Fractal Dimension of Paper Ball

An applied problem involving linear regression
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the Web site for *Fundamentals of Physics*
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Based on "Fractal Geometry in Crumpled Paper Balls," by M. A. F. Gomes, *American Journal of Physics*, **55**, 649-650 (1987), and "A Simple Experiment that Demonstrates Fractal Behavior," by R. H. Ko and C. P. Bean, *The Physics Teacher*, **29**, 78-79 (Feb. 1991).

A flat sheet of paper is two-dimensional (that is, its dimension is d = 2.0), and a solid cube of paper is three-dimensional (d = 3.0). According to fractal geometry, if we wad up the sheet into a ball, its two-dimensional surface is then embedded (contained within) three dimensions, and the sheet has a *fractal* dimension d that can lie between 2.0 and 3.0. A value that is close to 2.0 implies that the sheet tends to avoid itself when wadded up; a value that is close to 3.0 implies the opposite.

The mass m of the paper and the diameter D of the wadded-up ball are related by dimension d according to

$$m = kD^d, \qquad (1)$$

where k is an unknown constant. If we measure m for a range of D, using the same type of paper each time but wadding up different-sized balls, we can determine d from a plot of the data. However, instead of plotting the data according to Eq. 1, we want to transform that equation to a linear equation. Then we can do a linear regression on the data, that is, find the straight line that best fits a plot of the data.

Because dimension d is expressed as a power in Eq. 1, we transform Eq. 1 to a linear equation by taking the natural logarithm of both sides:

$$\begin{aligned} \ln m &= \ln kD^d \\ &= \ln k + \ln D^d \\ &= \ln k + d \ln D. \qquad (2) \end{aligned}$$

The result is in the form of the generic linear equation $y = a + bx$, where a is the y intercept and b is the slope. In Eq. 2, the variable y is ln m, the y intercept is ln k, the slope (which we want) is d, and the variable x is ln D.

We can now get the fractal dimension d if we do a linear regression of the ln m values versus the ln D values, to get the slope. We can do all this on a scientific calculator or with a math package on a computer.

To collect the data, we begin with a large sheet of paper, wad the sheet into a ball, measure its mass m on a balance, and then determine an average diameter D of the ball by averaging any two widths. After smoothing out the sheet, we tear it in half and repeat the process with each half. Then we tear one of those halves in half and repeat the process with each smaller half. We continue this process until we reach the limit of our ability to measure the mass or the diameter.

When I did such an experiment, using relatively stiff paper (with an initial area of about 0.80 m^2), the masses m were 112, 56.6, 55.5, 25.9, 30.0, 15.2, 14.8, 7.57, 7.71, 3.85, 3.89, 2.05, and 1.85 grams. The corresponding diameters D were 27.5, 20.0, 19.0, 14.5, 15.5, 10.0, 9.0, 7.8, 6.5, 6.0, 4.8, 4.9, and 4.8 cm.

What is the fractal dimension d of my wadded-up paper?

Hint: On a scientific calculator, first produce a list of the mass data and a list of the diameter data. Then take the natural logarithm of the two lists, to produce two new lists. Then use the two new lists with the linear regression capability of the calculator. The steps for producing lists and taking linear regression are explained for several calculators elsewhere on this Web site. Caution: some calculators use $y = a + bx$ as the generic linear equation and others use $y = ax + b$ as the generic equation, which makes a difference when you have a calculator make a linear regression.

The answer turns out to be closer to 2.0 than to 3.0. Interpret the result in terms of my paper’s tendency to avoid itself or not avoid itself when wadded up.

Find the fractal dimension for your own choice of paper or some other flat material, such as plastic food wrap, cheese slices, pita bread, or wet tortillas. (Hey, finding the fractal dimension of a wadded-up tortilla in a Mexican restaurant might be a real attention-grabber. Well, maybe not.)