On a sales rack of clothes at a department store, you see a shirt you like. The original price of the shirt was $100, but it has been discounted 30%. As a preferred shopper, you get an automatic additional 20% off the sale price at the register. How much will you pay for the shirt?

Naïve shoppers might be lured into thinking this shirt will cost $50 because they add the 20% and 30% to get 50% off, but they will end up paying more than that. Experienced shoppers know that they first take 30% off of $100, which results in a price of $70, and then they take an additional 20% off of the sale price, $70, which results in a final discounted price of $56. Experienced shoppers have already learned composition of functions.

A composition of functions can be thought of as a function of a function. One function takes an input (original price, $100) and maps it to an output (sale price, $70), and then another function takes that output as its input (sale price, $70) and maps that to an output (checkout price, $56).
In this chapter, you will find that functions are part of our everyday thinking: converting from degrees Celsius to degrees Fahrenheit, DNA testing in forensic science, determining stock values, and the sale price of a shirt. We will develop a more complete, thorough understanding of functions. First, we will establish what a relation is, and then we will determine whether a relation is a function. We will discuss common functions, domain and range of functions, and graphs of functions. We will determine whether a function is increasing or decreasing on an interval and calculate the average rate of change of a function. We will perform operations on functions and composition of functions. We will discuss one-to-one functions and inverse functions. Finally, we will model applications with functions using variation.

**Learning Objectives**

- Find the domain and range of a function.
- Sketch the graphs of common functions.
- Sketch graphs of general functions employing translations of common functions.
- Perform composition of functions.
- Find the inverse of a function.
- Model applications with functions using variation.
Relations and Functions

What do the following pairs have in common?

- Every person has a blood type.
- Temperature is some specific value at a particular time of day.
- Every working household phone in the United States has a 10-digit phone number.
- First-class postage rates correspond to the weight of a letter.
- Certain times of the day are start times of sporting events at a university.

They all describe a particular correspondence between two groups. A **relation** is a correspondence between two sets. The first set is called the **domain**, and the corresponding second set is called the **range**. Members of these sets are called **elements**.

**Definition**

A **relation** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to at least one element in the second set, called the **range**.

A relation is a set of ordered pairs. The domain is the set of all the first components of the ordered pairs, and the range is the set of all the second components of the ordered pairs.

<table>
<thead>
<tr>
<th>PERSON</th>
<th>BLOOD TYPE</th>
<th>ORDERED PAIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Michael</td>
<td>A</td>
<td>(Michael, A)</td>
</tr>
<tr>
<td>Tania</td>
<td>A</td>
<td>(Tania, A)</td>
</tr>
<tr>
<td>Dylan</td>
<td>AB</td>
<td>(Dylan, AB)</td>
</tr>
<tr>
<td>Trevor</td>
<td>O</td>
<td>(Trevor, O)</td>
</tr>
<tr>
<td>Megan</td>
<td>O</td>
<td>(Megan, O)</td>
</tr>
</tbody>
</table>

**Words**

- The domain is the set of all the first components.
- The range is the set of all the second components.

**Math**

- \{Michael, Tania, Dylan, Trevor, Megan\}
- \{A, AB, O\}

A relation in which each element in the domain corresponds to exactly one element in the range is a **function**.
Note that the definition of a function is more restrictive than the definition of a relation. For a relation, each input corresponds to at least one output, whereas, for a function, each input corresponds to exactly one output. The blood-type example given is both a relation and a function.

Also note that the range (set of values to which the elements of the domain correspond) is a subset of the set of all blood types. However, although all functions are relations, not all relations are functions.

For example, at a university, four primary sports typically overlap in the late fall: football, volleyball, soccer, and basketball. On a given Saturday, the following table indicates the start times for the competitions.

<table>
<thead>
<tr>
<th>Time of Day</th>
<th>Competition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:00 P.M.</td>
<td>Football</td>
</tr>
<tr>
<td>2:00 P.M.</td>
<td>Volleyball</td>
</tr>
<tr>
<td>7:00 P.M.</td>
<td>Soccer</td>
</tr>
<tr>
<td>7:00 P.M.</td>
<td>Basketball</td>
</tr>
</tbody>
</table>

Because an element in the domain, 7:00 P.M., corresponds to more than one element in the range, Soccer and Basketball, this is not a function. It is, however, a relation.

**Example 1: Determining Whether a Relation Is a Function**

Determine whether the following relations are functions.

a. \( \{(−3, 4), (2, 4), (3, 5), (6, 4)\} \)

b. \( \{(−3, 4), (2, 4), (3, 5), (2, 2)\} \)

c. Domain = Set of all items for sale in a grocery store; Range = Price

**Solution:**

a. No \( x \)-value is repeated. Therefore, each \( x \)-value corresponds to exactly one \( y \)-value. **This relation is a function.**

b. The value \( x = 2 \) corresponds to both \( y = 2 \) and \( y = 4 \). **This relation is not a function.**

c. Each item in the grocery store corresponds to exactly one price. **This relation is a function.**

**Your Turn** Determine whether the following relations are functions.

a. \( \{(1, 2), (3, 2), (5, 6), (7, 6)\} \)

b. \( \{(1, 2), (1, 3), (5, 6), (7, 8)\} \)

c. \( \{[11:00 \text{ A.M., } 83^\circ \text{F}], (2:00 \text{ P.M., } 89^\circ \text{F}), (6:00 \text{ P.M., } 85^\circ \text{F})\} \)

**Study Tip**

All functions are relations but not all relations are functions.
Functions Defined by Equations

Let’s start with the equation \( y = x^2 - 3x \), where \( x \) can be any real number. This equation assigns to each \( x \)-value exactly one corresponding \( y \)-value.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 - 3x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y = (1)^2 - 3(1) )</td>
<td>-2</td>
</tr>
<tr>
<td>5</td>
<td>( y = (5)^2 - 3(5) )</td>
<td>10</td>
</tr>
<tr>
<td>(-\frac{1}{3})</td>
<td>( y = \left(-\frac{1}{3}\right)^2 - 3\left(-\frac{1}{3}\right) )</td>
<td>( \frac{22}{9} )</td>
</tr>
<tr>
<td>1.2</td>
<td>( y = (1.2)^2 - 3(1.2) )</td>
<td>-2.16</td>
</tr>
</tbody>
</table>

Since the variable \( y \) depends on what value of \( x \) is selected, we denote \( y \) as the dependent variable. The variable \( x \) can be any number in the domain; therefore, we denote \( x \) as the independent variable.

