ cm/sec}^2 \approx 32 \text{ ft/sec}^2
\]

### EXAMPLE 1.5.2 Differential Equation for a Moving Ball: No Air Resistance

Suppose that \( y(t) \) measures the distance of a ball’s center above the ground at time \( t \), then the velocity of the ball at time \( t \) is \( v(t) = y'(t) \). The acceleration of the ball is

---

\(^3\)Willard Libby (1908–1980) was an American chemist who received the 1960 Nobel Prize for Chemistry for his work. His book *Radiocarbon Dating*, 2nd ed. (Chicago: University of Chicago Press, 1955) discusses the techniques used to do the dating. This process is most effective with material that is at least 200 years old, but not more than 70000 years old.

\(^4\)Galileo Galilei (1564–1642) was the first modeler of modern times. Although he is better known for his work in astronomy, his study of bodies dropped from a height and of motion on inclined planes led to a model for falling bodies relating the distance traveled to the time elapsed, and eventually to the law of acceleration. Later, Galileo turned his telescope toward the heavens and discovered four of the moons that orbit the planet Jupiter. His observations supported the Copernican model of the solar system. The connection between the physical and mathematical worlds is best said in Galileo’s verse, “Philosophy is written in this grand book of the universe, which stands continually open to our gaze... It is written in the language of mathematics.”
\[ a(t) = v'(t) = y''(t). \] Suppose the clock is started at \( t = 0 \), and at that instant \( y(0) = y_0 \) and \( v(0) = v_0 \). While the ball is in motion, Galileo’s experimental result implies that \( y''(t) = -g \) for all \( 0 \leq t \leq T \), where \( T \) is the time that the ball hits the ground. The minus sign arises because gravity acts downward in the direction of decreasing \( y \). Summarizing, we see that \( y(t) \) solves the IVP

\[
y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0
\]  

over the interval \( 0 \leq t \leq T \). Let’s call IVP (8) the Galilean model of vertical motion.

The ODE in (8) is not valid outside the interval \( 0 \leq t \leq T \) because we know nothing about the ball before \( t = 0 \) or after impact at \( t = T \). Now let’s solve IVP (8).

**Solution to Galileo’s Problem of the Moving Ball**

Suppose that \( y(t) \) solves IVP (8). Antidifferentiate each side of the ODE \( y'' = -g \) to obtain \( y' = -gt + C \) for some constant \( C \). The condition \( y'(0) = v_0 \) implies that \( C = v_0 \), and so

\[
y' = -gt + v_0
\]

Antidifferentiate each side of \( y' = -gt + v_0 \) to obtain \( y = -gt^2/2 + v_0 t + C \), where \( C \) is a constant. Since \( y = y_0 \) when \( t = 0 \), \( C \) must have the value \( y_0 \). The unique solution \( y(t) \) of IVP (8) is

\[
y(t) = y_0 + v_0 t - \frac{1}{2} gt^2, \quad 0 \leq t \leq T
\]  

Figure 1.5.2 shows the parabolic solution curves, where \( g = 9.8 \text{ m/sec}^2 \), \( y_0 = 10 \text{ m} \), and \( v_0 \) (in meters/second) takes on various values.

From formula (9) we see that if the earth had no atmosphere (and so no frictional forces to slow the body down), then all bodies moving vertically near the surface would move in the same way, regardless of their size, shape, or mass.

**The Newtonian Approach to Vertical Motion**

The Galilean model has flaws; for example, it ignores air resistance. When a vertically moving body is subjected to air resistance, Galileo’s experimental results do not provide a model that accurately describes the body’s motion. Actually, the motion of a body along the vertical is better governed by the following law:

**Newton’s Second Law (Restricted to Vertical Motion).** The product of the constant mass and the acceleration of a body moving vertically is the sum of the external forces acting vertically on the body.
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The celebrated general version of this law was formulated by Isaac Newton.\(^5\)

The gravitational force on a body of mass \(m\) near the earth’s surface is directed downward and has magnitude \(mg\). If the gravitational force is the only force acting on the body, Newton’s Second Law implies that the height \(y(t)\) of the body above the earth’s surface at time \(t\) satisfies the ODE \(m y'' = -mg\), since \(y''\) is the body’s acceleration. So \(y'' = -g\), the same result we obtained in Example 1.5.2.

But common sense tells us that air resistance will dampen the motion. So let’s build a better model that includes damping.

\(^5\)Isaac Newton (1642–1727) began his work in science and mathematics when he entered Trinity College in Cambridge, England in 1661. He graduated in 1665 without any special honors and returned home to avoid the plague, which was rapidly spreading through England that year. During the next two years, Newton discovered calculus, determined basic principles of gravity and planetary motion, and recognized that white light is composed of all colors—discoveries which he kept to himself. In 1667, Newton returned to Trinity College, obtaining a Master’s degree and staying on as a professor. Newton continued work on his earlier discoveries, formulated the law of gravitation, basic theories of light, thermodynamics, and hydrodynamics, and invented the first good reflecting telescope. In 1687 he was finally persuaded to publish *Philosophae Naturalis Principia Mathematica*, which contains his basic laws of motion and is considered one of the most influential scientific books ever written. Despite his great accomplishments, Newton said of himself that, “I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.”
Viscous Damping and the Motion of a Whiffle Ball

Experiments show that for a body of low density and extended rough surface (e.g., a feather, snowflake, or whiffle ball) the resistance of the air exerts a force on the body proportional to the magnitude of the velocity, but acting opposite to the direction of motion. This kind of resistive force is called \textit{viscous damping}, and the constant of proportionality is the \textit{viscous damping constant}. Suppose that \( y \) measures the distance along the local vertical, with up as the positive direction. Then \( y' = v \) is the velocity of the body. The resistive force is modeled by \(-k v\), where \( k \) is the viscous damping constant, and the minus sign reflects the fact that the force opposes motion.

Let’s suppose that a whiffle ball of constant mass \( m \) is thrown straight up from ground level with initial velocity \( v_0 \). Then by Newton’s Second Law for Vertical Motion, the location \( y(t) \) of the object solves the IVP

\[
my'' = -mg - k y', \quad y(0) = 0, \quad y'(0) = v_0 \tag{10}
\]

and we have the \textit{viscous damping model} for the motion of the ball.

\textbf{EXAMPLE 1.5.4 Tracking the Whiffle Ball}

Since \( y' = v \), we have \( y'' = v' \); rearranging IVP (10) we obtain the following linear IVP in \( v \):

\[
v' + \left( \frac{k}{m} \right) v = -g, \quad v(0) = v_0 \tag{11}
\]

Now IVP (11) can be solved using the integrating factor \( e^{k t/m} \) to obtain a formula for the whiffle ball’s velocity:

\[
v(t) = \left( v_0 + \frac{mg}{k} \right) e^{-kt/m} - \frac{mg}{k} \tag{12}
\]

We can find the location \( y(t) \) of the ball by replacing \( v(t) \) by \( y'(t) \) in formula (12) and then integrating. We obtain [using the fact that \( y(0) = 0 \)]

\[
y(t) = \int_0^t \left[ \left( v_0 + \frac{mg}{k} \right) e^{-ks/m} - \frac{mg}{k} \right] ds \\
= \left[ \frac{-m}{k} \left( v_0 + \frac{mg}{k} \right) e^{-kt/m} - \frac{mg}{k} s \right]_{s=0}^t \\
= \frac{m}{k} \left( v_0 + \frac{mg}{k} \right) (1 - e^{-kt/m}) - \frac{mg}{k} t, \quad 0 \leq t \leq T \tag{13}
\]

where \( T \) is the time the ball hits the ground.

From formula (12) we see that as \( t \to +\infty \), \( v(t) \) approaches the \textit{limiting velocity} \( v_\infty = -mg/k \). The whiffle ball’s velocity quickly gets very close to \( v_\infty \) because the ball’s mass is small and its damping constant is large. Just toss a whiffle ball into the air and watch its slow fall at an apparently constant speed.

Does the whiffle ball take longer to rise to its maximal height, or to fall back from that height? Formula (13) doesn’t help at all to answer this question.
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\[\begin{align*}
n' &= 0 \\
v' &= -9.8 - 3v, \quad v(0) = v_0
\end{align*}\]

**FIGURE 1.5.3** \(k/m = 3\): Longer to rise or to fall? (Example 1.5.5).

**FIGURE 1.5.4** \(k/m = 10\): Longer to rise or to fall? (Example 1.5.5).

**Longer to Rise or to Fall?**

To decide whether the whiffle ball takes longer to rise or to fall, we can use a computer to graph \(y = y(t)\) and answer the question by visual inspection. The graph can be produced by using graphing software and formula \((13)\) for \(y(t)\). A better way would be to use a numerical solver directly to solve the IVP

\[\begin{align*}
y'' &= -g - \frac{k}{m}y', \quad y(0) = 0, \quad y'(0) = v_0 \\
y' &= v, \quad y(0) = 0 \\
v' &= -g - \frac{k}{m}v, \quad v(0) = v_0
\end{align*}\]

Although some solvers accept second-order ODEs, many do not. All solvers accept an IVP for a first-order system of ODEs equivalent to IVP \((14)\). Here is how we convert IVP \((14)\) into such a first-order system. Set \(y' = v\) and use IVP \((14)\) to obtain

\[\begin{align*}
y' &= v, \quad y(0) = 0 \\
v' &= -g - \frac{k}{m}v, \quad v(0) = v_0
\end{align*}\]

If \(y = y(t)\) and \(v = v(t)\) solves IVP \((15)\), then we see that

\[\begin{align*}
y'' &= v' = -g - \frac{k}{m}y', \quad y(0) = 0, \quad y'(0) = v_0
\end{align*}\]

and so \(y(t)\) solves IVP \((14)\). In other words, IVPs \((14)\) and \((15)\) are equivalent. So let’s specify values for \(g, k/m,\) and \(v_0\) and apply a numerical solver to IVP \((15)\).

**EXAMPLE 1.5.5**

It Takes Longer to Fall!

With \(g = 9.8\) m/sec\(^2\), \(k/m = 3\) and then 10 sec\(^{-1}\), and using various values of \(v_0\), our solver produced the graphs shown in Figures 1.5.3 and 1.5.4. We see that the whiffle ball does indeed take longer to fall than it does to rise, and this is more apparent the higher the initial velocity. A whiffle ball with high initial velocity and high \(k/m\) ratio quickly reaches its maximal height, and then very slowly falls back. Use a straightedge...
and determine the slope of the flat side of the top graph in Figure 1.5.4. The measured slope is close to the limiting velocity \(-\frac{mg}{k}\), which has the value \(-0.98\) m/sec.

This doesn’t prove that the fall time is greater than the rise time, but the graphical evidence strongly suggests it. Toss a whiffle ball into the air, and check it out.

**Dynamical Systems**

These examples bring to light some essential components of ODE models.

**Elements of a Model**

**Natural Variables:** A natural process is described by a collection of variables called *natural variables* that depend on a single independent variable. For the viscous damping problem, the independent variable is time \(t\), and the natural variables are the position, velocity, and acceleration of the ball at time \(t\).

**Natural Laws:** A natural process evolves in time according to *natural laws* or *principles* involving the natural variables. Sometimes these laws arise empirically (e.g., the viscous damping law), and sometimes they have an intrinsic significance, such as Newton’s Second Law. Sometimes, a natural law is expressed in commonsense terms, such as the Balance Law in Section 1.4. Using appropriate notation, a mathematical structure can be given to the natural variables and the natural laws.

**Natural Parameters:** The natural laws often contain *parameters* that must be experimentally determined; for example, the viscous damping constant \(k\) is a parameter.

In many situations the natural laws describe a process that evolves over time, so we almost always think of time as the independent variable.

❖ **State Variables, Dynamical Systems.** State variables are natural variables whose given values at an instant, together with the natural laws of the process, uniquely determine values of these variables for all times when the laws apply. A natural process (or its mathematical representation) described by state variables is a *dynamical system*. Values of state variables at any instant are said to describe the *state* of the dynamical system at that instant.

For example, position and velocity are state variables for the dynamical system of a vertically moving body. The dynamical systems approach to modeling, involving as it does the evolution of the state of the system through time, is a concept that is modern in its scope but has been around in some form for centuries. It is our basic approach.

Some conventional terminology is used in speaking about a dynamical system. The effects of the external environment are referred to as *input data* to the system. Input data are often called *driving terms* or *source terms*, and such systems are said to be *driven*. Values of the state variables at a given initial time are *initial data*. The behavior of the state variables due to the input and initial data is the *response* (or
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Experimental determination of natural parameters

Initial data

Law governing evolution of the natural process

Input

Response

FIGURE 1.5.5 Schematic of a dynamical system.

output) of the dynamical system. Figure 1.5.5 summarizes the basic components of a dynamical system. We need to say something more about experimental determination of natural parameters, an often overlooked component of dynamical systems. The proportionality coefficients in our models play a very important role in the modeling process and someone has to determine their values. This important task usually falls upon scientists and engineers.

The Modeling Process

We need tools, variables, and natural laws to build a model, but one important step in the modeling process is often overlooked. This last step in the modeling process is usually called validation of the model. The modeler needs to make definitions and simplifying assumptions and discover some laws or principles that govern or explain the behavior of the phenomenon at hand. The goal of the modeler is to generate a model that is general enough to explain the phenomenon at hand, but not so complicated that analysis is impossible. There are trade-offs in this process. To have confidence in the model, the modeler often solves special problems and checks the results against experimental evidence. In this way the modeler learns something about the limits of applicability of the model. For example, if the differential equation \( y'' = -g \) were used to model the motion of a whiffle ball when dropped from a height of 20 meters, the inadequacy of the model would quickly be discovered. Modelers speak of ranges of validity to describe these limitations. Schematics such as Figure 1.5.6 describe the modeling process, but modelers rarely follow them rigidly.

As experienced practitioners know, models enjoy only a transitory existence. A model based on the empirical data of the day may become inadequate when better data are available. In any field, models are constantly being examined for accuracy in predicting the phenomena modeled. This involves a careful reconsideration of the basic assumptions that produced the model as well as an analysis of the mathematical approximations used in the course of computation. When models are found to be deficient they are modified or supplanted by other models. The progress of science depends on this process.
Initial Value Problems

State variables evolve over time, and rates of change are modeled by derivatives. After selecting state variables for a dynamical system, a mathematical model arises that consists of an ODE and conditions giving values of the state variables at a specific time $t_0$. As we have noted, these models are initial value problems (IVPs), and conditions involving the initial data are initial conditions. For example,

$$y'' = -g - \frac{k}{m} y', \quad y(0) = 0, \quad y'(0) = v_0$$

is an IVP. The number of initial conditions (two) in this IVP agrees with the order of the ODE, and this agreement is typical of IVPs.

The model for the motion of the whiffle ball tells you what happens after the ball is thrown upward. This is a forward initial value problem. The initial conditions in such problems are expressed in terms of the limiting value of the state variables as $t \to t_0^-$. Other phenomena lead to two-sided initial value problems where $t_0$ is in the interior of a time interval. To complete our characterization of initial value problems, we note that in a backward initial value problem, a description of the state variables is sought for $t < t_0$. Problem 10(a) has an example of a backward IVP.

Comments

A mathematical model of a natural process is a portrait in the language of mathematics. Like all portraits, the model will emphasize some features of the original and distort others. So modeling is as much an art as it is a logical procedure. A skillfully constructed model often can provide more insight than observation of the natural process itself. Accurate direct analysis of many natural processes is impossible, and our only perception of their realities is by mathematical or other models based on partial data, scientific common sense, experience, and intuition.
PROBLEMS

1. (Radioactive Decay). A substance decays at a rate proportional to the amount present, and in 25 years 1.1% of the initial amount \( N_0 \) has decomposed. What is the half-life of the substance?