Although functions are defined by equations, it is important to recognize that not all equations are functions. The requirement for an equation to define a function is that each element in the domain corresponds to exactly one element in the range. Throughout the ensuing discussion, we assume \( x \) to be the independent variable and \( y \) to be the dependent variable.

Equations that represent functions of \( x \):

- \( y = x^2 \)
- \( y = |x| \)
- \( y = x^3 \)

Equations that do not represent functions of \( x \):

- \( x = y^2 \)
- \( x = y \) \( x^2 + y^2 = 1 \) \( x = |y| \)

In the “equations that represent functions of \( x \),” every \( x \)-value corresponds to exactly one \( y \)-value. Some ordered pairs that correspond to these functions are

- \( y = x^2 \)\( : (-1, 1) \) \( (0, 0) \) \( (1, 1) \)
- \( y = |x| \)\( : (-1, 1) \) \( (0, 0) \) \( (1, 1) \)
- \( y = x^3 \)\( : (-1, -1) \) \( (0, 0) \) \( (1, 1) \)

The fact that \( x = -1 \) and \( x = 1 \) both correspond to \( y = 1 \) in the first two examples does not violate the definition of a function.

In the “equations that do not represent functions of \( x \),” some \( x \)-values correspond to more than one \( y \)-value. Some ordered pairs that correspond to these equations are

<table>
<thead>
<tr>
<th>Relation</th>
<th>Solve Relation for ( y )</th>
<th>Points That Lie on the Graph</th>
<th>( x = 1 ) maps to both ( y = -1 ) and ( y = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = y^2 )</td>
<td>( y = \pm x )</td>
<td>( (1, -1) ) ( (0, 0) ) ( (1, 1) )</td>
<td>( x = 1 ) maps to both ( y = -1 ) and ( y = 1 )</td>
</tr>
<tr>
<td>( x^2 + y^2 = 1 )</td>
<td>( y = \pm \sqrt{1 - x^2} )</td>
<td>( (0, -1) ) ( (0, 1) ) ( (-1, 0) ) ( (1, 0) )</td>
<td>( x = 0 ) maps to both ( y = -1 ) and ( y = 1 )</td>
</tr>
<tr>
<td>( x =</td>
<td>y</td>
<td>)</td>
<td>( y = \pm x )</td>
</tr>
</tbody>
</table>
Let’s take any value for \( x \), say \( x = a \). The graph of \( x = a \) corresponds to a vertical line. A function of \( x \) maps each \( x \)-value to exactly one \( y \)-value; therefore, there should be at most one point of intersection with any vertical line. We see in the three graphs of the functions above that if a vertical line is drawn at any value of \( x \) on any of the three graphs, the vertical line only intersects the graph in one place. Look at the graphs of the three equations that do not represent functions of \( x \).

Let’s look at the graphs of the three functions of \( x \):

\[
\begin{align*}
&y = x^2 \\
&y = |x| \\
&y = x^3
\end{align*}
\]

A vertical line can be drawn on any of the three graphs such that the vertical line will intersect each of these graphs at two points. Thus, there are two \( y \)-values that correspond to some \( x \)-value in the domain, which is why these equations do not define \( y \) as a function of \( x \).

**Definition**

**Vertical Line Test**

Given the graph of an equation, if any vertical line that can be drawn intersects the graph at no more than one point, the equation defines \( y \) as a function of \( x \). This test is called the **vertical line test**.

**Study Tip**

If any \( x \)-value corresponds to more than one \( y \)-value, then \( y \) is **not** a function of \( x \).
To recap, a function can be expressed one of four ways: verbally, numerically, algebraically, and graphically. This is sometimes called the Rule of 4.

### Expressing a Function

<table>
<thead>
<tr>
<th><strong>Verbally</strong></th>
<th><strong>Numerically</strong></th>
<th><strong>Algebraically</strong></th>
<th><strong>Graphically</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Every real number has a corresponding absolute value.</td>
<td>((-3, 3), (-1, 1), (0, 0), (1, 1), (5, 5))</td>
<td>(y =</td>
<td>x</td>
</tr>
</tbody>
</table>

**Classroom Example 3.1.2**

Use the vertical line test to determine whether these graphs of equations determine functions.

**a.**

**Answer:** yes

**b.**

**Answer:** no

**Classroom Example 3.1.2**

Let \(a\) be a positive real number.

Does the graph of \((x - a)^2 + (y + a)^2 = 4\) determine a function?

**Answer:** No, it's a circle.

**YOUR TURN** Determine whether the equation \((x - 3)^2 + (y + 2)^2 = 16\) is a function of \(x\).
Function Notation

We know that the equation \( y = 2x + 5 \) defines \( y \) as a function of \( x \) because its graph is a nonvertical line and thus passes the vertical line test. We can select \( x \)-values (input) and determine unique corresponding \( y \)-values (output). The output is found by taking 2 times the input and then adding 5. If we give the function a name, say, “\( f \)”, then we can use function notation:

\[
f(x) = 2x + 5
\]

The symbol \( f(x) \) is read “\( f \) evaluated at \( x \)” or “\( f \) of \( x \)” and represents the \( y \)-value that corresponds to a particular \( x \)-value. In other words, \( y = f(x) \).

<table>
<thead>
<tr>
<th>Input</th>
<th>Function</th>
<th>Output</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
<td>( f(x) = 2x + 5 )</td>
<td></td>
</tr>
</tbody>
</table>

The independent variable is also referred to as the argument of a function. To evaluate functions, it is often useful to think of the independent variable or argument as a placeholder. For example, \( f(x) = x^2 - 3x \) can be thought of as

\[
f(\square) = (\square)^2 - 3(\square)
\]

In other words, “\( f \) of the argument” is equal to the argument squared minus 3 times the argument.” Any expression can be substituted for the argument:

\[
\begin{align*}
f(1) &= (1)^2 - 3(1) \\
f(x + 1) &= (x + 1)^2 - 3(x + 1) \\
f(-x) &= (-x)^2 - 3(-x)
\end{align*}
\]

It is important to note:

- \( f(x) \) does not mean \( f \) times \( x \).
- The most common function names are \( f \) and \( F \) since the word function begins with an “\( f \)”. Other common function names are \( g \) and \( G \), but any letter can be used.
- The letter most commonly used for the independent variable is \( x \). The letter \( t \) is also common because in real-world applications it represents time, but any letter can be used.
- Although we can think of \( y \) and \( f(x) \) as interchangeable, the function notation is useful when we want to consider two or more functions of the same independent variable.
YOUR TURN
For the following graph of a function, find:

a. \( f(-1) \)  

b. \( f(0) \)  

c. \( 3f(2) \)  

d. the value of \( x \) that corresponds to \( f(x) = 0 \)

EXAMPLE 3 Evaluating Functions by Substitution
Given the function \( f(x) = 2x^3 - 3x^2 + 6 \), find \( f(-1) \).