2. (Radioactive Decay). What percentage of a substance remains after 100 years if its half-life is 1000 years?

3. (Radioactive Decay). Radioactive phosphorus with a half-life of 14.2 days is used as a tracer in biochemical studies. After an experiment with 8 grams of phosphorus, researchers must safely store the material until only \( 10^{-4} \) grams remain. How long must the contents be stored?

4. (Population Growth). A population grows exponentially for \( T \) months with growth constant 0.03 per month. Then the growth constant suddenly increases to 0.05 per month. After a total of 20 months, the population has doubled. At what time \( T \) did the growth constant change? [Hint: Solve \( y' = 0.03 y \), \( y(0) = y_0 \), over the interval \( 0 \leq t \leq T \). Then use \( y(T) \) as the initial value for a similar problem, replacing 0.03 by 0.05 and solving over the interval \( T \leq t \leq 20 \).

5. (No Damping). A body whose mass is 600 grams is thrown vertically upward with an initial velocity of 2000 cm/sec. Use \( g = 980 \) cm/sec\(^2\) and ignore air resistance.
   (a) Find the highest point and the time to reach that point. [Hint: See Example 1.5.3.]
   (b) Find the height of the body and the velocity after 3 sec. When does the body hit the ground?

6. (Vertical Motion of a Ball). Review Examples 1.5.2 and 1.5.3 and answer the questions.
   (a) Why do the solution curves in Figure 1.5.2 bend downward? [Hint: Recall the connection between the sign of the second derivative of a function and the concavity of its graph.]
   (b) Graph solution curves of the ODE \( y'' = -9.8 \) in the \( ty\)-plane for some values of \( y_0, v_0 \).
   (c) Thrown vertically from an initial height of 75 m, what is the initial velocity of the ball so that it stays in the air 5 sec before it hits the ground? Verify with a graph.
   (d) If \( y_0 = 0 \) and the ball reaches 30 m, find \( v_0 \). Verify with a graph.

7. (No Damping). A person drops a stone from the top of a building, waits 1.5 sec, then hurls a baseball downward with an initial speed of 20 m/sec. Ignore air resistance.
   (a) If the ball and stone hit the ground at the same time, how high is the building?
   (b) Show that if you wait too long before throwing the ball downward, the ball can’t catch up with the stone. Show that the maximum waiting time for a catch-up is independent of the building’s height.

8. (No Damping: Longer to Rise or to Fall?). Suppose that a ball is thrown vertically upward with initial velocity \( v_0 \) from ground level. Ignore air resistance. What is the time required for the ball to reach its maximum height? Does the ball spend as much time going up as it does coming down? What is the velocity of the ball on impact with the ground? Give reasons.

9. (Viscous Damping: Longer to Rise or to Fall?). Suppose that a whiffle ball of mass \( m \) is thrown straight up with velocity \( v_0 \) from height \( h \) and is subject to viscous air resistance.
   (a) Show that the IVP \( y' = v \), \( y(0) = h \), \( v' = -g - (k/m)v \), \( v(0) = v_0 \), governs the whiffle ball’s height \( y \) and velocity \( v \).
   (b) Suppose that \( h = 2 \) m, \( g = 9.8 \) m/sec\(^2\), and \( k/m = 5 \) sec\(^{-1} \) for the system in part (a). Use a numerical solver to estimate the rise time and the fall time (back to the initial height) for \( v_0 = 10, 30, 50, 70 \) m/sec. Repeat with \( k/m = 1, 10 \). Explain what you see.
   (c) (Proof That It Takes Longer to Fall than to Rise). Suppose that \( k/m = 1 \) in the IVP in part (a). Find the time \( T \) required for the whiffle ball to reach its highest point. Let the time \( \tau > 0 \) be such that \( y(\tau) = h \). Show that \( \tau > 2T \). Why do you think the fall-time is longer than the rise-time? Compare with the results of Problem 8. [Hint: Let \( f(v_0) = y(2T) - h \), express \( f(v_0) \) explicitly in terms of \( g \) and \( v_0 \), and show that \( f(0) = 0 \), \( df(v_0)/dv_0 > 0 \) for \( v_0 > 0 \).]
10. (Age of the Lascaux Cave Paintings). In 1950 a Geiger counter was used to measure the decay rate of $^{14}$C in charcoal fragments found in a cave near Lascaux, France, where there are prehistoric wall paintings of various animals. The counter recorded about 1.69 disintegrations per minute per gram of carbon, while for living tissue such as the wood in a tree the number of disintegrations was 13.5 per minute per gram of carbon. Follow the outline below to find out when the wood was burned to make the charcoal (and so determine the age of the cave paintings).

In any living organism the ratio of the amount of $^{14}$C to the total amount of carbon in the cells is the same as that in the air. After the organism is dead, ingestion of CO$_2$ ceases, and only the radioactive decay continues. The half-life $\tau$ of $^{14}$C is known to be about 5568 years. Suppose that $q(t)$ is the amount of $^{14}$C per gram of carbon at time $t$ in the charcoal sample; $q(t)$ is dimensionless because it is the ratio of masses. Suppose that $t = 0$ is now, and that $T < 0$ is the time that the wood was burned. Then $q(t) = q(T)$ for $t \leq T$.

(a) Suppose that $q_0$ is the amount of $^{14}$C per gram of carbon in the sample at $t = 0$. Verify that on the interval $T \leq t \leq 0$, $q(t)$ is the unique solution of the backward IVP

\[ q' = -kq, \quad q(0) = q_0, \quad T \leq t \leq 0 \]

(b) Solve the IVP in part (a) and show that

\[ T = -\frac{1}{k} \ln \frac{q(t)}{q_0} = -\frac{\tau}{\ln 2} \ln \frac{q(T)}{q(0)} \]

where $k$ is the rate constant and $\tau$ is the half-life for $^{14}$C.

(c) The reading of a Geiger counter at time $t$ is proportional to $q'(t)$, the rate of decay of radioactive nuclei in a sample. Using the given data in the problem statement, find $T$.

11. (Radiocarbon Dating: Scaling). Follow the outline below to obtain a graphical solution for the radiocarbon dating problem of the Lascaux paintings. [Hint: Read Problem 10.]

(a) Define the new state variable $Q(t) = q(t)/q_0$. Show that this change of variables converts the IVP of that example into the form $Q' = -kQ$. $Q(0) = 1$. Show that the dating problem may be reformulated as follows: Find a value $T < 0$ such that $Q(T) = 13.5/1.69$.

(b) Plot the solution of the IVP of part (a) backward in time and find the value of $T$.

12. (Dating Stonehenge). In 1977, the rate of $^{14}$C radioactivity of a piece of charcoal found at Stonehenge in southern England was 8.2 disintegrations per minute per gram of carbon. Given that in 1977 the rate of $^{14}$C radioactivity of a living tree was 13.5 disintegrations per minute per gram, estimate the date of construction. [Hint: Read Problem 10.]

13. (Dating a Sea Shell). An archeologist finds a sea shell that contains 60% of the $^{14}$C of a living shell. How old is the shell? [Hint: Read Problem 10.]

14. (The Bones of Olduvai: Potassium-Argon Dating). Olduvai Gorge, in Kenya, cuts through volcanic flows, volcanic ash, and sedimentary deposits. It is the site of bones and artifacts of early hominids, considered by some to be precursors of man. In 1959, Mary and Louis Leakey uncovered a fossil hominid skull and primitive stone tools of obviously great age. Carbon-14 dating methods being inappropriate for a specimen of that age and nature, dating had to be based on the ages of the underlying and overlying volcanic strata. The method used was that of potassium-argon decay. The potassium-argon clock is an accumulation clock, in contrast to the $^{14}$C dating method. Follow the outline below to model this accumulation clock.

The potassium-argon method depends on measuring the accumulation of “daughter” argon atoms, which are decay products of radioactive potassium atoms. Specifically, potassium-40 ($^{40}$K) decays to argon-40 ($^{40}$Ar) and to calcium-40 ($^{40}$Ca) at rates proportional to the amount of potassium but with respective constants of proportionality $k_1$ and $k_2$. 

### Mathematical Model

The potassium-argon method is based on the following equations:

\[ \frac{dQ}{dt} = -k_1 Q, \quad \frac{dR}{dt} = k_1 Q - k_2 R \]

where $Q$ is the amount of potassium, $R$ is the amount of argon-40, $k_1$ and $k_2$ are the decay constants. The initial conditions are $Q(0) = Q_0$ and $R(0) = R_0$.
The model for this decay process may be written in terms of the amounts $K(t)$, $A(t)$, and $C(t)$ of potassium, argon, and calcium in a sample of rock. Using the Balance Law, we have

$$K' = -(k_1 + k_2)K, \quad A' = k_1 K, \quad C' = k_2 K$$

(16)

where time $t$ is measured forward from the time the volcanic ash was deposited around the skull.

(a) Solve the system to find $K(t)$, $A(t)$, and $C(t)$ in terms of $k_1$, $k_2$, and $k = k_1 + k_2$. Set $K(0) = K_0$, $A(0) = C(0) = 0$. Why is $K(t) + A(t) + C(t) = K_0$ for all $t \geq 0$? Show that $K(t) \to 0$, $A(t) \to k_1 K_0/k$, and $C(t) \to k_2 K_0/k$ as $t \to \infty$.

(b) The age $T$ of the volcanic strata is the current value of the time variable $t$ because the potassium-argon clock started when the volcanic material was laid down. This age is estimated by measuring the ratio of argon to potassium in a sample. Show that this ratio is $A/K = (k_1/k)(e^{kT} - 1)$. Show that the age of the sample (in years) is $(1/k)\ln[(k/k_1)(A/K) + 1]$.

(c) When the actual measurements were made at the University of California at Berkeley, $T$ (the age of the bones) was estimated to be 1.75 million years. The values of the constants of proportionality are known to be $k_1 = 5.76 \times 10^{-11}/yr$ and $k_2 = 4.85 \times 10^{-10}/yr$. What was the value of the measured ratio $A/K$?

15. (Shoveling Snow). During a steady snowfall, a man starts clearing a sidewalk at noon, shoveling the snow at a constant rate and clearing a path of constant width. He shovels two blocks by 2 P.M., one block more by 4 P.M. When did the snow begin to fall? Explain your modeling process. [Hint: Additional assumptions are required to solve the problem.]

### 1.6 Separable Differential Equations

We have just about wrapped up the topic of first-order linear ODEs in that we know how to find and interpret a formula for all solutions and how to interpret solution curves produced with a numerical solver. Now it’s time to look at first-order nonlinear ODEs. In this section we will look at ODEs of the form, $N(y)y'(t) + M(t) = 0$, which are called separable ODEs because the variables $y$ and $t$ are separated as indicated.

As we will see later on, there are advantages to using $x$ as a variable name instead of $t$. So let’s find a solution formula for the separable ODE

$$N(y)y'(x) + M(x) = 0$$

(1)

This ODE is often written as $N(y)y' = -M(x)$ with the equal sign “separating” the $y$-terms and the $x$-terms. Let’s suppose that $N(y)$ and $M(x)$ are continuous on the respective $y$ and $x$ intervals $J$ and $I$.

Here is a procedure for solving ODE (1).

**Solving a Separable ODE**

1. **Separate variables** to write the ODE in the form $N(y)y' + M(x) = 0$, and identify the coefficients $N(y)$ and $M(x)$.

2. **Find any antiderivative** $G(y)$ for $N(y)$ and any antiderivative $F(x)$ for $M(x)$.

3. **All solutions** $y = y(x)$ satisfy the equation $G(y) + F(x) = C$, where $C$ is constant.
After you use this procedure a few times, it will become second nature and you won’t have to refer to it again.

Here is an explanation of why the procedure works. Suppose that $G(y)$ is any antiderivative of $N(y)$ on $J$ and that $F(x)$ is any antiderivative of $M(x)$ on $I$. In other words,

$$dG/dy = N(y), \quad \text{for all } y \text{ in } J, \quad dF/dx = M(x), \quad \text{for all } x \text{ in } I$$

If $y(x)$ is any solution of ODE (1) whose solution curve remains in the $xy$-rectangle defined by $I$ and $J$, then ODE (1) becomes

$$N(y)y'(x) + M(x) = [G(y(x)) + F(x)]' = 0$$

(2)

because the Chain Rule says that

$$\frac{dG(y(x))}{dx} = \frac{\partial G}{\partial y} \frac{dy}{dx} = N(y)y'(x)$$

Antidifferentiation of (2) then implies that

$$G(y(x)) + F(x) = C$$

(3)

for some constant $C$ and for all $x$ on the interval where $y(x)$ is defined.

On the other hand, suppose that $y(x)$ is a continuously differentiable function whose graph lies in the rectangle determined by $I$ and $J$ and that $y(x)$ satisfies equation (3) for some constant $C$. Then differentiation of (3) via the Chain Rule shows that $y(x)$ is a solution of ODE (1). So equation (3) can be regarded as implicitly defining the general solution of ODE (1) for a range of values of the constant $C$. Formula (3) is implicit because it has not been solved to express $y$ explicitly as a function of $x$.

Here’s an example that uses the separable variables method.

**EXAMPLE 1.6.1 Separating the Variables and Solving**

The ODE

$$y' = -\frac{x}{y}$$

in separated form is

$$yy' + x = 0$$

and so $N(y) = y$ and $M(x) = x$. Then $G(y) = y^2/2$ and $F(x) = x^2/2$ are antiderivatives of $N$ and $M$. Using formula (3), the general solution of the ODE is given by

$$\frac{[y(x)]^2}{2} + \frac{x^2}{2} = C$$

(4)

where $C$ is a constant. For any value $C > 0$, we can solve equation (4) for $y$ in terms of $x$. There are two solutions

$$y = \pm \sqrt{2C - x^2}$$

(5)

each defined on the interval $|x| < \sqrt{2C}$. Direct substitution in ODE (4) shows that formula (5) does define the solutions.
It’s worth noting that if we think of \( x \) as a function of \( y \), instead of the other way around, then we can write ODE (1) as

\[
N(y) + M(x) \frac{dx}{dy} = 0
\]

If we don’t want to choose between \( x \) and \( y \) we can write ODE (1) in the differential form

\[
N(y) \, dy + M(x) \, dx = 0
\]

Now let’s look at the geometric structure of the solution curves of a separable ODE.

### Integrals and Integral Curves

Our approach to solving separable ODEs suggests the need for some new terms.

\( \blacktriangleright \) **Integrals and Integral Curves.** A nonconstant function \( H(x, y) \) on a rectangle \( R \) is an integral of the ODE \( Ny' + M = 0 \) if for any solution \( y(x) \) whose graph lies in \( R \), \( H(x, y(x)) = C \) for some constant \( C \). The set of points in \( R \) satisfying \( H(x, y) = C \) is an integral curve for the ODE. If \( C \) is an unspecified constant, then \( H(x, y) = C \) is the general implicit solution of the ODE.

For the antiderivatives \( G(y) \) and \( F(x) \) defined above, the function \( H(x, y) = G(y) + F(x) \) is an integral for ODE (1), so the equation \( H(x, y) = C \) defines an integral curve of ODE (1). These integral curves are level sets or contours of \( H \). Every solution curve of the ODE is an arc of some integral curve for that ODE.

**EXAMPLE 1.6.2**

**Integral Curves of** \( yy' + x = 0 \)

As we saw in Example 1.6.1, \( H(x, y) = y^2/2 + x^2/2 \) is an integral for the separable ODE \( yy' + x = 0 \). The integral curves are circles defined by \( y^2/2 + x^2/2 = \text{constant} \).

The ODE of Examples 1.6.1 and 1.6.2 shows that solving the equation \( H(x, y) = C \) for \( y \) in terms of \( x \) may lead to more than one solution of the ODE for each value of \( C \). The next example shows that integral curves may have several branches.

**EXAMPLE 1.6.3**

**Four Solution Curves on the Two Branches of an Integral Curve**

Let’s look at the IVP

\[
yy' - x = 0, \quad y(2) = -1
\]

Following the procedure above with \( N(y) = y \) and \( M(x) = -x \), we have (after carrying out the integrations)

\[
y^2/2 - x^2/2 = C
\]

(6)

The function \( H = y^2/2 - x^2/2 \) is an integral of the ODE. The constant \( C \) is determined from the initial condition \( x = 2 \) when \( y = -1 \), and we see that the implicit solution is
\[ y' - x = 0, \quad y(2) = -1 \]

\[ y^2/2 - x^2/2 = -3/2 \]

which defines a hyperbola through the initial point \((2, -1)\). The explicit solution \(y(x)\) is given by
\[ y = -(x^2 - 3)^{1/2}, \quad x > 3^{1/2} \]

which defines the solid arc passing through \((2, -1)\) and shown in Figure 1.6.1. The hyperbolic integral curve defined by (6) contains three other solution curves that are also shown in Figure 1.6.1: \(y = (x^2 - 3)^{1/2}, \quad x > 3^{1/2}\) (upper right), and \(y = \pm(x^2 - 3)^{1/2}, \quad x < -3^{1/2}\) (on the left).

Sometimes it is not practical to find \(y\) as an explicit function of \(x\).

**EXAMPLE 1.6.4**  
A Solution Curve on an Integral Curve

The separable ODE
\[ (1 - y^2)y' + x^2 = 0 \]

has the integral \(H(x, y) = y - y^3/3 + x^3/3\) and the general solution
\[ y - y^3/3 + x^3/3 = C \quad (7) \]

where \(C\) is a constant. There is no simple way to solve (7) for \(y\) in terms of \(x\) and \(C\).

Suppose now that we want to find the integral curve through the point \((-1, 1/2)\). Inserting \(x = -1\) and \(y = 1/2\) into formula (7), we see that \(C = 1/8\). The solution \(y(x)\) of the IVP
\[ (1 - y^2)y' + x^2 = 0, \quad y(-1) = 1/2 \quad (8) \]

is defined implicitly by the formula
\[ y - y^3/3 + x^3/3 = 1/8 \quad (9) \]
1.6/ Separable Differential Equations

See Figure 1.6.2 for the S-shaped integral curve defined by formula (9). The solid arc on this integral curve is the graph of the solution \( y(x) \) of IVP (8) extended forward and backward from \( x = -1 \) as far as it will go.

To find the largest \( x \)-interval on which the graph of \( y(x) \) in Example 1.6.4 is defined, note that \( y' \) is infinite when \( y = \pm 1 \) [because \( y' = x^2/(y^2 - 1) \)]. Putting \( y = \pm 1 \) into the equation

\[
y - y^3/3 + x^3/3 = 1/8
\]

we obtain \( x^3 = -13/8, 19/8 \). So, \(-13^{1/3}/2 \) and \( 19^{1/3}/2 \) are the endpoints of the largest \( x \)-interval on which this solution curve is defined.

One small mystery still remains: How was the integral curve in Figure 1.6.2 plotted? We did not use a contour plotter! In Section 1.7 all will be revealed.

Examples 1.6.3 and 1.6.4 illustrate the power of the integral curves to help us to visualize the solution curves of an ODE. In fact, if we plot all of the integral curves that lie in a rectangle, then we can immediately see all of the solution curves in the rectangle. In this sense integral curves give a global description of the solution curves.

We end the section with two applications that involve separable ODEs.

The Logistic ODE

In Section 1.1 we looked at a model ODE

\[
dy/dt = ay - cy^2
\]

where \( a \) and \( c \) are positive constants. This ODE was used to describe a fish population, taking overcrowding into account. The more common form of this ODE arises by putting \( r = a \) and \( K = a/c \) to obtain the logistic equation

\[
dy/dt = ry(1 - y/K)
\]  

(10)

The constant \( r \) is called the intrinsic growth constant and measures the difference between the birth and death rates per population unit if there is no overcrowding. For example, \( r = 0.05 \) corresponds to a net growth rate of 5% per unit time. The positive constant \( K \) is the saturation constant or carrying capacity. As we will soon see, every solution \( y(t) \) approaches \( K \) as \( t \to +\infty \) if \( y(0) \) is positive. Let’s derive a solution formula for the logistic ODE (10).

**EXAMPLE 1.6.5** Solving the Logistic ODE

Let’s write the logistic equation in the differential form

\[
dy - ry(1 - y/K) dt = 0
\]  

(11)

This ODE isn’t linear, but it is separable. Assume that \( y \) is not 0 or \( K \) to avoid division by 0, and separate the variables:

\[
\frac{K}{(K - y)} dy - r dt = 0
\]  

(12)
In the notation of formula (3), we need an antiderivative $G(y)$ of $K/[y(K - y)]$ and the antiderivative $F(t) = -rt$ of the function $-r$. We use partial fractions to find $G(y)$:

$$\frac{K}{(K - y)y} = \frac{1}{K - y} + \frac{1}{y}$$

Integrate to obtain

$$G(y) = -\ln |K - y| + \ln |y|$$

We have the general solution of ODE (11):

$$-\ln |K - y| + \ln |y| - rt = c, \quad \text{or} \quad \ln \left| \frac{y}{K - y} \right| = rt + c$$

where $c$ is any constant. Exponentiate to obtain

$$\left| \frac{y}{K - y} \right| = Ce^{rt}, \quad \text{where} \quad C = e^c$$  (13)

If the absolute value sign in (13) is dropped, then $C$ can be either positive or negative. Now let’s solve $y/(K - y) = Ce^{rt}$ for $y$:

$$y = (K - y)Ce^{rt}$$

$$y(1 + Ce^{rt}) = KCe^{rt}$$

$$y = \frac{KCe^{rt}}{1 + Ce^{rt}}$$  (14)

Evaluating $C$ from the initial condition $y(0) = y_0 \geq 0$, we have from (13) that $y_0 = KC/(1 + C)$. Solving for $C$ in terms of $y_0$ we have $C = y_0/(K - y_0)$. Inserting this value for $C$ in (14) we have (after a lot of algebra)

$$y(t) = \frac{y_0K}{y_0 + (K - y_0)e^{-rt}}$$  (15)

which is the formula for the solutions of the logistic ODE (10). Note that if $y_0 > 0$, then $y(t) \to K$ as $t \to +\infty$ since $e^{-rt} \to 0$.

Let’s put in some numbers for $r$ and $K$.

**EXAMPLE 1.6.6 Logistic Population Change**

Suppose that the rate coefficient $r$ is 1 (corresponding to a 100% growth rate per unit time) and the carrying capacity is 12. From formula (15), the solution formula for the IVP $y' = (1 - y/12)y, \ y(0) = y_0$, is

$$y(t) = \frac{12y_0}{y_0 + (12 - y_0)e^{-t}}$$

Since $e^{-t} \to 0$ as $t \to +\infty$, we see that $y(t) \to 12y_0/(y_0 + 0) = 12$, so the population tends to the carrying capacity $K = 12$. Turn back to Figure 1.1.3 for a picture of the solution curves.
1.6/ Separable Differential Equations

The S-shaped population curves of the logistic equation are called *logistic curves*. Figure 1.1.3 illustrates the appropriateness of the terms “carrying capacity” and “saturation population” for \( y = K \). The resources of the community can support a population of size \( K \), which is the asymptotic limit of the population curves as \( t \to +\infty \).

**Newtonian Damping and a Sky Diver**

Careful measurements reveal that when a dense body rises or falls through the air the magnitude of the damping force is proportional to the square of the magnitude of the velocity. This is known as *Newtonian damping*; compare this with viscous damping defined in Section 1.5. Suppose that \( y \) measures the distance along the vertical with “up” as the positive direction. Taking into account the forces acting on the sky diver, we have the initial value problem

\[
my'' = -mg - kv|v|, \quad y(0) = h, \quad v(0) = v_0 \tag{16}
\]

where \( v = y' \) and the minus sign in the term \(-kv|v|\) indicates that the drag acts opposite to the motion. The positive constant \( k \) in (16) is the *Newtonian damping constant*. Note that for a falling body the velocity \( v = y' \) is negative so \(-kv|v| = kv^2\).

### Example 1.6.7 Newtonian Damping: Limiting Velocity

A free-falling sky diver jumps from a plane \( h \) feet above the ground, and is acted upon by Newtonian air resistance. Because the sky diver is falling, the Newtonian damping force acts upward (in the positive direction) and so is given by \( kv^2 \). Replacing \( y'' \) by \( v' \) in IVP (16) and dividing by \( m \), we obtain the first-order IVP for the velocity

\[
v' = -g + \left( \frac{k}{m} \right) v^2, \quad v(0) = 0 \tag{17}
\]

The ODE in (17) is separable. After separating the variables and integrating, it can be shown (Problem 7) that the solution of IVP (17) is

\[
v(t) = \left( \frac{mg}{k} \right)^{1/2} \frac{e^{-At} - 1}{e^{-At} + 1} \tag{18}
\]

where \( A = 2(gk/m)^{1/2} \). As \( t \to +\infty \), we see that \( v(t) \) tends to the *limiting velocity* \( v_\infty = -(mg/k)^{1/2} \). This velocity is not actually reached because the sky diver hits the ground at some finite time, and the model IVP (17) loses its validity.

The motion of a sky diver has been studied in detail, and Newtonian damping provides a good model. If the mass of the equipped sky diver is 120 kg, then the Newtonian damping constant \( k \) is about 0.1838 kg/m, and the limiting velocity has magnitude about 80 m/sec. This value compares favorably with the observed limiting velocities of free-falling sky divers.
Chapter 1/First-Order Differential Equations and Models

Comments

Historically, the ODE \( N(dy/dx) + M = 0 \) was usually written in the differential form \( Ndy + Mdx = 0 \), and that is why the subject is named “differential equations” rather than “derivative equations.” Because of the convenience of the differential form, we sometimes use it, especially in the problems.

The case where \( N \) and \( M \) are functions of both \( x \) and \( y \) is taken up in Problem 10 for the special case where \( \partial N/\partial x = \partial M/\partial y \).

PROBLEMS

1. (Losing a Solution). Sometimes when you rewrite an ODE to separate the variables you may inadvertently lose a solution. Find all solutions of the ODE \( y' = 2xy^2 \), but watch out that you don’t lose a solution.

2. Find all solutions \( y = y(x) \) of each ODE. [Hint: Watch out for solutions that are lost when you separate the variables. See Problem 1.]
   \( \begin{align*}
   (a) & \quad dy/dx = -4xy \\
   (b) & \quad 2y dx + 3x dy = 0 \\
   (c) & \quad y' = -xe^{-xy} \\
   (d) & \quad (1-x)y' = y^2 \\
   (e) & \quad y' = -y/(x^2 - 4) \\
   (f) & \quad y' = xe^{3x^2}
   \end{align*} \)

3. For each IVP find a solution formula and the largest \( x \)-interval on which it is defined. For (b) and (e), find solution formulas and also solve the IVPs with a numerical solver.
   \( \begin{align*}
   (a) & \quad y' = (y+1)/(x+1), \quad y(1) = 1 \\
   (b) & \quad y' = y^2/x, \quad y(1) = 1 \\
   (c) & \quad y' = ye^{-x}, \quad y(0) = e \\
   (d) & \quad y' = 3x^2/(1 + x^3), \quad y(0) = 1 \\
   (e) & \quad y' = -x/y, \quad y(1) = 2 \\
   (f) & \quad 2xy' = 1 + y^2, \quad y(2) = 3
   \end{align*} \)

4. For each ODE find the general solution formula, but don’t try to solve that formula for \( y \) as an explicit function of \( x \).
   \( \begin{align*}
   (a) & \quad y' = (x^2 + 2)(y+1)/xy \\
   (b) & \quad (1 + \sin x) dx + (1 + \cos y) dy = 0 \\
   (c) & \quad (\tan^2 y) dy = (\sin^3 x) dx \\
   (d) & \quad (3y^2 + 2y + 1)y' = 3\sin(x^2)
   \end{align*} \)

5. (Logistic Change). The problems below involve logistic growth and decay.
   \( \begin{align*}
   (a) & \quad \text{Solve the IVPs } y' = (1 - y/20)y, \quad y(0) = 5, 10, 20, 30. \quad [\text{Hint: See Formula (15).}] \\
   (b) & \quad \text{Plot the solution curves in (a) on the interval } 0 \leq t \leq 10 \text{ and highlight the carrying capacity.} \\
   (c) & \quad \text{(Harvesting). The ODE } y' = 3(1 - y/12)y - 8 \text{ models population changes for a harvested, logistically changing species. Find the two equilibrium levels, and discuss the fate of the species if } y(0) = 2, 4, 6, 8, \text{ or } 10. \quad [\text{Hint: Follow the analysis of IVP (12) in Section 1.1}] \\
   (d) & \quad \text{Plot the solution curves in (c) on the interval } 0 \leq t \leq 5 \text{ and highlight the equilibrium levels.} \\
   (e) & \quad \text{A colony of bacteria grows according to the logistic law, with a carrying capacity of } 5 \times 10^9 \text{ individuals and natural growth coefficient } r = 0.01 \text{ day}^{-1}. \text{ What will the population be after } 2 \text{ days if it is initially } 1 \times 10^9 \text{ individuals?}
   \end{align*} \)

6. (Projectile Motion: Newtonian Damping). A spherical projectile weighing 100 lb is observed to have a limiting velocity of \(-400 \text{ ft/sec.} \)
   \( \begin{align*}
   (a) & \quad \text{Show that the velocity } v \text{ of the projectile undergoing either upward or downward vertical motion and acted upon by Newtonian air resistance is given by } v' = -g - (gk/w)v|v|, \text{ where the magnitude of } k \text{ is } 1/1600 \text{ and } w = mg \text{ is the weight of the projectile.} \\
   (b) & \quad \text{Model the projectile motion as a differential system in state variables } y \text{ and } v, \text{ where } v = y'.
   \end{align*} \)
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(c) If the projectile is shot straight upward from the ground with an initial velocity of 500 ft/sec, what is its velocity when it hits the ground? [Hint: Use a numerical solver to solve the system of part (b) with suitable initial conditions. Use the solver to plot the solution \( v = v(t) \) and \( y = y(t) \) as a parametric curve in the \( vy \)-plane. Estimate the impact velocity from the graph.]

(d) (Longer to Fall?). Does the projectile in (c) take longer to rise or to fall? Explain. Repeat for \( v_0 = 100, 200, \ldots, 1000 \) ft/sec. [Hint: Use a numerical solver and graph \( y(t) \).]

(www) 7. (The Newtonian Sky Diver’s Velocity). In Example 1.6.7 it was shown that the sky diver’s velocity \( v(t) \) satisfies the IVP \( v' = -g + kv^2/m \), \( v(0) = 0 \). Separate variables, use partial fractions and algebraic manipulation to derive the solution formula

\[
v(t) = \left( \frac{mg}{k} \right)^{1/2} e^{-kt} - 1 e^{-kt} + 1, \quad \text{where} \quad A = 2 \left( \frac{gk}{m} \right)^{1/2}
\]

8. (Sky Diver: Newtonian Damping). A sky diver and equipment weigh 240 lb. [Note: weight = \( mg \).] In free fall the sky diver reaches a limiting velocity of 250 ft/sec. Some time after the parachute opens the sky diver reaches the limiting velocity of 17 ft/sec. Suppose that later the sky diver jumps out of an airplane at 10000 feet. Use \( g = 32.2 \) ft/sec², and answer the following questions about the second jump.