Solution:
Consider the independent variable \( x \) to be a placeholder.

\[
f(\Box) = 2(\Box)^3 - 3(\Box)^2 + 6
\]

To find \( f(-1) \), substitute \( x = -1 \) into the function.

\[
f(-1) = 2(-1)^3 - 3(-1)^2 + 6
\]

Evaluate the right side.

\[
f(-1) = -2 - 3 + 6
\]

Simplify.

\[
f(-1) = 1
\]

EXAMPLE 4 Finding Function Values from the Graph of a Function
The graph of \( f \) is given on the right.

a. Find \( f(0) \).

b. Find \( f(1) \).

c. Find \( f(2) \).

d. Find \( 4f(3) \).

e. Find \( x \) such that \( f(x) = 10 \).

f. Find \( x \) such that \( f(x) = 2 \).

Solution (a): The value \( x = 0 \) corresponds to the value \( y = 5 \).

\[ f(0) = 5 \]

Solution (b): The value \( x = 1 \) corresponds to the value \( y = 2 \).

\[ f(1) = 2 \]

Solution (c): The value \( x = 2 \) corresponds to the value \( y = 1 \).

\[ f(2) = 1 \]

Solution (d): The value \( x = 3 \) corresponds to the value \( y = 2 \).

\[ 4f(3) = 4 \cdot 2 = 8 \]

Solution (e): The value \( y = 10 \) corresponds to the value \( x = 5 \).

\[ x = 5 \]

Solution (f): The value \( y = 2 \) corresponds to the values \( x = 1 \) and \( x = 3 \).

\[ x = 1 \text{ and } x = 3 \]
EXAMPLE 5 Evaluating Functions with Variable Arguments (Inputs)

For the given function \( f(x) = x^2 - 3x \), evaluate \( f(x + 1) \) and simplify if possible.

**Common Mistake**

A common misunderstanding is to interpret the notation \( f(x + 1) \) as a sum:

\[ f(x + 1) \neq f(x) + f(1) \]

**Correct**

Write the original function.

\[ f(x) = x^2 - 3x \]

Replace the argument \( x \) with a placeholder.

\[ f(\Box) = (\Box)^2 - 3(\Box) \]

Substitute \( x + 1 \) for the argument.

\[ f(x + 1) = (x + 1)^2 - 3(x + 1) \]

Eliminate the parentheses.

\[ f(x + 1) = x^2 + 2x + 1 - 3x - 3 \]

Combine like terms.

\[ f(x + 1) = x^2 - x - 2 \]

**YOUR TURN** For the given function \( g(x) = x^2 - 2x + 3 \), evaluate \( g(x - 1) \).

**Example 6 Evaluating Functions: Sums**

For the given function \( H(x) = x^2 + 2x \), evaluate:

a. \( H(x + 1) \)

b. \( H(x) + H(1) \)

**Solution (a):**

Write the function \( H \) in placeholder notation.

\[ H(\Box) = (\Box)^2 + 2(\Box) \]

Substitute \( x + 1 \) for the argument of \( H \).

\[ H(x + 1) = (x + 1)^2 + 2(x + 1) \]

Eliminate the parentheses on the right side.

\[ H(x + 1) = x^2 + 2x + 1 + 2x + 2 \]

Combine like terms on the right side.

\[ H(x + 1) = x^2 + 4x + 3 \]

**Solution (b):**

Write \( H(x) \).

Evaluate \( H \) at \( x = 1 \).

\[ H(1) = (1)^2 + 2(1) = 3 \]

Evaluate the sum \( H(x) + H(1) \).

\[ H(x) + H(1) = x^2 + 2x + 3 \]

\[ H(x) + H(1) = x^2 + 2x + 3 \]

**Note:** Comparing the results of part (a) and part (b), we see that

\[ H(x + 1) \neq H(x) + H(1) \]
CHAPTER 3 Functions and Their Graphs

Technology Tip

Use a graphing utility to display graphs of \( y_1 = G(x) = (−x)^3 − (−x) \) and \( y_2 = −G(x) = −(x^3 − x) \).

The graphs are not the same.

\[ \text{Plot1 Plot2 Plot3} \]
\[ V_1: (−x)^3 − (−x) \]
\[ V_2: −(x^3 − x) \]
\[ V_3 = \]

CAUTION

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EXAMPLE 7 Evaluating Functions: Negatives

For the given function \( G(t) = t^2 - t \), evaluate:

a. \( G(t) \)  

Solution (a):

Write the function \( G \) in placeholder notation.

\( G(\square) = (\square)^2 - (\square) \)

Substitute \(-t\) for the argument of \( G \).

\( G(-t) = (-t)^2 - (-t) \)

Eliminate the parentheses on the right side.

\[ G(-t) = t^2 + t \]

Solution (b):

Write \( G(t) \).

\( G(t) = t^2 - t \)

Multiply by \(-1\).

\( -G(t) = -(t^2 - t) \)

Eliminate the parentheses on the right side.

\[ -G(t) = -t^2 + t \]

Note: Comparing the results of part (a) and part (b), we see that \( G(-t) \neq -G(t) \).

EXAMPLE 8 Evaluating Functions: Quotients

For the given function \( F(x) = 3x + 5 \), evaluate:

a. \( F\left(\frac{1}{2}\right) \)  
b. \( \frac{F(1)}{F(2)} \)

Solution (a):

Write \( F \) in placeholder notation.

\( F(\square) = 3(\square) + 5 \)

Replace the argument with \( \frac{1}{2} \).

\[ F\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) + 5 \]

Simplify the right side.

\[ F\left(\frac{1}{2}\right) = \frac{13}{2} \]

Solution (b):

Evaluate \( F(1) \).

\( F(1) = 3(1) + 5 = 8 \)

Evaluate \( F(2) \).

\( F(2) = 3(2) + 5 = 11 \)

Divide \( F(1) \) by \( F(2) \).

\[ \frac{F(1)}{F(2)} = \frac{8}{11} \]

Note: Comparing the results of part (a) and part (b), we see that \( F\left(\frac{1}{2}\right) \neq \frac{F(1)}{F(2)} \).