(a) How much time must elapse before the free-falling sky diver falls at a speed of 100 ft/sec? [Hint: Use the data to determine the coefficient \( k \) in IVP (16); then use formula (18).]

(b) When falling at 100 ft/sec the sky diver pulls the rip cord and the chute opens instantaneously. How much longer does it take for the speed of the descent to drop to 25 ft/sec?

(c) How long does the jump last? What is the velocity on landing?

9. (A Harvested Logistic Population). An initial value problem for a logistic population harvested at a constant rate is given by

\[
y' = y(1 - y/10) - 9/10, \quad y(0) = y_0
\]

(a) Find the solution formula for this IVP. [Hint: Allow enough time for a lot of algebraic manipulation.]

(b) Plot some solution curves for the IVP.

10. (Solution Formula for Exact ODEs). The first-order ODE

\[
N(x, y)y' + M(x, y) = 0
\]

is exact in a rectangle \( R \) of the \( xy \)-plane if \( M \) and \( N \) are continuously differentiable in \( R \) and if there is a function \( H(x, y) \) such that \( \partial H/\partial x = M, \ \partial H/\partial y = N \) for all \( (x, y) \) in \( R \). Such a function \( H \) is called an integral of the ODE. If the ODE is exact, then its general solution is \( H(x, y) = C \), where the constant \( C \) is chosen appropriately. The level curves of an integral are called integral curves. It can be shown that the exactness condition

\[
\partial N/\partial x = \partial M/\partial y
\]

for all \( (x, y) \) in \( R \) guarantees that a function \( H \) with the desired properties exists. So if the ODE satisfies the exactness condition, all we need to do to construct the general solution \( H(x, y) = C \) is to calculate \( H \). This can be accomplished by completing the following steps: (1) Check that the exactness condition holds. (2) Think of \( y \) as a constant and evaluate the \( x \)-antiderivative \( \int M(x, y) \, dx \); write \( H = \int M(x, y) \, dx + g(y) \) in terms of an unknown function \( g \). (3) Then

\[
g'(y) = \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \int M(x, y) \, dx = N - \frac{\partial}{\partial y} \int M(x, y) \, dx
\]

which follows from the fact that \( \partial H/\partial y = N \) (definition of exactness). (4) The exactness con-
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11. (Newtonian versus Viscous Damping). Design an experiment to determine whether Newtonian or viscous damping better describes the motion of the sky diver of Problem 8. Carry out a computer simulation and discuss the results. [Hint: See Examples 1.5.4, 1.5.5, and 1.6.7. The damping constant can be determined from the terminal velocity.]

1.7 Planar Systems and First-Order ODEs

Most of the dynamical systems considered so far have been modeled by single, first-order ODEs in one state variable. Some dynamical systems require two state variables, say $x$ and $y$, and the governing laws lead to two rate equations:

\[
\frac{dx}{dt} = x' = f(t, x, y), \quad \frac{dy}{dt} = y' = g(t, x, y)
\]  

(1)

The pair of differential equations is called a first-order planar differential system; the $xy$-plane is the state space. When the rate functions $f$ and $g$ do not depend on $t$, system (1) is said to be autonomous.

Systems first appeared in (15) of Section 1.5, where we modeled a whiffle ball of mass $m$ moving along the local vertical, using position $y$ and velocity $v$ as state variables, to obtain the planar autonomous system

\[
y' = v, \quad v' = -g - \frac{k}{m}v
\]  

(2)
1.7/ Planar Systems and First-Order ODEs

where $g$ is the earth’s gravitational constant and $k$ is the viscous damping constant. We find solutions of system (2) by observing that the second ODE (the $v'$ equation) decouples from the first. We can solve the decoupled first-order ODE for $v(t)$ and insert it into the other first-order ODE, which can then be solved for $y(t)$.

It is not always the case that one ODE of a first-order planar system decouples from the other ODE, so we need to search for other solution techniques. But first, here is some important terminology for planar systems.

❖ Solutions, Orbits, Component Curves. The functions $x = x(t), \ y = y(t)$ define a solution of system (1) if $x'(t) = f(t, x(t), y(t)), \ y'(t) = g(t, x(t), y(t))$ for all $t$ in an interval $I$. The parametric plot of $x(t)$ versus $y(t)$ in the $xy$-plane is the orbit of the solution. The graphs of $x(t)$ versus $t$ in the $tx$-plane and $y(t)$ versus $t$ in the $ty$-plane are the component curves of the solution.

Examples of orbits and component curves appear in the figures that follow.

Now we come to one reason why $x$ is often used instead of $t$ in a first-order ODE.

The System Way to Plot Solution Curves

Solution curves of the first-order ODE

$$N(x, y) \frac{dy}{dx} + M(x, y) = 0 \quad (3)$$

can be obtained by first plotting orbits of the planar autonomous system

$$\frac{dx}{dt} = N(x, y), \quad \frac{dy}{dt} = -M(x, y) \quad (4)$$

We can explain this process by reasoning as follows: The parametric plot in $xy$-space of a solution $x = x(t), \ y = y(t)$ of system (4) is an orbit of the system. The slope $dy/dx$ of the orbit at a point $(x, y)$ is $-M(x, y)/N(x, y)$ since $dy/dx = (dy/dt)/(dx/dt) = -M/N$. Solution curves of ODE (3), and so of $dy/dx = -M/N$, are arcs of orbits of the system (4), which contain no vertical tangents.

The ODE $dy/dx = -M/N$ may be hard to solve numerically in regions where the denominator $N(x, y)$ vanishes. The technique described above gets around this problem of the vanishing denominator, but at the cost of introducing another state variable. The examples below illustrate this approach.

EXAMPLE 1.7.1 Solution Curves of a First-Order ODE as Arcs of Orbits of a System

Let’s find the solution curve through the point $(-1, 1/2)$ of the ODE

$$(1 - y^2) \frac{dy}{dx} + x^2 = 0 \quad (5)$$

where $N = 1 - y^2$ and $M = x^2$, by first plotting the orbit of the system IVP

$$\frac{dx}{dt} = N = 1 - y^2, \quad x(0) = -1$$

$$\frac{dy}{dt} = -M = -x^2, \quad y(0) = 1/2 \quad (6)$$

This approach beats most contour plotting software. Problem 12 has more.
using the initial point \((-1, 1/2)\). This is done by using a numerical solver to solve IVP (6) and letting \(t\) run forward and then backward from \(t = 0\) until the orbit exits the rectangle \(|x| \leq 2, |y| \leq 3\).

The orbit is the S-shaped curve shown in Figure 1.6.2. The solution curve of ODE (5) that passes through \((-1, 1/2)\) is the longest arc of this orbit that contains the point \((-1, 1/2)\) and has no vertical tangents. The solution curve is the solid arc shown in Figure 1.6.2.

Here’s a second example that uses the same approach, but with many initial points and orbits.

**EXAMPLE 1.7.2 Teddy Bears**

Consider the first-order ODE

\[
\left(\sin y - 2\sin x^2 \sin 2y\right) \frac{dy}{dx} + \cos x + 2x \cos x^2 \cos 2y = 0
\]  

(7)

The first-order system (4) that corresponds to ODE (7) is

\[
\frac{dx}{dt} = \sin y - 2\sin x^2 \sin 2y
\]

\[
\frac{dy}{dt} = -\cos x - 2x \cos x^2 \cos 2y
\]

(8)

The orbits of system (8) shown in Figure 1.7.1 were produced by a numerical solver using several initial points \((x_0, y_0)\) in the rectangle \(|x| \leq 6, |y| \leq 10\). For example, the curve outlining the lower torso and legs of the teddy bears is the orbit of system (8) that passes through the point \((0, \pi/2)\) at \(t = 0\). In each case, the orbit is run forward and backward in time from the initial point selected until the orbit exits the rectangle or returns to its starting point (as happens with orbits inside a bear).

In this example we have not singled out the arcs on each orbit that correspond to solutions \(y = y(x)\) of the original ODE (7). The fact that the rate functions for \(x(t)\) and \(y(t)\) are periodic functions of \(y\) with period \(2\pi\) suggests that orbital shapes might repeat with every increase of \(2\pi\) in \(y\), so that’s why we see the teddy bear triplets.

The teddy bear example shows how the system approach helps us to visualize solution curves of a first-order ODE over an extended region, that is, globally. Since solution curves are arcs of orbits with no vertical tangents, we clearly see from Figure 1.7.1 that some solution curves live only over very short \(x\)-intervals (see, for example, the many “eyes” in a teddy bear).

Here’s a way to use the systems approach to construct formulas for some second-order ODEs that occur quite often in the applications.

**Reduction Methods and Planar Systems**

Some second-order differential equations can be solved by reducing them to first-order systems by a suitable choice of new variables. For example, the second-order ODE

\[
y'' = F(t, y')
\]  

(9)
1.7 Planar Systems and First-Order ODEs

\[(\sin y - 2 \sin x \sin 2y) y' + \cos x + 2 \cos x \cos 2y = 0\]

\[
y', v' = F(t, v)
\]

\[\text{FIGURE 1.7.1 Orbits of the first-order system (8) (Example 1.7.2).}\]

where the derivatives are with respect to \(t\), is transformed to the system

We lucked out here. Solving this ODE involves just solving two first-order ODEs in succession.

\[\text{EXAMPLE 1.7.3 Solving } y'' = F(t, y')\]

The ODE

\[y'' = y' - t\]

can be solved by setting \(v = y'\) and rewriting the ODE as

\[v' - v = -t\]

which is a linear first-order ODE in \(v\) with integrating factor \(e^{-t}\). We have \((ve^{-t})' = -te^{-t}\), which can be solved to obtain \(v = C_1e^t + 1 + t\), where \(C_1\) is an arbitrary constant. Recalling that \(y' = v\), an antidifferentiation yields the solutions

\[y = C_1e^t + t^2/2 + C_2\]

where \(C_1\) and \(C_2\) are arbitrary constants.
Another method applies when the second-order ODE is autonomous:

$$y'' = F(y, y') \quad (11)$$

Since $t$ does not appear explicitly in the differential equation, an independent variable other than $t$ might be suitable. We introduce $y$ itself as the new independent variable and $v = y'$ as a new dependent variable! Using the Chain Rule, we have

$$y'' = \frac{d^2 y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} \cdot v$$

and ODE (11) is transformed to a pair of first-order ODEs quite different from (10):

$$\begin{align*}
\frac{dv}{dy} &= F(y, v) \\
\frac{dy}{dt} &= v(y)
\end{align*} \quad (12)$$

The first ODE in (12) decouples from the second, so it can be solved for $v(y)$ separately as a first-order ODE. Once the solution $v(y)$ is known, the second ODE in (12) can be solved for $y(t)$ by separating variables and integrating. Here is an example.

**EXAMPLE 1.7.4 Solving $y'' = F(y, y')$**

The IVP

$$y'' = \frac{1}{y}(y')^2 - \frac{y'}{y}, \quad y = 1 \quad \text{and} \quad y' = 2 \quad \text{when} \quad t = 0$$

can be solved for $y(t)$ by the method given above. Comparing this ODE with (11) we see that $F(y, y') = [(y')^2 - y']/y$. We assume that $y > 0$. Introducing $y$ and $v = y'$ as variables, we have from (12) and the initial data the two first-order IVPs:

$$\begin{align*}
\frac{dv}{dy} &= \frac{1}{y}v^2 - \frac{v}{y}, \quad v = 2 \quad \text{when} \quad y = 1 \\
\frac{dy}{dt} &= v(y), \quad y = 1 \quad \text{when} \quad t = 0
\end{align*} \quad (13a)$$

The solution $v = 0$ of the ODE in (13a) is not of interest, so we divide out the factor $v$ and solve the resulting linear IVP $dv/dy = (v - 1)/y$ by the integrating factor technique to find

$$v = 1 + y$$

So IVP (13b) becomes

$$\frac{dy}{dt} = v(y) = 1 + y, \quad y = 1 \quad \text{when} \quad t = 0$$

This linear ODE can be solved by using an integrating factor to obtain $y = Ce^t - 1$. Using the initial data $y = 1$ when $t = 0$, we have the desired solution:

$$y(t) = 2e^t - 1$$

We must have that $t > -\ln 2$ because we assumed at the start that $y$ is positive.

Next, we present a surprising application of this reduction technique.
Inverse Square Law of Gravitation: Escape Velocities

Using the extensive astronomical work and empirical laws of Tycho Brahe (1546–1601) and Johannes Kepler (1571–1630) concerning the orbits of the moon and the planets, Newton focused attention on just one force: gravity. His law of universal gravitation deals with the gravitational effect of one body upon another.

Newton’s Law of Universal Gravitation. The force $F$ between two particles having masses $m_1$ and $m_2$ and separated by a distance $r$ is attractive, acts along the line joining the particles, and has magnitude

$$|F| = \frac{m_1 m_2 G}{r^2}$$

where $G$ is a constant. This is the Inverse Square Law of Gravitation.

$G$ is a universal constant independent of the masses $m_1$ and $m_2$; in SI units,

$$G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2$$

Newton showed that bodies affect one another as if the mass of each were concentrated at its center of mass, provided that the mass of each body is distributed in a spherically symmetric way. In this case $r$ is the distance between the centers of mass.

Suppose that a spherical projectile of mass $m$ is hurled straight up from the surface of the earth. Can it escape into outer space? As it will turn out, we don’t need a solution formula to answer this question. Suppose that the projectile moves along a $y$-axis perpendicular to the earth’s surface with the positive direction upward ($y = 0$ at the surface). Suppose that air resistance is negligible, so the only significant force acting on the body is the gravitational attraction of the earth. This force acts downward and, according to Newton’s Law of Universal Gravitation, is given by

$$F = \frac{-mMG}{(y + R)^2}$$

where $M$ is the mass and $R$ the radius of the earth. For values of $y$ close to 0, the gravitational force $F$ is close to the constant $-mMG/R^2 = -mg$, where $g = MG/R^2$. This approximation to the gravitational force is quite good near the ground, but it is not appropriate here because we want to see what happens when the projectile is far above the earth’s surface.

Using Newton’s Second Law (see Section 1.5), and the information given above, the projectile’s upward motion is modeled by the IVP

$$my'' = \frac{-mMG}{(y + R)^2}, \quad y(0) = 0, \quad y'(0) = v_0 > 0$$

If the initial velocity $v_0$ is small enough, we know from experience that the body will rise to a high point and then fall back to earth. Is there a smallest value for $v_0$ such that the body does not fall back?
Since the differential equation in IVP (15) doesn’t involve \( t \) explicitly, we may use the second method of reduction of order. If we set \( y'' = v rac{dv}{dy} \), we have that

\[
\frac{dv}{dy} = -\frac{MG}{(y + R)^2}, \quad v = v_0 \quad \text{when} \quad y = 0
\]

(16)

When the variables are separated and IVP (16) is solved, we have

\[
v^2 = \left( v_0^2 - \frac{2MG}{R} \right) + \frac{2MG}{y + R}
\]

(17)

From formula (17) we see that \( v^2 \) remains positive for all \( y \geq 0 \) as long as \( v_0^2 \geq \frac{2MG}{R} \).

So for any value \( v_0 > 0 \) such that \( v_0 \geq (2MG/R)^{1/2} \), it follows that \( v(y) > 0 \) for all \( y \geq 0 \). Reason: if \( v(\bar{y}) \leq 0 \) for some value \( \bar{y} > 0 \), then there would be a height \( h \) between 0 and \( \bar{y} \) such that \( v(h) = 0 \). But formula (17) tells us that \( v^2 \) is the sum of a nonnegative term \( [v_0^2 - (2MG/R)] \) and a positive term \( [2MG/(y + R)] \) so \( v \) can’t be zero. On the other hand, if \( 0 < v_0 < (2MG/R)^{1/2} \), then from (17) there is a value of \( y \) that makes \( v^2 = 0 \) and from that point on the projectile falls back toward the earth.

The velocity \( v_0 = (2MG/R)^{1/2} \) (18)

is called the escape velocity of the body because it is the smallest value of \( v_0 \) for which the body never falls back. Try to show this fact with a numerical solver! Note that we have answered our original question about the existence of an escape velocity without completing all the steps in the reduction process.

The escape velocity \((2MG/R)^{1/2}\) from the surface of a body depends only on the body’s mass and radius. The escape velocity from the earth’s surface turns out to be roughly 11.179 km/sec. Escape velocities for all the larger satellites in the solar system have been calculated and are listed in Handbook of Chemistry and Physics, 75th ed. (D. R. Lide, ed., Boca Raton, Fla.: CRC Press, 1994).

Sometimes we can work this process backwards and solve a system by looking at a first-order ODE. We illustrate this process by a very different (and controversial) model.

Destructive Competition

In 1916, Lanchester\(^6\) described some mathematical models for air warfare. These have been extended to a general combat situation, not just air warfare. We look at a particular case of the models.

\(^6\)Frederick William Lanchester (1868–1946) was an English engineer, mathematician, inventor, poet, and musical theorist who designed and produced some of the earliest automobiles and wrote the first theoretical treatise of any substance on flight, the book Aerial Flight, (Constable, London, 1907–08). His book Aircraft in Warfare (Constable, London, 1916), contains the square law of conventional combat discussed in this section. Twenty-five years later (during the Second World War) Lanchester’s scientific approach to military questions became the basis of a new science, Operations Research, which is now applied to problems of industrial management, production, and government procedure, as well as to military problems.
An x-force and a y-force are engaged in combat. Suppose that \( x(t) \) and \( y(t) \) denote the respective strengths of the forces at time \( t \), where \( t \) is measured in days from the start of the combat. It is not easy to quantify strength, including, as it does, the numbers of combatants, their battle readiness, the nature and number of the weapons, the quality of leadership, and a host of psychological and other intangible factors difficult even to describe, much less to turn into state variables. Nevertheless, we will suppose that the strengths can be quantified, that \( x(t) > y(t) \) means the x-force is stronger than the y-force, and that \( x(t) \) and \( y(t) \) are differentiable functions. The pair of values \( x(t), y(t) \) defines the state of this system at time \( t \).

Suppose that we can estimate the noncombat loss rate of the x-force (i.e., the loss rate due to diseases, desertions, and other noncombat mishaps), the combat loss rate due to encounters with the y-force, and the reinforcement rate. Then the net rate of change in \( x(t) \) is given by the Balance Law:

\[
x'(t) = \text{Reinforcement rate} - \text{Noncombat loss rate} - \text{Combat loss rate}
\]

A similar ODE applies to the y-force. The problem is to analyze the solutions \( x(t) \) and \( y(t) \) of the resulting system to determine who wins the combat.

According to Lanchester, a model system of ODEs for a pair of conventional combat forces operating in the open with negligible noncombat losses is

\[
\begin{align*}
x'(t) &= R_1(t) - by(t) \\
y'(t) &= R_2(t) - ax(t)
\end{align*}
\]  

(19)

where \( a \) and \( b \) are positive constants and \( R_1 \) and \( R_2 \) are the rates of reinforcement. The reinforcement rates are assumed to depend only on time and not on the strength of either force (a dubious assumption). The combat loss rates \( by(t) \) and \( ax(t) \) introduce actual combat into the model. Lanchester argues for the specific form of the combat loss rate terms in system (19) as follows. Every member of the conventional force is assumed to be within range of the enemy. It is also assumed that as soon as a conventional force suffers a loss, fire is concentrated on the remaining combatants. This implies that the combat loss rate of the x-force is proportional to the number of the enemy, and so is given by \( by(t) \). The coefficient \( b \) is a measure of the average effectiveness in combat of each member of the y-force. A similar argument applies to the term \( ax(t) \).

Earlier in this section we showed how the solution curves of the ODE \( N(y) y' + M(x) = 0 \) can be characterized in terms of the orbits of the differential system

\[
\frac{dx}{dt} = N(y), \quad \frac{dy}{dt} = -M(x)
\]

The next example shows the usefulness of the reverse process.

**EXAMPLE 1.7.5**

**Conventional Combat: From a System to a First-Order ODE**

Combat between two forces with no reinforcements may be modeled by the system

\[
\begin{align*}
x' &= -by \\
y' &= -ax
\end{align*}
\]  

(20)

Let \( x = x(t) \) and \( y = y(t) \) solve system (20), and suppose that the orbit lies inside the
first quadrant of the $xy$-plane (i.e., $x > 0$, $y > 0$). We have from (20) that
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{ax}{by}
\]
or
\[
by \frac{dy}{dt} - ax \frac{dx}{dt} = 0 \tag{21}
\]
which is a separable ODE.

So the integral curves of the “reduced” ODE (21) are just the orbits of system (20). The only assumptions made are that $x > 0$ and $y > 0$. Now let’s solve ODE (21) and see what happens.

**Example 1.7.6 Who Wins?**

Let’s suppose that $x_0 > 0$ and $y_0 > 0$ are the strengths of the two forces at the start of combat. Integrating (21), we have
\[
by^2(t) - ax^2(t) = by_0^2 - ax_0^2 \tag{22}
\]
Equation (22) is known as the *square law of conventional combat*. Although we can’t tell just how $x(t)$ and $y(t)$ change in time, (22) implies that the point $(x(t), y(t))$ moves along an arc of a hyperbola as time advances. These hyperbolas are orbits of system (20) and are plotted in Figure 1.7.2 for $a = 0.064$, $b = 0.1$, $x_0 = 10$, and various values of $y_0$. The arrowheads on the curves show the direction of changing strengths as time passes. Since $x'(t) = -by(t)$, $x(t)$ decreases as time advances if $y(t)$ is positive. Similarly, $y(t)$ decreases with the advance of time if $x(t)$ is positive.

One force wins if the other force vanishes first. For example, $y$ wins if $by_0^2 > ax_0^2$, since from (22), $y(t)$ would never vanish, while the $x$-force is gone by the time $y(t)$ has decreased to $\left( \frac{(by_0^2 - ax_0^2)}{b} \right)^{1/2}$. So the $y$-force wants a combat setting in which $by_0^2 > ax_0^2$; that is, the $y$-force needs a large enough initial strength $y_0$ so that
\[
y_0 > \left( \frac{a}{b} \right)^{1/2} x_0 \tag{23}
\]
Figure 1.7.3 shows components of the top orbit in Figure 1.7.2 ($a = 0.064$, $b = 0.1$). The combatants start out even in Figure 1.7.3 at $x_0 = y_0 = 10$, but the $y$-force wins because $b = 0.1$, which is larger than $a = 0.064$.

The simplified model solved above is unrealistic. If noncombat loss rates and reinforcement rates are included, a certain element of realism enters and one might actually compare the model with historical battles. Studies along these lines for the Battle of the Ardennes and the Battle of Iwo Jima in the Second World War have been carried out. These studies give results reasonably close to the actual combat statistics once the coefficients $a$ and $b$ are determined. Whether these coefficients could ever be estimated with any accuracy before combat is an open question.

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\[ x' = -0.1y, \quad x(0) = 10 \]
\[ y' = -0.064x, \quad y(0) = 6, 7, 8, 9, 10 \]

\[ x' = -0.1y, \quad x(0) = 10 \]
\[ y' = -0.064x, \quad y(0) = 10 \]

**FIGURE 1.7.2** Arcs of hyperbolas of the square law of conventional combat (Example 1.7.6).

**FIGURE 1.7.3** Component curves of the top orbit in Figure 1.7.2.

**The Dark Side of Modeling**

Combat models raise ethical questions about the uses of mathematics. G. H. Hardy, a noted mathematician of the early twentieth century, wrote:

*So a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is... a harmless and innocent occupation.*

One wonders if it is that simple. Mathematics is a part of the cultures and societies of the mathematicians who create or discover it. Since war and the preparation for war continue to be a preoccupation of humankind, it is not surprising that mathematics is applied to its study and analysis.

**PROBLEMS**

1. *(The System Way to Plot Integral Curves).* Find an integral for each ODE and plot representative integral curves (which in this case are also orbits) in the given rectangle. *[Hint: Follow the method used in Example 1.7.1 to plot curves, choose initial points in the rectangle, and solve forward and backward in time.]

   (a) \( (1 - y^2)y' + x^2 - 1 = 0; \quad |x| \leq 3, \quad |y| \leq 3 \)

   (b) \( (1 - y^2)y' + x^2 = 0; \quad |x| \leq 1.5, \quad |y| \leq 2 \)

---

2. (The System Way to Plot Solution Curves). Plot the solution curve of each IVP on the largest possible interval.
   (a) \( x^2 - 1 + (1 - y^2)y' = 0; \quad y(-1) = -2 \)
   (b) \( (1 - y^2)y' + x^2 = 0; \quad y(-1) = 0.5 \)
   (c) \( 2(x^2 + 2xy)y' = 2xy + y^2; \quad y(-0.5) = 1 \)

3. (Combat Model: Who Wins the Battle?).
   (a) Reproduce Figure 1.7.2 as closely as possible. Plot the orbit with initial condition \( x_0 = 10, y_0 = 7 \). Does \( y \) win or lose?
   (b) How much time does it take for the conflict to be resolved?

4. (Reduction of Order). Solve the following ODEs by reduction of order. Plot solution curves \( y = y(t) \) in the \( t,y \)-plane using various sets of initial data. [Hint: If \( y \) does not appear explicitly, set \( y' = v, \quad y'' = v' \). If \( t \) does not appear explicitly, set \( y' = v \) and \( y'' = v dv/dy \).]
   (a) \( ty'' - y' = 3t^2 \)
   (b) \( y'' - y = 0 \)
   (c) \( y'' + (y')^2 = 1 \)
   (d) \( y'' + 2ty' = 2t \)

5. (Reduction of Order). Solve the given initial value problems.
   (a) \( 2yy'' + (y')^2 = 0, \quad y(0) = 1, \quad y'(0) = -1 \)
   (b) \( y'' = y'(1+4/y^2), \quad y(0) = 4, \quad y'(0) = 3 \)
   (c) \( y'' = -g - y', \quad y(0) = h, \quad y'(0) = 0, \quad g, h \) are positive constants

6. (Reduction of Order by Varied Parameters). Suppose that \( z(t) \) is a solution of the linear second-order undriven ODE \( z'' + a(t)z' + b(t)z = 0 \), where \( a(t) \) and \( b(t) \) are continuous on an interval \( I \). The function \( z(t) \) can be used to find solutions of the driven ODE \( y'' + a(t)y' + b(t)y = f(t) \), where \( f(t) \) is also continuous on \( I \).
   (a) Show that \( y(t) = u(t)z(t) \) is a solution on \( I \) of the ODE \( y'' + a(t)y' + b(t)y = f(t) \) if \( u(t) \) solves the first-order linear ODE \( zu'' + (2z' + az)u' = f(t) \).
   (b) If \( z(t) \neq 0 \) on \( I \), show that the linear ODE for \( u' \) in part (a) can be put in normal linear form by dividing by \( z \). Find \( u'(t) \) by using the integrating factor \( z^2e^{A(t)} \), where \( A(t) \) is an antiderivative of \( a(t) \).
   (c) Find another solution of \( tz'' - (t+2)z' + 2z = 0 \) for \( t > 0 \), given that \( z = e^t \) is one solution.

7. Using the method described in Problem 6, find a solution of the ODE \( t^2y'' + 4ty' + 2y = \sin t \), \( t > 0 \), where it is known that \( z(t) = t^{-2} \) solves the ODE \( t^2z'' + 4tz' + 2z = 0 \). [Hint: First put the ODE in normal form by dividing by \( t^2 \).]

8. A tunnel is bored through the earth and an object of mass \( m \) is dropped into the tunnel.
   (a) Neglecting friction and air resistance, write an ODE which describes the motion of the object given that the component of the force of gravity in the direction of motion is given by \( f = GmM \sin \theta/(b^2 + y^2) \).
   (b) Find the general solution of the equation of motion. [Hint: Use a reduction-of-order technique. Your answer should be left in implicit form with an integral.]

9. (Escape Velocity in an Inverse Cube Universe). Suppose that on a planet in another universe the magnitude of the force of gravity obeys the Inverse Cube Law:
   \[ |F| = \frac{\tilde{G}mM}{(y+R)^3} \]
   where \( m \) is the mass of the object, \( y \) its location above the surface of the planet, \( M \) the mass of the planet, \( \tilde{G} \) a new universal constant, and \( R \) the radius of the planet.
   (a) What is the escape velocity \( \tilde{v}_0 \) from this planet?
   (b) What is the ratio of \( \tilde{v}_0 \) to the Inverse Square Law escape velocity \( v_0 \) [equation (18)]?
1.8 Cold Pills

What do you do when you catch a cold? If you are like many of us, you take cold pills. The pills contain a decongestant to relieve stuffiness and an antihistamine to stop the sneezing and to dry up a runny nose. The pill dissolves and releases the medications into the gastrointestinal tract. The medications diffuse from there into the blood, and the bloodstream takes each medication to the site where it has therapeutic effect. Both medications are gradually cleared from the blood by the kidneys and the liver.

Pharmaceutical companies do a lot of testing to determine the flow of a medication through the body. This flow is modeled by treating the parts of the body as compartments, and then tracking the medication as it enters and leaves each compartment. A typical cold medication leaves one compartment (e.g., the GI tract) and moves into another (such as the bloodstream) at a rate proportional to the amount present in the first compartment. The constant of proportionality depends upon the medication, the compartment, and the age and general health of the individual.
Now let’s build a system of ODEs that models the passage of one of the medications, say antihistamine, through the body compartments.9

### One Dose of Antihistamine

Let’s see what happens to a dose of antihistamine once it lands in the GI tract.

#### Modeling the Rates

Suppose that there are $A$ units of antihistamine in the GI tract at time 0 and that $x(t)$ is the number of units remaining at any later time $t$. The Balance Law applies:

\[
\text{Net rate} = \text{Rate in} - \text{Rate out}
\]

Since we start with $A$ units and the medication moves out of the GI tract and into the blood at a rate proportional to the amount in the GI tract, we have the IVP

\[
\frac{dx(t)}{dt} = -k_1 x(t), \quad x(0) = A
\]

where $k_1$ is a positive constant. The top part of the margin sketch pictures the IVP. Time is measured in hours and $k_1$ in hours$^{-1}$.

We assume that the compartment labeled “blood” includes the tissues where the medication does its work. The level $y(t)$ of antihistamine in the blood builds up from zero but then falls as the kidneys and liver do their job of clearing foreign substances from the blood. The Balance Law applied to the antihistamine in the blood compartment leads to the IVP

\[
\frac{dy(t)}{dt} = k_1 x(t) - k_2 y(t), \quad y(0) = 0
\]

The first term on the right side of the rate equation in (2) models the fact that the exit rate of antihistamine from the GI tract equals the entrance rate into the blood. The second rate term models the clearance of antihistamine from the blood. The clearance constant $k_2$ is measured in hours$^{-1}$. The lower part of the margin sketch illustrates IVP (2).