YOUR TURN

Given the function \( G(t) = 3t - 4 \), evaluate:

a. \( G(t - 2) \)  
b. \( G(t) - G(2) \)  
c. \( \frac{G(1)}{G(3)} \)  
d. \( G\left(\frac{1}{3}\right) \)

Examples 6, 7, and 8 illustrate the following:

\[ f(a + b) \neq f(a) + f(b) \quad f(-t) \neq -f(t) \quad f\left(\frac{a}{b}\right) \neq \frac{f(a)}{f(b)} \]
Now that we have shown that \( f(x + h) \neq f(x) + f(h) \), we turn our attention to one of the fundamental expressions in calculus: the **difference quotient**.

\[
\frac{f(x + h) - f(x)}{h} \quad h \neq 0
\]

Example 9 illustrates the difference quotient, which will be discussed in detail in Section 3.2. For now, we will concentrate on the algebra involved when finding the difference quotient. In Section 3.2, the application of the difference quotient will be the emphasis.

**Example 9**  
**Evaluating the Difference Quotient**

For the function \( f(x) = x^2 - x \), find \( \frac{f(x + h) - f(x)}{h} \), \( h \neq 0 \).

**Solution:**

Use placeholder notation for the function \( f(x) = x^2 - x \).

\( f(\square) = (\square)^2 - (\square) \)

Calculate \( f(x + h) \).

\( f(x + h) = (x + h)^2 - (x + h) \)

Write the difference quotient.

\[
\frac{f(x + h) - f(x)}{h}
\]

Let \( f(x + h) = (x + h)^2 - (x + h) \) and \( f(x) = x^2 - x \).

\[
\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^2 - (x + h)] - [x^2 - x]}{h} \quad h \neq 0
\]

Eliminate the parentheses inside the first set of brackets.

\[
= \frac{[x^2 + 2hx + h^2 - x - h] - [x^2 - x]}{h}
\]

Eliminate the brackets in the numerator.

\[
= \frac{x^2 + 2hx + h^2 - x - h - x^2 + x}{h}
\]

Combine like terms.

\[
= \frac{2hx + h^2 - x}{h}
\]

Factor the numerator.

\[
= \frac{h(2x + h - 1)}{h}
\]

Divide out the common factor, \( h \).

\[
= 2x + h - 1 \quad h \neq 0
\]

**Your Turn**  
Evaluate the difference quotient for \( f(x) = x^2 - 1 \).

**Answer:** \( 2x + h \)

**Domain of a Function**

Sometimes the domain of a function is stated *explicitly*. For example,

\[
f(x) = |x| \quad \text{for } x < 0
\]

Here, the **explicit domain** is the set of all negative real numbers, \((-\infty, 0)\). Every negative real number in the domain is mapped to a positive real number in the range through the absolute value function.
If the expression that defines the function is given but the domain is not stated explicitly, then the domain is implied. The **implicit domain** is the largest set of real numbers for which the function is defined and the output value \( f(x) \) is a real number. For example, 
\[
 f(x) = \sqrt{x}
\]
does not have the domain explicitly stated. There is, however, an implicit domain. Note that if the argument is negative, that is, if \( x < 0 \), then the result is an imaginary number. In order for the output of the function, \( f(x) \), to be a real number, we must restrict the domain to nonnegative numbers, that is, if \( x \geq 0 \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Implicit Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \sqrt{x} )</td>
<td>([0, \infty))</td>
</tr>
</tbody>
</table>

In general, we ask the question, “what can \( x \) be?” The implicit domain of a function excludes values that cause a function to be undefined or have outputs that are not real numbers.

**Technology Tip**
To visualize the domain of each function, ask the question: What are the excluded \( x \)-values in the graph?

**EXAMPLE 10**
**Determining the Domain of a Function**

State the domain of the given functions.

a. \( F(x) = \frac{3}{x^2 - 25} \)

b. \( H(x) = \sqrt[4]{9 - 2x} \)

c. \( G(x) = \sqrt{x - 1} \)

**Solution (a):**
Write the original equation.
\[
 F(x) = \frac{3}{x^2 - 25}
\]
Determine any restrictions on the values of \( x \).
\[
 x^2 - 25 \neq 0
\]
Solve the restriction equation.
\[
 x^2 \neq 25 \text{ or } x \neq \pm \sqrt{25} = \pm 5
\]
State the domain restrictions.
\[
 (-\infty, -5) \cup (-5, 5) \cup (5, \infty)
\]
Write the domain in interval notation.
Solution (b):
Write the original equation.
\[ H(x) = \sqrt{9 - 2x} \]
Determine any restrictions on the values of \( x \).
\[ 9 - 2x \geq 0 \]
Solve the restriction inequality.
\[ 9 \geq 2x \]
State the domain restrictions.
\[ x \leq \frac{9}{2} \]
Write the domain in interval notation.
\[ (-\infty, \frac{9}{2}] \]

Solution (c):
Write the original equation.
\[ G(x) = \sqrt{x - 1} \]
Determine any restrictions on the values of \( x \).
no restrictions
State the domain.
\( \mathbb{R} \)
Write the domain in interval notation.
\[ (-\infty, \infty) \]

Your Turn: State the domain of the given functions.

A. \( f(x) = \sqrt{x - 3} \)  
B. \( g(x) = \frac{1}{x^2 - 4} \)

Applications

Functions that are used in applications often have restrictions on the domains due to physical constraints. For example, the volume of a cube is given by the function \( V(x) = x^3 \), where \( x \) is the length of a side. The function \( f(x) = x^3 \) has no restrictions on \( x \), and therefore the domain is the set of all real numbers. However, the volume of any cube has the restriction that the length of a side can never be negative or zero.

Example 11 Price of Gasoline

Following the capture of Saddam Hussein in Iraq in 2003, gas prices in the United States escalated and then finally returned to their precapture prices. Over a 6-month period, the average price of a gallon of 87 octane gasoline was given by the function \( C(x) = -0.05x^2 + 0.3x + 1.7 \), where \( C \) is the cost function and \( x \) represents the number of months after the capture.

A. Determine the domain of the cost function.
B. What was the average price of gas per gallon 3 months after the capture?

Solution (a):
Since the cost function \( C(x) = -0.05x^2 + 0.3x + 1.7 \) modeled the price of gas only for 6 months after the capture, the domain is \( 0 \leq x \leq 6 \) or \([0, 6]\).