Putting (1) and (2) together, we have a system of two first-order ODEs with initial data:

\[
\frac{dx}{dt} = -k_1 x, \quad x(0) = A
\]

\[
\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0
\]

9The cold pill model is based on the work of the contemporary applied mathematician, Edward Spitznagel, Professor of Mathematics at Washington University, St. Louis, Missouri. His current research areas are pharmacokinetics (study of the flow of medications through the body) and bioequivalence (study of the efficacy of medications), fields that use compartment models extensively. Professor Spitznagel started off his career in mathematics because he could not limit his interest to just one area of science. He does consulting for a wide variety of clients, principally the pharmaceutical industry and medical schools. His advice to aspiring applied mathematicians: Learn as much mathematics as you can because it will make you versatile and able to pick up new ideas easily. Seek out opportunities to learn how to apply mathematics in the real world by on-the-job training.
1.8/ Cold Pills

IVP (3) is our mathematical model for the flow of a single dose of $A$ units of medication through the GI tract and blood compartments.

Now let’s solve IVP (3) and find formulas for the amounts of medication in each compartment.

**EXAMPLE 1.8.2 Growth and Decay of Antihistamine Levels**

Integrating factors are used to solve the linear differential equations of IVP (3) one at a time, starting with the first. Rearrange the first rate equation, multiply by the integrating factor $e^{k_1 t}$, integrate, and then use the initial data to obtain

$$x(t) = Ae^{-k_1 t}$$

Insert this formula for $x(t)$ into the second rate equation, which becomes

$$\frac{dy}{dt} + k_2 y = k_1 Ae^{-k_1 t}, \quad y(0) = 0$$

This linear IVP is solved using the integrating factor $e^{k_2 t}$. After some calculation, we obtain the formula

$$y(t) = \frac{k_1 A}{k_1 - k_2} (e^{-k_2 t} - e^{-k_1 t})$$

Summarizing, we see that the antihistamine levels in the GI tract and blood are given by the formulas

$$x(t) = Ae^{-k_1 t}, \quad y(t) = \frac{k_1 A}{k_1 - k_2} (e^{-k_2 t} - e^{-k_1 t})$$

We have assumed that $k_1 \neq k_2$, an assumption that is justified by the pharmaceutical data, as we will soon see.

From the formulas in (4), we see that the antihistamine levels in the GI tract and the blood tend to zero as time increases. The level in the blood reaches a maximum value at some positive time (see Problem 4) and then drops back.

The symbolic solution formulas (4) are abstract, so let’s introduce some numbers.

**EXAMPLE 1.8.3 The Rate Constants**

One pharmaceutical company estimates that the values of the rate constants for the antihistamine in the cold pills it makes are

$$k_1 = 0.6931 \text{ (hour)}^{-1}, \quad k_2 = 0.0231 \text{ (hour)}^{-1}$$

Because $k_2$ is so much smaller than $k_1$, antihistamine stays at a high level a lot longer in the blood than in the GI tract. Figure 1.8.1 shows the levels over a six-hour period as predicted by system (3) if $A = 1$ and $k_1$ and $k_2$ are given by (5).

The values of the rate constants given in (5) are for the “average” person, but what about someone who isn’t average?
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\[ x' = -0.6931x, \quad x(0) = 1 \]
\[ y' = 0.6931x - 0.0231y, \quad y(0) = 0 \]

**FIGURE 1.8.1** Effect of one dose of antihistamine (Example 1.8.3).

**FIGURE 1.8.2** Sensitivity to the clearance coefficient (Example 1.8.4).

**EXAMPLE 1.8.4 Sensitivity to the Clearance Coefficient**

The clearance coefficient \( k_2 \) of medication from the blood is often much lower for the old and sick than it is for the young and healthy. This means that for some people medication levels in the blood may become and then remain excessively high, even with a standard dosage. Figure 1.8.2 displays the results of a parameter study in which antihistamine levels in the blood are plotted over a 24-hour period. This is done for \( k_1 = 0.6931 \) and five values of \( k_2 \), corresponding to five people of different ages and states of health. The variations in the levels are a measure of the sensitivity of the medication levels to changes in the value of \( k_2 \).

Now let’s determine the effect of repeating the dose.

**Repeated Doses**

Few of us stop with taking just one cold pill. We take several doses until we feel better. Most cold pills dissolve quickly and release their medications in the GI tract at a constant rate over half an hour. The dose is then repeated every six hours to maintain the medication levels in the blood. Typically, the release rate of antihistamine from the pill into the GI tract is constant for a short time, but then cuts off entirely until the next dose. We will use an on-off function to model this kind of repeated dosage.

**EXAMPLE 1.8.5 One Dose Every Six Hours: Square Wave Delivery Rate**

Let’s suppose that 6 units of antihistamine are delivered to the GI tract at a constant
rate and over a half-hour time span, then repeated every six hours. The model IVP is

\[ \frac{dx}{dt} = I(t) - k_1 x, \quad x(0) = 0 \]

\[ \frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0 \]

(6)

The periodic on-off function \( 12 \text{sqw}(t, 25/3, 6) \) denotes the square wave of amplitude 12 and period 6 hours, which is “on” for half an hour at the start of each period (and so delivers 6 units of medication in that time span), and is “off” otherwise. Note that one half hour is \( \frac{100}{12} = \frac{25}{3} \% \) of the six-hour period. See Figure 1.8.3 for the graph of \( I(t) \).

Each ODE in (6) is linear, but the strange form of the input function \( I(t) \) is somewhat daunting. Is there a way to study solutions of IVP (3) without having to find solution formulas?

**EXAMPLE 1.8.6 Rising Levels of Antihistamine: Using a Numerical Solver**

Although the integrating factor technique used before also applies to system (6), there is an awkward integration to carry out because \( I(t) \) repeatedly turns on and off. So we used a numerical solver that handles on-off functions to produce Figure 1.8.4. The figure shows that the amount of antihistamine in the GI tract rises quite rapidly as the cold pill dissolves, but then drops back almost to the zero-level before the next dose is taken. It is a different story in the blood, where the low value of the clearance coefficient \( k_2 \) keeps antihistamine levels high. Even after 48 hours the levels in the blood show little sign of approaching any kind of equilibrium.
\[ x' = 12 \sqrt{y(t, 25/3, 6)} - 0.6931 \cdot x, \] 
\[ y' = 0.6931 \cdot x - k_2 \cdot y, \] 
\[ x(0) = 0, \quad y(0) = 0 \]

FIGURE 1.8.5 Therapeutic zone for antihistamine (Example 1.8.7).

When you take cold pills, you want the medication in the bloodstream to reach therapeutic levels quickly and to stay in a safe range. The rising levels in the blood shown in Figure 1.8.4 are alarming. What happens if they get too high?

EXAMPLE 1.8.7 Asleep in Math Class

High levels of antihistamine cause drowsiness, but low levels are ineffective. Suppose that the therapeutic but safe range for antihistamine in the blood is from 5 to 40 units. Figure 1.8.5 shows the amounts of antihistamine in the blood of three math students with quite different clearance coefficients \( k_2 \). From the graphs we see that each of the three gets relief from cold symptoms within six hours. But with repeated doses the levels are markedly different. Which of the three students fall asleep in math class?

The cold pill model is just one of many examples of a compartment model. Let’s take a brief look at some general compartment models.

Compartment Models

A compartment model consists of a finite number of compartments (or boxes) connected by arrows. Each arrow means that the substance being tracked leaves the box at the foot of the arrow and enters the box at the arrowhead.
There are also nonlinear cascades.

The simplest compartment models are the linear cascades. A compartment model is a linear cascade if

(a) no directed chain of arrows and boxes begins and ends at the same box, and
(b) the substance exits a box at a rate proportional to the amount in the box and, if it enters another box, does so at that same rate.

An arrow pointing toward one box, but not out of another, indicates an external source of the substance (i.e., an input). These arrows are labeled with an “I,” where \( I \) is the input rate of the substance. An arrow that points away from a box, but not toward another box, means that the substance exits the system from that box. The state variable \( x_i(t) \) denotes the amount of the substance in box \( i \) at time \( t \). The symbols \( k_{ij} \) by an arrow leaving box \( i \) and entering box \( j \) mean that the substance exits \( i \) and enters \( j \) at the rate \( k_{ij} \).

The models for the flow of cold medication are examples of linear cascades. Here is a more complex example.

**EXAMPLE 1.8.8** From Boxes and Arrows to a System of ODEs

Figure 1.8.6 shows a linear cascade with two inputs. The corresponding ODEs can be constructed directly from the boxes and arrows. The first-order linear system of ODEs is based on the Balance Law applied to each box:

\[
egin{align*}
    x_1' &= I_1 - k_{1x_1} \\
    x_2' &= -k_{2x_2} \\
    x_3' &= I_3 + k_{1x_1} + k_{2x_2} - k_{3x_3} - k_{4x_3} \\
    x_4' &= k_{3x_3} \\
    x_5' &= k_{4x_3} - k_{5x_5}
\end{align*}
\]  

(7)

System (7) can be solved from the top down, one ODE at a time.

Turning this around, we can construct a linear cascade model of boxes and arrows as in Figure 1.8.6 from a system of linear ODEs like (7).

The ODEs that model a linear cascade are called a linear cascade of ODEs; the system can always be solved from the top down, as in Example 1.8.8.

Linear cascades may be used to model other physical phenomena besides flow through compartments. For example, in a radioactive decay process one element de-
cays into another, which in turn decays into a third, and so on, until the process terminates in a stable, nonradioactive element. In this case each compartment corresponds to a distinct element.

**Comments**

A numerical solver is useful in solving a cascade of IVPs, particularly if some intake rates are on-off functions such as square waves. If you do use a solver in this situation, you may have to set the maximal internal step-size quite low so that the solver detects each of the on-off times.

Although on-off intake rate functions are discontinuous, the solution formulas of Sections 1.3 and 1.4 for first-order linear IVPs still apply. Imagine a sequence of IVPs, one for each interval between the break points of the rate function. The final value for one interval becomes the initial value for the next. The resulting solution curves may have corners, but the curves are continuous. See Figure 1.8.4 for an example.

**PROBLEMS**

1. *(From Boxes and Arrows to ODEs).* Write the system of first-order linear IVPs that models each of the following linear cascades, solve, and describe what happens in each compartment as \( t \to \infty \). [Hint: See Example 1.8.8.]

   **(a)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 \\
   x' = 5 - x, \quad y = x - 5y \\
   x(0) = 1, \quad y(0) = z(0) = 0
   \end{array}
   \]

   \[
   \begin{array}{c}
   \text{I} = 1 \\
   x' = 5 - x, \quad y = x - 5y \\
   x(0) = 1, \quad y(0) = z(0) = 0
   \end{array}
   \]

   **(b)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 \\
   x' = -x/2, \quad y' = 1 - y/3, \quad z' = x/2 + y/3 \\
   x(0) = y(0) = z(0) = 0
   \end{array}
   \]

   **(c)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 + \sin t \\
   x' = -x, \quad y' = x/2 - 3y, \quad z' = x/2 + 3y - 2z \\
   x(0) = y(0) = z(0) = 0
   \end{array}
   \]

   **(d)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 \\
   x' = -x, \quad y' = x/2 - 3y, \quad z' = x/2 + 3y - 2z \\
   x(0) = y(0) = z(0) = 0
   \end{array}
   \]

2. *(From ODEs to Boxes and Arrows).* In each case sketch and label the boxes and arrows diagram.

   **(a)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 \\
   x' = -x/2, \quad y' = 1 - y/3, \quad z' = x/2 + y/3 \\
   x(0) = y(0) = z(0) = 0
   \end{array}
   \]

   **(b)**
   
   \[
   \begin{array}{c}
   \text{I} = 1 + \sin t \\
   x' = -x, \quad y' = x/2 - 3y, \quad z' = x/2 + 3y - 2z \\
   x(0) = y(0) = z(0) = 0
   \end{array}
   \]
3. *(One Dose of Antihistamine).* Suppose that $A$ units of antihistamine are present in the GI tract and $B$ units in the blood at time 0.

(a) Solve IVP (3) with the condition $y(0) = 0$ replaced by $y(0) = B$.

(b) Use a numerical solver to solve IVP (3) but with $x(0) = 1$, $y(0) = 1$, $k_1 = 0.6931$, $k_2 = 0.0231$, $0 \leq t \leq 6$. Graph the levels of antihistamine in the GI tract and in the blood. Estimate the highest level of the antihistamine in the blood and the time it reaches that level.

4. *(Time of Maximum Dosage).* Show that the medication in the bloodstream reaches its maximum after a single dose when $t = (\ln k_1 - \ln k_2)/(k_1 - k_2)$, $k_1 \neq k_2$. [Hint: Use the model developed in Examples 1.8.1 and 1.8.2.]

5. *(One Dose: Sensitivity to $k_1$).* Let $A = 1$, keep $k_2 = 0.0231$ as in Example 1.8.3, but let $k_1$ vary.

(a) Using a numerical solver for system (3), display the effects on the antihistamine levels in the bloodstream if $k_1 = 0.06931$, 0.11, 0.3, 0.6931, 1.0, and 1.5. Plot the graphs over a 24-hour period. Why do the graphs for larger values of $k_1$ cross the graphs for smaller values?

(b) You need to keep medication levels within a fixed range in order to be both therapeutic and safe. Suppose that the desired range for antihistamine levels in the blood is from 0.2 to 0.8 for a unit dose taken once. With $k_2 = 0.0231$, find upper and lower bounds on $k_1$ so that the antihistamine levels in the blood reach 0.2 within 2 hours and stay below 0.8 for 24 hours.

6. *(Cold Medication: Decongestant).* Most cold pills contain a decongestant as well as an antihistamine. The form of the rate equations for the flow of decongestant is the same as that for antihistamine, but the rate constants are different. The values of these rate constants for one brand of cold pills now on the market have been determined by the manufacturer to be $k_1 = 1.386$ (hour)$^{-1}$ for the passage from the GI tract into the blood, and $k_2 = 0.1386$ (hour)$^{-1}$ for clearance from the blood. The respective amounts of decongestant in the GI tract and the blood are denoted by $x(t)$, $y(t)$. Use Examples 1.8.1–1.8.6 as guides as you solve the following problems.

(a) *(A Single Dose of Decongestant).* Suppose that $A$ units of decongestant are in the GI tract at time 0, while the blood is free of decongestant. Construct a labeled boxes and arrows diagram and the corresponding system of IVPs for the flow of decongestant.

(b) Solve the IVPs of part (a). Then set $A = 1$, plot $x(t)$ and $y(t)$ over a six-hour time span, and describe what happens to the decongestant levels. Compare with Figure 1.8.1.

(c) *(Sensitivity to Clearance Coefficient).* Plot decongestant levels in the blood if $A = 1$, keeping the value of $k_1$ fixed at 1.386 but setting $k_2 = 0.01386$, 0.06386, 0.1386, 0.6386, 1.386. Describe what you see.

(d) *(One Dose Every Six Hours).* Suppose decongestant is released in the GI tract at the constant rate of 12 units/hour for 1/2 hour, and then repeated every 6 hours. Construct the model system of IVPs and plot the decongestant levels in the GI tract and blood over 48 hours (use $k_1 = 1.386$, $k_2 = 0.1386$). Compare and contrast your plots with Figure 1.8.4.

(e) *(Take the Pills for Five Days).* Using the data of part (d), plot decongestant levels in the GI tract and in the blood over a 5-day period. Repeat with $k_2 = 0.06386$, 0.01386. Will there be excessive decongestant accumulation in the blood of the old and the sick?

7. *(Continuous Doses of Medication).* If a medication is embedded in beads of resins that dissolve at varying rates, a constant flow rate $I$ of medication can be assured. See the margin sketch.

(a) Use the information in the compartment diagram to construct IVPs for $x(t)$ and $y(t)$.

(b) Solve the system found in part (a) from the top down (assume that $k_1 \neq k_2$).

(c) *(Approach to Equilibrium).* What happens to the levels of medication in the GI tract and in the blood as $t \to \infty$?

(d) *(Antihistamines and Decongestants).* Use a numerical solver to plot $x(t)$ and $y(t)$ for
200 hours. Use \( I = 1 \text{ unit/hour} \), \( k_1 = 0.6931 \text{ (hour)}^{-1} \), and \( k_2 = 0.0231 \text{ (hour)}^{-1} \) for antihistamine. Are the antihistamine levels in the blood close to equilibrium as \( t \to 200 \)? Repeat with the decongestant: \( k_1 = 1.386 \text{ (hour)}^{-1} \), \( k_2 = 0.1386 \text{ (hour)}^{-1} \), \( 0 \leq t \leq 100 \).

\( \square \) (e) (The Old and the Sick). The coefficients \( k_1 \) and \( k_2 \) for the old and sick may be much less than those for the young and healthy. Plot decongestant and antihistamine levels in the GI tract and the blood if the values of \( k_1 \) and \( k_2 \) are one-third of those in (d). Interpret your graphs.

\( \square \) (f) (Safe and Effective Zone). Assume that the clearance coefficients \( k_1 \) and \( k_2 \) have one-third of the values given in part (d). Suppose that the levels of antihistamine in the blood are designed to reach and remain between 25 and 50 units for a dose taken continuously at the rate of 1 unit per hour. Is this possible during the second through the fifth days?

### 1.9 Change of Variables and Pursuit Models

A change of variable may convert an apparently intractable ODE into another that can be solved by one of the techniques of this chapter. We will show how to convert an ODE with a certain kind of rate function (defined below) to a separable ODE, which can be solved by the techniques of Section 1.6. Problems 9 and 10 show how to reduce some other nonlinear first-order ODEs to linear form. We will also show how scaling the variables in an ODE has all sorts of computational and modeling pay-offs.

#### Homogeneous Rate Functions of Order Zero

Some types of rate functions appear so often in applications that it is worthwhile singling them out for special consideration. One of these types is defined here. A continuous function \( f(x, y) \) is said to be **homogeneous of order zero** if

\[
    f(kx, ky) = f(x, y)
\]

for all \( k > 0 \), \( x, y \) for which \( f(x, y) \) and \( f(kx, ky) \) are defined. Here’s an example.

**EXAMPLE 1.9.1 A Homogeneous Function of Order Zero**

The function \( f = (x^2 + y^2)^{1/2}/(x + y) \) is homogeneous of order zero because we have \( f(kx, ky) = ((kx)^2 + (ky)^2)^{1/2}/(kx + ky) = f(x, y) \). But \( g = (x + 1)/(x + y) \) is not homogeneous of order zero because \( g(kx, ky) = (kx + 1)/(kx + ky) \neq g(x, y) \).

Now consider the ODE \( y' = f(x, y) \), where \( f(x, y) \) is homogeneous of order zero. The change of variable \( y = xz \) converts \( y' \) to \( (xz)' = xz' + z \) and \( f \) to

\[
    f(x, y) = f(x, xz) = f(x \cdot 1, x \cdot z) = f(1, z)
\]

since \( f \) is homogeneous of order zero (\( x \) plays the role of \( k \) in this setting). We have a separable ODE in \( x \) and \( z \):

\[
    xz' + z = f(1, z), \quad \text{or} \quad xz' = f(1, z) - z, \quad \text{or} \quad \frac{1}{f(1, z) - z} z' = \frac{1}{x} \tag{2}
\]
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The last ODE in (2) can be solved by integrating each side: \( F(z) = \ln |x| + C \), where \( F(z) \) is an antiderivative of \( [f(1, z) - z]^{-1} \) and \( C \) is any constant. Since \( z = y/x \),

\[
F\left(\frac{y}{x}\right) = \ln |x| + C \quad (3)
\]

So, a solution \( y(x) \) of the first-order ODE \( y' = f(x, y) \) satisfies (3) on some \( x \)-interval and for some choice of \( C \). Let’s see how this works in practice.

**EXAMPLE 1.9.2 An ODE with a Homogeneous Rate Function of Order Zero**

The function \( f(x, y) = \frac{x^2 + y^2}{2xy} \) is homogeneous of order zero. To solve

\[
y' = \frac{x^2 + y^2}{2xy}, \quad x \neq 0, \quad y \neq 0 \quad (4)
\]

we introduce the new variable \( z \) by setting \( y = xz \) to obtain the equation

\[
xz' + z = \frac{x^3 + x^2z^2}{2x^2} = \frac{1 + z^2}{2z}
\]

Separating the variables, we have

\[
\frac{2z}{1-z^2}z' = \frac{1}{x}, \quad z \neq \pm 1
\]

Antidifferentiate both sides to obtain

\[-\ln |1 - z^2| = \ln |x| + C, \quad z \neq \pm 1\]
where $C$ is the constant of integration. Rearranging and returning to the original variables $y$ and $x$ by setting $z = y/x$, we have

$$\ln |x| + \ln \left| 1 - \frac{y^2}{x^2} \right| = \ln \left| x \left( 1 - \frac{y^2}{x^2} \right) \right| = -C, \quad y \neq \pm x \quad (5)$$

Exponentiate each side of the last equality in (5) to get

$$\left| x \left( 1 - \frac{y^2}{x^2} \right) \right| = e^{-C}, \quad y \neq \pm x \quad (6)$$

We remove the absolute value signs from (6), replace $e^{-C}$ by a constant $K$ (which can have any real value), multiply by $x$, and obtain the following equation:

$$x^2 - y^2 = Kx, \quad x \neq 0, \quad y \neq 0 \quad (7)$$

Figure 1.9.1 shows some of the hyperbolic curves defined by (7) for various values of $K$. For $K \neq 0$, solution curves $y = y(x)$ of ODE (4) are the hyperbolic arcs above or below the $x$-axis. Finally, note that $y = \pm x$ are also solutions of ODE (4).

Ordinary differential equations with homogeneous rate functions of order zero come up in pursuit problems. Let’s see how this happens.

**Curves of Pursuit**

In models of pursuit, the pursuer chases a target (whose motion is known) by using a predetermined strategy, for example, by deliberately aiming toward it. Let’s say a ferryboat is set to sail across the river to a dock, but a current complicates the captain’s decision making. The captain decides to aim the ferry toward the dock at all times. Will the ferry make it? Although not at all obvious, the problem of finding the path of pursuit where the target is at rest eventually comes down to solving the first-order ODE $dy/dx = f(x, y)$, where $f(x, y)$ is a homogeneous function of order zero. Let’s see why this is so by looking at a specific example.

**EXAMPLE 1.9.3 A Goose Flies to Its Nest: The Mathematical Model**

A goose attempts to fly back to its nest, which is directly west of its position, but a steady wind is blowing from the south. The goose keeps heading to its nest, and the wind blows it off course. What is the goose’s flight path? Can the goose get home?

Suppose that the path of the goose is given by the parametric equations $x = x(t), \quad y = y(t)$, where $t$ is time. $(x(t), y(t))$ is the location of the goose at time $t$. $(x(0), y(0)) = (a, 0)$, and $(x(T), y(T)) = (0, 0)$. Time $T$ is the unknown time of arrival at the nest. Suppose that the bird can fly at a rate of $b$ miles per hour, and that the south wind is blowing at a rate of $w$ miles per hour. The heading angle $\theta$ (see Figure 1.9.2) will change as the bird’s position changes.

The rate $dx/dt$ is the component of the goose’s velocity in the $x$ direction:

$$\frac{dx(t)}{dt} = -b \cos \theta = \frac{-bx}{(x^2 + y^2)^{1/2}} \quad (8)$$
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1.9.3 Change of Variables and Pursuit Models

\[ \frac{dy}{dt} = -b \sin \theta + w = \frac{-by}{(x^2 + y^2)^{1/2}} + w \]  

(9)

The differential system (8), (9) can be treated as we did the combat model system in Section 1.7 because the rate functions don’t contain time explicitly.

Dividing ODE (9) by ODE (8), we have a first-order ODE in \( x \) and \( y \), where \( x \) is now the independent variable. Setting the constant \( c = w/b \), we have the IVP

\[ \frac{dy}{dx} = \frac{by - w(x^2 + y^2)^{1/2}}{bx} = \frac{y - c(x^2 + y^2)^{1/2}}{x}, \quad y(a) = 0 \]  

(10)

The ODE in (10) has a rate function \( f(x, y) \) that is homogeneous of order zero, since

\[ f(kx, ky) = \frac{ky - c(k^2x^2 + k^2y^2)^{1/2}}{kx} = \frac{y - c(x^2 + y^2)^{1/2}}{x} = f(x, y) \]

So now that we have the model, let’s solve ODE (10) and see what happens to the goose.

**EXAMPLE 1.9.4 Going Home, or Gone with the Wind?**

We set \( y = xz \) and from ODE (10) we get a separable ODE

\[ \frac{dy}{dx} = x \frac{dz}{dx} + z = \frac{zx - c(x^2 + x^2z^2)^{1/2}}{x} = z - c(1 + z^2)^{1/2}, \quad \text{or} \]

\[ x \frac{dz}{dx} = -c(1 + z^2)^{1/2} \]

\[ (1 + z^2)^{-1/2} \frac{dz}{dx} = -\frac{c}{x} \]
with solutions defined by
\[ \ln[z + (1 + z^2)^{1/2}] = -c \ln x + C \]
where \( C \) is a constant of integration. We do not need absolute value signs inside the logarithms since \( x > 0 \) and \( z > 0 \). Note that \( C = c \ln a \) because \( z = y = 0 \) if \( x = a \). So, \( \ln[z + (1 + z^2)^{1/2}] = -c \ln x + c \ln a = \ln(x/a)^{-c} \); exponentiating, we obtain
\[ z + (1 + z^2)^{1/2} = \left( \frac{x}{a} \right)^{-c} \]  
(11)
Write (11) as \( (1 + z^2)^{1/2} = (x/a)^{-c} - z \), square, and solve for \( z \):
\[ z = \frac{1}{2} \left[ \left( \frac{x}{a} \right)^{-c} - \left( \frac{x}{a} \right)^c \right] \]
Since \( z = y/x \), the equation of the path followed by the goose is
\[ y = \frac{a}{2} \left[ \left( \frac{x}{a} \right)^{1-c} - \left( \frac{x}{a} \right)^{1+c} \right] \]  
(12)
In Figure 1.9.3 this path is plotted for \( a = 10 \) and several values of \( c = w/b \). If the wind’s speed is less than the bird’s (i.e., if \( c < 1 \)), the bird will reach its nest (solid curves) because the terms in formula (12) have positive exponents and so tend to 0 as \( x \to 0^+ \). But if \( c > 1 \), then the exponent of the first term inside the brackets is negative, and so that term blows up as \( x \to 0^+ \): the goose is gone with the wind.

Preparing for Computation: Scaling the Variables

Sometimes we change variables, not to reduce an ODE to a form where there is a known solution formula, but to reduce the number of symbolic coefficients appearing in the ODE. This is usually done by scaling the variables, that is, replacing \( y \) and \( t \) in the ODE \( y' = f(t, y) \) by \( s = t/t_1, \ w = y/y_1 \), for suitably chosen constants \( y_1 \) and \( t_1 \). The example below shows how this is done.

**EXAMPLE 1.9.5**

This ODE was derived in Section 1.6.

More on the chain rule in Theorem B.5.7 in Appendix B.5.

**Scaling the Velocity and Time Variables**

The velocity \( v(t) \) of a dense body of mass \( m \) falling along a vertical line against a Newtonian damping force is the solution of the IVP
\[ \frac{dv}{dt} = -g - \frac{k}{m} v|v|, \quad v(0) = v_0 \]  
(13)
where \( g \) and \( k \) are the gravitational and damping constants. Let’s show that by scaling time and velocity we can change the number of parameters in IVP (13) from four (\( g, k, m, \) and \( v_0 \)) to just one.

Say that \( t = t_1 s \) and \( v = v_1 w \) (the positive constants \( t_1 \) and \( v_1 \) to be determined). Then the chain rule converts IVP (13) to
\[ \frac{dv}{dt} = \frac{dv}{dw} \frac{dw}{ds} \frac{ds}{dt} = \frac{v_1}{t_1} \frac{dw}{ds} = -g - \frac{k}{m} v_1 w|v_1 w|, \quad w(0) = \frac{v(0)}{v_1} \]
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\[ \frac{dw}{ds} = -1 - w|w| \]

\[ w(0) = -3, -2, \ldots, 3 \]

Multiply by \( t_1/v_1 \) and replace \( |v_1 w| \) by \( v_1|w| \) (valid since \( v_1 > 0 \)):

\[ \frac{dw}{ds} = -\frac{t_1 g}{v_1} - \frac{t_1 k v_1}{m}|w|, \quad w(0) = \frac{v(0)}{v_1} \]  

(14)

The IVP (14) looks more complicated than IVP (13), but we haven’t yet chosen values for \( t_1 \) and \( v_1 \). In particular, let’s choose the scaling constants \( t_1 \) and \( v_1 \) so that the coefficients in the ODE have value 1. To find the values of \( t_1 \) and \( v_1 \) that will do the job, we work backwards:

\[ t_1 g/v_1 = 1, \quad t_1 k v_1/m = 1, \quad \text{and so} \]

\[ t_1 = v_1/g, \quad k v_1^2/mg = 1, \quad \text{and so} \]

\[ v_1 = (mg/k)^{1/2}, \quad t_1 = (m/kg)^{1/2} \]

With these values of \( t_1 \) and \( v_1 \), the only parameter left in the scaled IVP

\[ \frac{dw}{ds} = -1 - w|w|, \quad w(0) = w_0 = (k/mg)^{1/2} v_0 \]  

(15)

is \( w_0 \). Scaling has reduced IVP (13) with four parameters to IVP (15) with only one!

Some solution curves for ODE (15) are plotted in Figure 1.9.4 for various values of \( w_0 \). All the solution curves tend to the equilibrium solution \( w = -1 \) as \( s \) increases. This means that for any values of \( v_0, g, k, \) and \( m \), the solution of IVP (13) tends toward the limiting value \( v = -(mg/k)^{1/2} \).
There are various reasons for scaling an ODE before solving. Sometimes only combinations of parameters can be measured, not the individual parameters. In Example 1.9.5, rescaling has reduced the set of four original parameters, \( g, k, m, \) and \( v_0 \) to the single parameter \((k/2mg)^{1/2}v_0\). There is another advantage of working with scaled variables: you don’t have to worry about units. For example, in Example 1.9.5 the scaled variable \( s \) has no units (i.e., it is dimensionless). This follows from the fact that \((mg)^{1/2}\) has units of time since \( k \) has units of mass/distance and \( g \) has units of distance/(time)². Similarly, \( w \) has no units, which means that whether the various quantities in IVP (13) are measured in feet or meters, hours or seconds, grams or slugs, it makes no difference at all if we use IVP (15) for our analysis and our computing.