Solution (b):
Write the cost function.
\[ C(x) = -0.05x^2 + 0.3x + 1.7 \quad 0 \leq x \leq 6 \]
Find the value of the function when \( x = 3 \).
\[ C(3) = -0.05(3)^2 + 0.3(3) + 1.7 \]
Simplify.
\[ C(3) = 2.15 \]
The average price per gallon 3 months after the capture was $2.15.
EXAMPLE 12  The Dimensions of a Pool

Express the volume of a 30 ft \times 10 \text{ ft} rectangular swimming pool as a function of its depth.

Solution:
The volume of any rectangular box is $V = lwh$, where $V$ is the volume, $l$ is the length, $w$ is the width, and $h$ is the height. In this example, the length is 30 ft, the width is 10 ft, and the height represents the depth $d$ of the pool.

Write the volume as a function of depth $d$.

| $V(d)$ = (30)(10)d |

Simplify.

$V(d) = 300d$

Determine any restrictions on the domain.

$d > 0$

### SECTION 3.1 SUMMARY

#### Relations and Functions (Let $x$ represent the independent variable and $y$ the dependent variable.)

<table>
<thead>
<tr>
<th>TYPE</th>
<th>MAPPING/CORRESPONDENCE</th>
<th>EQUATION</th>
<th>GRAPH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relation</td>
<td>Every $x$-value in the domain maps to at least one $y$-value in the range.</td>
<td>$x = y^2$</td>
<td></td>
</tr>
<tr>
<td>Function</td>
<td>Every $x$-value in the domain maps to exactly one $y$-value in the range.</td>
<td>$y = x^2$</td>
<td>Passes vertical line test</td>
</tr>
</tbody>
</table>

All functions are relations, but not all relations are functions. Functions can be represented by equations. In the following table, each column illustrates an alternative notation.

<table>
<thead>
<tr>
<th>INPUT</th>
<th>CORRESPONDENCE</th>
<th>OUTPUT</th>
<th>EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>Function</td>
<td>$y$</td>
<td>$y = 2x + 5$</td>
</tr>
<tr>
<td>Independent variable</td>
<td>Mapping</td>
<td>Dependent variable</td>
<td>Mathematical rule</td>
</tr>
<tr>
<td>Argument</td>
<td>$f$</td>
<td>$f(x)$</td>
<td>$f(x) = 2x + 5$</td>
</tr>
</tbody>
</table>

The domain is the set of all inputs ($x$-values), and the range is the set of all corresponding outputs ($y$-values). Placeholder notation is useful when evaluating functions.

$f(x) = 3x^2 + 2x$

$f(□) = 3(□)^2 + 2(□)$

Explicit domain is stated, whereas implicit domain is found by excluding $x$-values that:

- make the function undefined (denominator = 0),
- result in a nonreal output (even roots of negative real numbers).
In Exercises 1–24, determine whether each relation is a function. Assume that the coordinate pair \((x, y)\) represents the independent variable \(x\) and the dependent variable \(y\).

1. \(\{(0, 1), (1, 0), (2, 1), (3, 4)\}\)
2. \(\{(2, 2), (2, 2), (5, 5)\}\)
3. \(\{(0, 1), (1, 1), (2, 1), (3, 1)\}\)
4. \(\{(-5, 0), (0, 0), (5, 0)\}\)
5. \(\{(-3, 0), (3, 0), (-3, 0), (3, 0)\}\)
6. \(\{(-1, -1), (-2, -8), (1, 1), (2, 8)\}\)
7. \(\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}\)
8. \(\{(2, -2), (2, 2), (5, -5), (5, 5)\}\)
9. \(\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}\)
10. \(\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}\)
11. \(\{(0, 1), (1, 0), (2, 1), (-2, 1), (5, 4), (-3, 4)\}\)
12. \(\{(0, 1), (1, 1), (2, 1), (3, 1)\}\)
13. \(x^2 + y^2 = 9\)
14. \(x = |y|\)
15. \(x = y^2\)
16. \(y = x^3\)
17. \(y = |x - 1|\)
18. \(y = 3\)
19. \(\{(0, 5), (-5, 0), (5, 0), (0, -5)\}\)
20. \(\{(x, y)\}\)
21. \(\{(x, y)\}\)
22. \(\{(x, y)\}\)
23. \(\{(x, y)\}\)
24. \(\{(x, y)\}\)
In Exercises 25–32, use the given graphs to evaluate the functions.

25. \( y = f(x) \)

\[
\begin{array}{c}
\text{a. } f(2) \quad \text{b. } f(0) \quad \text{c. } f(-2)
\end{array}
\]

26. \( y = g(x) \)

\[
\begin{array}{c}
\text{a. } g(-3) \quad \text{b. } g(0) \quad \text{c. } g(5)
\end{array}
\]

27. \( y = p(x) \)

\[
\begin{array}{c}
\text{a. } p(-1) \quad \text{b. } p(0) \quad \text{c. } p(1)
\end{array}
\]

28. \( y = r(x) \)

\[
\begin{array}{c}
\text{a. } r(-4) \quad \text{b. } r(-1) \quad \text{c. } r(3)
\end{array}
\]

29. \( y = C(x) \)

\[
\begin{array}{c}
\text{a. } C(2) \quad \text{b. } C(0) \quad \text{c. } C(-2)
\end{array}
\]

30. \( y = q(x) \)

\[
\begin{array}{c}
\text{a. } q(-4) \quad \text{b. } q(0) \quad \text{c. } q(2)
\end{array}
\]

31. \( y = S(x) \)

\[
\begin{array}{c}
\text{a. } S(-3) \quad \text{b. } S(0) \quad \text{c. } S(2)
\end{array}
\]

32. \( y = T(x) \)

\[
\begin{array}{c}
\text{a. } T(-5) \quad \text{b. } T(-2) \quad \text{c. } T(4)
\end{array}
\]

33. Find \( x \) if \( f(x) = 3 \) in Exercise 25.

34. Find \( x \) if \( g(x) = -2 \) in Exercise 26.

35. Find \( x \) if \( p(x) = 5 \) in Exercise 27.

36. Find \( x \) if \( C(x) = -7 \) in Exercise 29.

37. Find \( x \) if \( C(x) = -5 \) in Exercise 29.

38. Find \( x \) if \( q(x) = -2 \) in Exercise 30.

39. Find \( x \) if \( S(x) = 1 \) in Exercise 31.

In Exercises 41–56, evaluate the given quantities applying the following four functions.

\[
\begin{align*}
f(x) &= 2x - 3 \\
F(t) &= 4 - t^2 \\
g(t) &= 5 + t \\
G(x) &= x^2 + 2x - 7
\end{align*}
\]

41. \( f(-2) \)

42. \( G(-3) \)

43. \( g(1) \)

44. \( F(-1) \)

45. \( f(-2) + g(1) \)

46. \( G(-3) - F(-1) \)

47. \( 3f(-2) - 2g(1) \)

48. \( 2F(-1) - 2G(-3) \)

49. \( \frac{f(-2)}{g(1)} \)

50. \( \frac{G(-3)}{F(-1)} \)

51. \( \frac{f(0) - f(-2)}{g(1)} \)

52. \( \frac{G(0) - G(-3)}{F(-1)} \)

53. \( f(x + 1) - f(x - 1) \)

54. \( F(t + 1) - F(t - 1) \)

55. \( g(x + a) - f(x + a) \)

56. \( G(x + b) + F(b) \)

In Exercises 57–64, evaluate the difference quotients using the same \( f, F, G \), and \( g \) given for Exercises 41–56.