Comments

Changing variables to simplify an ODE is an art form and has been around for a long time. To see several hundred examples, check out the handbook *Exact Solutions for Ordinary Differential Equations* by Andrei D. Polyanin and Valentin Zaitsev (Boca Raton, Fla.: CRC Press, 1995).

PROBLEMS

1. *(Homogeneous Rate Functions of Order Zero).* For each ODE find a formula that defines solution curves implicitly. [Hint: Change variables from \( y \) to \( z \) by \( y = xz \), and solve as in Example 1.9.2.]
   
   (a) \( y' = (y + x)/x \)  
   (b) \( (x - y)dx + (x - 4y)dy = 0 \)  
   (c) \( (x^2 - xy - y^2)dx - xydy = 0 \)  
   (d) \( (x^2 - 2y^2)dx + xydy = 0 \)  
   (e) \( x^2y' = 4x^2 + 7xy + 2y^2 \)

2. Use the approach in Example 1.7.2 to plot solution curves for parts (a)–(e) of Problem 1. Highlight arcs of orbits that are solution curves.

3. *(From Nonseparable to Separable).* Nonseparable ODEs may become separable by changing a variable. For each of the following cases, demonstrate this process and solve the new ODE. Then find the solution of the original ODE. [Hint: Let \( z = x + y \) in parts (a), (d); let \( z = 2x + y \) in part (b).]
   
   (a) \( dy/dx = \cos(x + y) \)  
   (b) \( (2x + y + 1)dx + (4x + 2y + 3)dy = 0 \)  
   (c) \( (x + 2y - 1)dx + (3x + 2y)dy = 0 \)  
   (d) \( e^{x^2}(y^2 + 1) = xe^x \)

4. Use the substitution \( y = z^{1/2} \) to solve the IVP \( yy'' + (y')^2 = 1, \ y(0) = 1, \ y'(0) = 0 \).

5. *(Alternative Derivation of Solution Formula (15) in Section 1.6).* The logistic ODE given by \( y' = r(1 - y/K)y \) can be solved by making the change of dependent variable \( z = 1/y \), which transforms the logistic ODE into the linear ODE \( z' = -rz + r/K \).
   
   (a) Show that the ODE for \( z \) is as claimed.
   
   (b) Solve the ODE for \( z \), and then show that \( y(t) \) becomes solution formula (15) in Section 1.6 if \( y(0) = y_0 \) and \( y = 1/z \).
6. (Reduction to Linear Form). Look at the ODE
\[ y'(t) = (a + by)(c(t) + d(t)y) \]
where \(a\) and \(b\) are constants, \(b \neq 0\), and \(c(t)\) and \(d(t)\) are continuous on some \(t\)-interval \(I\).

(a) Show that the variable change \(y = (1/z - a)/b\) converts the given ODE into the linear ODE
\[ dz/dt = [ad(t) - bc(t)]z - d(t) \]
(b) Find all solutions of the ODE
\[ y'(t) = (3 - y)(2t + ty) \]

7. (Rescaling the Logistic ODE). Show that the logistic ODE \(P'(t) = r(1 - P(t)/K)P(t)\) can be
rescaled to \(x'(s) = (1 - x(s))x(s)\) if we set \(x = P/K\) and \(t = s/r\). Why would you want to do
this rescaling before using a computer to examine the long term behavior of some logically
changing population?

8. (Scaling the Whiffle Ball IVP). The velocity of a vertically moving whiffle ball subject to viscous
damping is given by the ODE \(v' = -g - (k/m)v\), where \(g\), \(k\), and \(m\) are positive constants.
Rescale the state and time variables so that the scaled ODE is free of these constants. Solve the
scaled ODE and draw a conclusion about the limiting velocity for the original ODE. [Hint: Let
\(t = aT\), \(v = bV\); see Example 1.9.5.]

9. (Bernoulli’s ODE). The ODE \(dy/dt + p(t)y = q(t)y^b\) is Beroulli’s ODE.\(^{10}\)

(a) Show that the change of variable \(z = y^{1-b}\) changes the Bernoulli ODE, where \(b\) is a constant,
\(b \neq 0, 1\), to the linear ODE \(dz/dt + (1 - b)p(t)z = (1 - b)q(t)\).
(b) Show that the logistic ODE \(y' = r(1 - y/K)y\) is a Bernoulli ODE with \(b = 2\).
(c) Find all solutions of \(dy/dt + r^{-1}y = y^{-d}, t > 0\). Plot solutions for \(t > 0, |y| \leq 5\).
(d) Find all solutions of \(dy/dt - r^{-1}y = -y^{-d}/2, t > 0\). Plot solutions for \(t > 0, 1 \leq y \leq 2\).

10. (Riccati’s ODE). The Riccati \(^{11}\) ODE is \(dy/dt = a(t)y + b(t)y^2 + F(t)\). If \(F(t) = 0\), the ODE
is a special case of Bernoulli’s ODE (Problem 9). Riccati’s ODE may be reduced to a first-order
linear ODE if one solution is known [see part (a) below]. Parts (b)–(e) contain examples.
(a) Let \(g(t)\) be one solution of Riccati’s ODE. Let \(z = [y - g]^{-1}\). Show that \(dz/dt + (a +
2bg)z = -b\), which is a first-order linear ODE in \(z\). If \(z(t)\) is the general solution of the linear
ODE, show that the general solution \(y(t)\) of the Riccati ODE is \(y = g + 1/z\).
(b) Show that the ODE \(dy/dt = (1 - 2t)y + ty^2 + t - 1\) has a solution \(y = 1\). Let \(z = (y - 1)^{-1}\)
and show that \(dz/dt = -z - t\). Find the general solution \(y(t)\) of the original ODE.
(c) Find all solutions of \(dy/dt = e^{-t}y^2 + y - e\). [Hint: First show that \(y = e^t\) is a solution.]
(d) Show that \(y = t\) is a solution of \(dy/dt = t(y - t)^2 + yt^{-1}, t > 0\), and then find all solutions.
(e) Show that the harvesting model of Problem 9, Section 1.6, is a Riccati ODE and solve it.

\(^{10}\)Jacques Bernoulli (1654–1705) introduced the ODE and his brother Jean (1667–1748) solved it, but Gottfried
Leibniz (1646–1717) solved it the same way we do. The two Bernoullis were members of a remarkable Swiss
family which produced eight famous mathematicians over a span of four generations. Leibniz was a German
mathematician and philosopher who discovered the calculus independently of Newton, but at roughly the same
time. The two later quarreled over who should get the credit for the discovery.

\(^{11}\)The Italian mathematician Jacopo Riccati (1676–1754) discussed particular cases of the ODE now named in
his honor, but it was the Bernoulli brothers who actually worked out the solutions.
11. (Flight Path of a Goose, Wind from the Southeast).
   (a) Set up the flight path problem for the goose, flying at speed \( b \), if the wind is blowing from the southeast at a speed of \( w = b/\sqrt{2} \) and the goose starts at \( x = a > 0, \ y = 0 \). Solve the IVP in implicit form (don’t attempt to solve for \( y \) in terms of \( x \)). [Hint: See Example 1.9.3. Note that \( x' = -b \cos \theta - b/2, \ y' = -b \sin \theta + b/2 \). Use a table of integrals.]

   (b) Set \( b = 1, \ a = 1, 2, \ldots, 9 \) and plot the paths. [Hint: Use a numerical solver on the system \( x' = \ldots, \ y' = \ldots \) that models the flight path.] Does the goose reach the nest? Does it overshoot?

12. (The Goose and a Moving Nest). On a windless day a goose sees its gosling aboard a raft in the middle of a river that is moving at 8 yd/sec. When the raft is directly opposite the goose, the raft is 30 yd distant, and the goose instantly takes flight to save her gosling from going over a waterfall 60 yd downstream. If the goose flies directly toward the raft at the constant speed of 10 yd/sec, does she rescue her gosling before it tumbles over the falls? Follow the outline below for this problem.

   At \( t = 0 \) place the raft and the goose in the \( xy \)-plane at the origin and at \((30, 0)\), respectively. Let the river flow in the positive \( y \)-direction. The parametric path \((x(t), y(t))\) followed by the goose in the \( xy \)-plane has the following properties: the goose’s velocity vector at time \( t \), \((x'(t), y'(t))\), always points toward the raft, and \((x')^2 + (y')^2 = 10 \) yd/sec at all times. So, if the goose is at \((x, y)\) at time \( t \), then the raft is at \((0, 8t)\), and there is a factor \( k > 0 \) (which may depend on \( x \), \( y \), and \( t \)) such that \( x' = k(-x), \ y' = k(8t - y) \). Since \((x')^2 + (y')^2 = 100\), we find that \( k = 10/((x^2 + (8t - y)^2)^{1/2}) \), and we have the IVP

   \[
   \begin{align*}
   x' &= -\frac{10x}{(x^2 + (8t - y)^2)^{1/2}}, \quad x(0) = 30 \\
   y' &= \frac{10(8t - y)}{(x^2 + (8t - y)^2)^{1/2}}, \quad y(0) = 0
   \end{align*}
   \]

   Since the rate functions in this system depend on \( t \), we can’t directly apply the technique used in Example 1.9.3. But the system is written in normal form, so a numerical solver can be used to solve and plot an orbit of this system to determine if the goose rescues the gosling. If the goose does reach the gosling in time, how long does it take?
Solution Formula Techniques Involving First-Order ODEs

Explicit Techniques

1. **Linear ODE**
   \[ y' + p(t)y = q(t) \]
   Multiply the ODE through by an integrating factor \( e^{\int p(t) \, dt} \), where \( P(t) = \int p(t) \, dt \), and then use the Antiderivative Theorem. (See Example 1.3.3 and the procedure on page 19.)

2. **Linear Cascade**
   \[
   \begin{align*}
   x' &= k_1 x + f(t) \\
   y' &= k_2 x + k_3 y + g(t)
   \end{align*}
   \]
   The coefficients \( k_1, k_2, \) and \( k_3 \) may depend on \( t \). Solve the first linear ODE for \( x(t) \), insert \( x(t) \) into second linear ODE, which then can be solved as a linear ODE for \( y(t) \). (See Examples 1.8.1, 1.8.2.)

Implicit Techniques

1. **Variables Separate**
   \[ N(y)y' + M(x) = 0 \]
   Find antiderivatives \( F(x) = \int M(x) \, dx \), \( G(y) = \int N(y) \, dy \). The level curves defined by \( F(x) + G(y) = C \), for a constant \( C \), are integral curves of the ODE in the \( xy \)-plane. An arc of an integral curve with no vertical tangents is a solution curve of the ODE. (See Examples 1.6.1 and 1.6.3. Also see the procedure on page 47.) For another way to plot integral curves see Example 1.7.1.

2. **Exact ODE**
   \[ N(x, y)y' + M(x, y) = 0 \]
   Find a function \( F(x, y) \) with \( \frac{\partial F}{\partial x} = M \), \( \frac{\partial F}{\partial y} = N \) in a rectangle \( R \); then put \( F(x, y) = C \), a constant, and solve for \( y \). Such a function \( F(x, y) \) is an integral of the ODE and solution curves can be visualized using integral curves. (See Problem 10, Section 1.6.)

3. **Differential Form of ODE**
   \[ M(x, y) \, dx + N(x, y) \, dy = 0 \]
   Convenient way to write as an ODE for \( y(x) \), or as an ODE for \( x(y) \):
   \[
   N \frac{dy}{dx} + M = 0, \quad \text{or} \quad M \frac{dx}{dy} + N = 0
   \]

System Techniques

1. **Planar Autonomous System**
   \[
   \begin{align*}
   \frac{dx}{dt} &= N(x, y) \\
   \frac{dy}{dt} &= -M(x, y)
   \end{align*}
   \]
   Arcs of orbits of the system in the \( xy \)-plane with no vertical tangent lines are solution curves of the first-order ODE \( dy/dx = -M/N \). (See Example 1.7.5.)

2. **General ODE**
   \[ N(x, y)y' + M(x, y) = 0 \]
   Convert the first-order ODE to a planar autonomous system
   \[
   \begin{align*}
   \frac{dx}{dt} &= N(x, y) \\
   \frac{dy}{dt} &= -M(x, y)
   \end{align*}
   \]
   whose orbits are composed of solution curves of the ODE. (See Example 1.7.2.)
### Change of Variables

1. **Bernoulli Equation**
   \[ y' + p(t)y = q(t)y^b \]
   \( b \neq 0, 1 \)
   Change the state variable to \( z = y^{1-b} \) to obtain the new ODE \( z' + (1-b)p z = (1-b)q \), which is linear in \( z \). (See Problem 9, Section 1.9.)

2. **Riccati Equation**
   \[ y' = a(t)y + b(t)y^2 + F(t) \]
   If one solution \( g(t) \) of the Riccati equation is known, then every solution \( y(t) \) has the form \( y(t) = g(t) + 1/z(t) \), where \( z(t) \) solves the linear ODE \( z' + (a + 2b)z = -b \). (See Problem 10, Section 1.9.)

3. The ODE
   \[ y' = (a + by)(\alpha(t) + \beta(t)y) \]
   where \( a, b \) are constants, \( b \neq 0 \)
   Change the state variable to \( z = (a + by)^{-1} \) to obtain the new ODE \( z' = [a\beta(t) - b\alpha(t)]z - \beta(t) \), which is linear in \( z \). (See Problem 6, Section 1.9.)

4. **Homogeneous Rate Function of Order Zero**
   \[ y' = f(x, y) \]
   \( f(kx, ky) = f(x, y) \)
   Change the variables \( x \) and \( y \) to polar form using \( x = r \cos \theta \), \( y = r \sin \theta \). If \( r = r(\theta) \) is a solution curve of the ODE, then \( r(\theta) \) satisfies the ODE (see The Student Resource Manual, Section 1.9):
   \[
   \frac{dr}{d\theta} \sin \theta + r \cos \theta = f(r \cos \theta, r \sin \theta) \left( \frac{dr}{d\theta} \cos \theta - r \sin \theta \right)
   \]

### Reduction to First-Order ODEs

1. **Method of Varied Parameters**
   \[ y'' + a(t)y' + b(t)y = f(t) \]
   Let \( z(t) \) be a known solution of \( z'' + a(t)z' + b(t)z = 0 \). Then every solution \( y(t) \) of \( y'' + a(t)y' + b(t)y = f(t) \) is given by \( y = uz \), where \( u(t) \) solves the ODE \( zu'' + (2z' + az)u' = f \), which is a linear first-order ODE for \( u = u' \). (See Problem 6, Section 1.7.)

2. \( y'' = F(t, y') \)
   \( F \) is independent of \( y \)
   The state variable \( v = y' \) solves the first-order ODE \( v' = F(t, v) \). The state variable \( y = \int v(t) \, dt \). (See Example 1.7.3.)

3. \( y'' = F(y, y') \)
   \( F \) is independent of \( t \)
   Introduce \( y \) as a new independent variable and consider the state variable \( v = y' \) as a function of \( y \): \( v = v(y) \). Since \( y'' = (dv/dy)v \), it follows that \( v \) solves the first-order ODE \( v(dv/dy) = F(y, v) \). Note that \( y(t) \) solves the ODE \( dy/dt = v(y) \). (See Example 1.7.4.)

### Forced Oscillation

\[ y' + p_0 y = q(t) \]
\( 0 \neq p_0 = \text{constant}, q(t) \) is piecewise continuous, periodic with period \( T \)
Has a unique periodic solution with period \( T \) generated by the initial condition \( y(0) = y_0 \), given by (see Theorem 2.2.4.)
\[
y_0 = [e^{p_0 T} - 1]^{-1} \int_0^T e^{p_0 s} q(s) \, ds
\]