57. \( \frac{f(x + h) - f(x)}{h} \)

58. \( \frac{F(t + h) - F(t)}{h} \)

59. \( \frac{g(t + h) - g(t)}{h} \)

60. \( \frac{G(x + h) - G(x)}{h} \)

61. \( \frac{f(-2 + h) - f(-2)}{h} \)

62. \( \frac{F(-1 + h) - F(-1)}{h} \)

63. \( \frac{g(1 + h) - g(1)}{h} \)

64. \( \frac{G(-3 + h) - G(-3)}{h} \)
In Exercises 65-96, find the domain of the given function. Express the domain in interval notation.

65. \( f(x) = 2x - 5 \) 
66. \( f(x) = -2x - 5 \) 
67. \( g(t) = t^2 + 3t \) 
68. \( h(x) = 3x^2 - 1 \) 
69. \( P(x) = \frac{x + 5}{x - 5} \) 
70. \( Q(t) = \frac{2 - t}{t + 3} \) 
71. \( T(x) = \frac{2}{x^2 - 4} \) 
72. \( R(x) = \frac{1}{x^2 - 1} \) 
73. \( F(x) = \frac{1}{x^3 + 1} \) 
74. \( G(t) = \frac{2}{t^2 + 4} \) 
75. \( q(x) = \sqrt{7} - x \) 
76. \( k(t) = \sqrt{t - 7} \) 
77. \( f(x) = \sqrt{2x + 5} \) 
78. \( g(x) = \sqrt{5} - \sqrt{2x} \) 
79. \( G(t) = \sqrt{t^2 - 4} \) 
80. \( F(x) = \sqrt{x^2 - 25} \) 
81. \( F(x) = \frac{1}{\sqrt{x - 3}} \) 
82. \( G(x) = \frac{2}{\sqrt{5} - x} \) 
83. \( f(x) = \sqrt[3]{1 - 2x} \) 
84. \( g(x) = \sqrt[5]{7 - 5x} \) 
85. \( P(x) = \frac{1}{\sqrt{x + 4}} \) 
86. \( Q(x) = \frac{x}{\sqrt{x^2 - 9}} \) 
87. \( R(x) = \frac{x + 1}{\sqrt{3} - 2x} \) 
88. \( p(x) = \frac{x^2}{\sqrt{25 - x^2}} \) 
89. \( H(t) = \frac{t}{\sqrt{t^2 - t - 6}} \) 
90. \( f(t) = \frac{t - 3}{\sqrt{t^2 + 9}} \) 
91. \( f(x) = (x^2 - 16)^{1/2} \) 
92. \( g(x) = (2x - 5)^{1/3} \) 
93. \( r(x) = x^3 - (3 - 2x)^{1/2} \) 
94. \( p(x) = (x - 1)^2 (x^2 - 9)^{3/5} \) 
95. \( f(x) = \frac{3}{2}x - \frac{2}{3} \) 
96. \( g(x) = \frac{3}{5}x^2 - \frac{1}{2}x - \frac{3}{4} \)

97. Let \( g(x) = x^2 - 2x - 5 \) and find the values of \( x \) that correspond to \( g(x) = 3 \).
98. Let \( g(x) = \frac{5}{6}x - 3 \) and find the value of \( x \) that corresponds to \( g(x) = \frac{2}{3} \).
99. Let \( f(x) = 2x(x - 5)^3 - 12(x - 5)^2 \) and find the values of \( x \) that correspond to \( f(x) = 0 \).
100. Let \( f(x) = 3x(x + 3)^2 - 6(x + 3)^3 \) and find the values of \( x \) that correspond to \( f(x) = 0 \).

### Applications

101. **Budget: Event Planning.** The cost associated with a catered wedding reception in Florida, in the springtime is given by the function
\[
T(x) = -0.7x^2 + 16.8x - 10.8,
\]
where \( T \) is the temperature in degrees Fahrenheit and \( x \) is the time of day in military time and is restricted to \( 6 \leq x \leq 18 \) (sunrise to sunset). What is the temperature at 6 A.M.? What is the temperature at noon?

102. **Budget: Long-Distance Calling.** The cost of a local home phone plan is $35 for basic service and $0.10 per minute for any domestic long-distance calls. Write the cost of monthly phone service in terms of the number of monthly long-distance minutes and state any domain restrictions.

103. **Temperature.** The cost associated with a catered wedding reception in Florida, in the springtime is given by the function
\[
T(x) = -0.7x^2 + 16.8x - 10.8,
\]
where \( T \) is the temperature in degrees Fahrenheit and \( x \) is the time of day in military time and is restricted to \( 6 \leq x \leq 18 \) (sunrise to sunset). What is the temperature at 6 A.M.? What is the temperature at noon?

104. **Falling Objects: Firecrackers.** A firecracker is launched straight up, and its height is a function of time,
\[
h(t) = -16t^2 + 128t,
\]
where \( h \) is the height in feet and \( t \) is the time in seconds with \( t = 0 \) corresponding to the instant it launches. What is the height 4 seconds after launch? What is the domain of this function?

105. **Collectibles.** The price of a signed Alex Rodriguez baseball card is a function of how many are for sale. When Rodriguez was traded from the Texas Rangers to the New York Yankees in 2004, the going rate for a signed baseball card on eBay was \( P(x) = 10 + \sqrt{400,000 - 100x} \), where \( x \) represents the number of signed cards for sale. What was the value of the card when there were 10 signed cards for sale? What was the value of the card when there were 100 signed cards for sale?
Chapter 3 Functions and Their Graphs

106. Collectibles. In Exercise 105, what was the lowest price on eBay, and how many cards were available then? What was the highest price on eBay, and how many cards were available then?

107. Volume. An open box is constructed from a square 10-inch piece of cardboard by cutting squares of length x inches out of each corner and folding the sides up. Express the volume of the box as a function of x, and state the domain.

108. Volume. A cylindrical water basin will be built to harvest rainwater. The basin is limited in that the largest radius it can have is 10 feet. Write a function representing the volume of water as a function of height h. How many additional gallons of water will be collected if you increase the height by 2 feet? Hint: 1 cubic foot = 7.48 gallons.

For Exercises 109–110, refer to the following:
The weekly exchange rate of the U.S. dollar to the Japanese yen is shown in the graph as varying over an 8-week period. Assume the exchange rate E(t) is a function of time (week); let E(1) be the exchange rate during Week 1.

109. Economics. Approximate the exchange rates of the U.S. dollar to the nearest yen during Weeks 4, 7, and 8.

110. Economics. Find the increase or decrease in the number of Japanese yen to the U.S. dollar exchange rate, to the nearest yen, from (a) Week 2 to Week 3 and (b) Week 6 to Week 7.

For Exercises 111–112, refer to the following:
An epidemiological study of the spread of malaria in a rural area finds that the total number P of people who contracted malaria t days into an outbreak is modeled by the function

\[ P(t) = \frac{1}{4} t^2 + 7t + 180 \quad 1 \leq t \leq 14 \]

111. Medicine/Health. How many people have contracted malaria 14 days into the outbreak?

112. Medicine/Health. How many people have contracted malaria 6 days into the outbreak?

113. Environment: Tossing the Envelopes. The average American adult receives 24 pieces of mail per week, usually of some combination of ads and envelopes with windows. Suppose each of these adults throws away a dozen envelopes per week.

a. The width of the window of an envelope is 3.375 inches less than its length x. Create the function A(x) that represents the area of the window in square inches. Simplify, if possible.

b. Evaluate A(4.5) and explain what this value represents.

c. Assume the dimensions of the envelope are 8 inches by 4 inches. Evaluate A(8.5). Is this possible for this particular envelope? Explain.

114. Environment: Tossing the Envelopes. Each month, Jack receives his bank statement in a 9.5 inch by 6 inch envelope. Each month, he throws away the envelope after removing the statement.

a. The width of the window of the envelope is 2.875 inches less than its length x. Create the function A(x) that represents the area of the window in square inches. Simplify, if possible.

b. Evaluate A(5.25) and explain what this value represents.

c. Evaluate A(10). Is this possible for this particular envelope? Explain.

Refer to the table below for Exercises 115 and 116. It illustrates the average federal funds rate for the month of January (2000 to 2008).

<table>
<thead>
<tr>
<th>Year</th>
<th>Fed. Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>5.45</td>
</tr>
<tr>
<td>2001</td>
<td>5.98</td>
</tr>
<tr>
<td>2002</td>
<td>1.73</td>
</tr>
<tr>
<td>2003</td>
<td>1.24</td>
</tr>
<tr>
<td>2004</td>
<td>1.00</td>
</tr>
<tr>
<td>2005</td>
<td>2.25</td>
</tr>
<tr>
<td>2006</td>
<td>4.50</td>
</tr>
<tr>
<td>2007</td>
<td>5.25</td>
</tr>
<tr>
<td>2008</td>
<td>3.50</td>
</tr>
</tbody>
</table>

115. Finance. Is the relation whose domain is the year and whose range is the average federal funds rate for the month of January a function? Explain.

116. Finance. Write five ordered pairs whose domain is the set of even years from 2000 to 2008 and whose range is the set of corresponding average federal funds rate for the month of January.
For Exercises 117 and 118, use the following figure:

![Graph showing U.S. health care costs for family plan premiums, adjusted for inflation, from 1990 to 2005.](image)


117. Health Care Costs. Fill in the following table. Round dollars to the nearest $1000.

<table>
<thead>
<tr>
<th>Year</th>
<th>Total Health Care Cost for Family Plans</th>
</tr>
</thead>
<tbody>
<tr>
<td>1989</td>
<td></td>
</tr>
<tr>
<td>1993</td>
<td></td>
</tr>
<tr>
<td>1997</td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td></td>
</tr>
</tbody>
</table>

Write the five ordered pairs resulting from the table.

### CATCH THE MISTAKE

In Exercises 121–126, explain the mistake that is made.

121. Determine whether the relationship is a function.

![Graph showing U-shaped graph with a horizontal line test applied.](image)

Solution:

Apply the horizontal line test.
Because the horizontal line intersects the graph in two places, this is not a function.

This is incorrect. What mistake was made?

122. Given the function \( H(x) = 3x - 2 \), evaluate the quantity \( H(3) - H(-1) \).

Solution: \( H(3) - H(-1) = H(3) + H(1) = 7 + 1 = 8 \)
This is incorrect. What mistake was made?

123. Given the function \( f(x) = x^2 - x \), evaluate the quantity \( f(x + 1) \).

Solution: \( f(x + 1) = f(x) + f(1) = x^2 - x + 0 \)
\( f(x + 1) = x^2 - x \)
This is incorrect. What mistake was made?

124. Determine the domain of the function \( g(t) = \sqrt{3 - t} \) and express it in interval notation.

Solution:

What can \( t \) be? Any nonnegative real number.
\( 3 - t > 0 \)
\( 3 > t \) or \( t < 3 \)
Domain: \((-\infty, 3)\)
This is incorrect. What mistake was made?
125. Given the function \( G(x) = x^2 \), evaluate \( \frac{G(-1 + h) - G(-1)}{h} \).

Solution:
\[
\frac{G(-1 + h) - G(-1)}{h} = \frac{G(-1) + G(h) - G(-1)}{h} = \frac{G(h)}{h} = \frac{h^2}{h} = h.
\]
This is incorrect. What mistake was made?

126. Given the functions \( f(x) = |x - A| - 1 \) and \( f(1) = -1 \), find \( A \).

Solution:
Since \( f(1) = -1 \), the point \((-1, 1)\) must satisfy the function. Add 1 to both sides of the equation. The absolute value of zero is zero, so there is no need for the absolute value signs: \(-1 - A = 0 \Rightarrow A = -1\). This is incorrect. What mistake was made?

**CONCEPTUAL**

In Exercises 127–130, determine whether each statement is true or false.

127. If a vertical line does not intersect the graph of an equation, then that equation does not represent a function.

128. If a horizontal line intersects a graph of an equation more than once, the equation does not represent a function.

129. If \( f(-a) = f(a) \), then \( f \) does not represent a function.

130. If \( f(-a) = f(a) \), then \( f \) may or may not represent a function.

131. If \( f(x) = Ax^2 - 3x \) and \( f(1) = -1 \), find \( A \).

132. If \( g(x) = \frac{1}{b-x} \) and \( g(3) \) is undefined, find \( b \).

**CHALLENGE**

133. If \( F(x) = \frac{C-x}{D-x} \), \( F(-2) \) is undefined, and \( F(-1) = 4 \), find \( C \) and \( D \).

134. Construct a function that is undefined at \( x = 5 \) and whose graph passes through the point \((1, -1)\).

135. If \( f(x) = \frac{-100}{x^2 - a^2} \), find the domain of each function, where \( a \) is any positive real number.

136. \( f(x) = -5\sqrt{x^2 - a^2} \)

**TECHNOLOGY**

137. Using a graphing utility, graph the temperature function in Exercise 103. What time of day is it the warmest? What is the temperature? Looking at this function, explain why this model for Tampa, Florida, is valid only from sunrise to sunset (6 to 18).

138. Using a graphing utility, graph the height of the firecracker in Exercise 104. How long after liftoff is the firecracker airborne? What is the maximum height that the firecracker attains? Explain why this height model is valid only for the first 8 seconds.

139. Using a graphing utility, graph the price function in Exercise 105. What are the lowest and highest prices of the cards? Does this agree with what you found in Exercise 106?

140. The makers of malted milk balls are considering increasing the size of the spherical treats. The thin chocolate coating on a malted milk ball can be approximated by the surface area, \( S(r) = 4\pi r^2 \). If the radius is increased 3 mm, what is the resulting increase in required chocolate for the thin outer coating?

141. Let \( f(x) = x^2 + 1 \). Graph \( y_1 = f(x) \) and \( y_2 = f(x - 2) \) in the same viewing window. Describe how the graph of \( y_2 \) can be obtained from the graph of \( y_1 \).

142. Let \( f(x) = 4 - x^2 \). Graph \( y_1 = f(x) \) and \( y_2 = f(x + 2) \) in the same viewing window. Describe how the graph of \( y_2 \) can be obtained from the graph of \( y_1 \).
Recognizing and Classifying Functions

Common Functions

Point-plotting techniques were introduced in Section 2.2, and we noted there that we would explore some more efficient ways of graphing functions in Chapter 3. The nine main functions you will read about in this section will constitute a “library” of functions that you should commit to memory. We will draw on this library of functions in the next section when graphing transformations are discussed. Several of these functions have been shown previously in this chapter, but now we will classify them specifically by name and identify properties that each function exhibits.

In Section 2.3, we discussed equations and graphs of lines. All lines (with the exception of vertical lines) pass the vertical line test, and hence are classified as functions. Instead of the traditional notation of a line, \( y = mx + b \), we use function notation and classify a function whose graph is a line as a linear function.

**Linear function**

\[ f(x) = mx + b \]

\( m \) and \( b \) are real numbers.

The domain of a linear function \( f(x) = mx + b \) is the set of all real numbers \( \mathbb{R} \). The graph of this function has slope \( m \) and y-intercept \( b \).

<table>
<thead>
<tr>
<th>Linear Function: ( f(x) = mx + b )</th>
<th>Slope: ( m )</th>
<th>y-Intercept: ( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2x - 7 )</td>
<td>( m = 2 )</td>
<td>( b = -7 )</td>
</tr>
<tr>
<td>( f(x) = -x + 3 )</td>
<td>( m = -1 )</td>
<td>( b = 3 )</td>
</tr>
<tr>
<td>( f(x) = x )</td>
<td>( m = 1 )</td>
<td>( b = 0 )</td>
</tr>
<tr>
<td>( f(x) = 5 )</td>
<td>( m = 0 )</td>
<td>( b = 5 )</td>
</tr>
</tbody>
</table>
One special case of the linear function is the constant function \((m = 0)\).

**Constant function**

\[
f(x) = b \quad b \text{ is any real number.}
\]

The graph of a constant function \(f(x) = b\) is a horizontal line. The \(y\)-intercept corresponds to the point \((0, b)\). The domain of a constant function is the set of all real numbers \(\mathbb{R}\). The range, however, is a single value \(b\). In other words, all \(x\)-values correspond to a single \(y\)-value.

Points that lie on the graph of a constant function \(f(x) = b\) are

\[
(-5, b) \\
(-1, b) \\
(0, b) \\
(2, b) \\
(4, b) \\
\ldots \\
(x, b)
\]

Another specific example of a linear function is the function having a slope of one \((m = 1)\) and a \(y\)-intercept of zero \((b = 0)\). This special case is called the identity function.

**Identity function**

\[
f(x) = x
\]

The graph of the identity function has the following properties: It passes through the origin, and every point that lies on the line has equal \(x\)- and \(y\)-coordinates. Both the domain and the range of the identity function are the set of all real numbers \(\mathbb{R}\).

A function that squares the input is called the square function.

**Square function**

\[
f(x) = x^2
\]

The graph of the square function is called a parabola and will be discussed in further detail in Chapters 4 and 8. The domain of the square function is the set of all real numbers \(\mathbb{R}\). Because squaring a real number always yields a positive number or zero, the range of the square function is the set of all nonnegative numbers. Note that the intercept is the origin and the square function is symmetric about the \(y\)-axis. This graph is contained in quadrants I and II.
A function that cubes the input is called the *cube function*.

**Cube function**

\[ f(x) = x^3 \]

The domain of the cube function is the set of all real numbers \( \mathbb{R} \). Because cubing a negative number yields a negative number, cubing a positive number yields a positive number, and cubing 0 yields 0, the range of the cube function is also the set of all real numbers \( \mathbb{R} \). Note that the only intercept is the origin and the cube function is symmetric about the origin. This graph extends only into quadrants I and III.

The next two functions are counterparts of the previous two functions: square root and cube root. When a function takes the square root of the input or the cube root of the input, the function is called the *square root function* or the *cube root function*, respectively.

**Square root function**

\[ f(x) = \sqrt{x} \quad \text{or} \quad f(x) = x^{1/2} \]

In Section 3.1, we found the domain to be \([0, \infty)\). The output of the function will be all real numbers greater than or equal to zero. Therefore, the range of the square root function is \([0, \infty)\). The graph of this function will be contained in quadrant I.

**Cube root function**

\[ f(x) = \sqrt[3]{x} \quad \text{or} \quad f(x) = x^{1/3} \]

In Section 3.1, we stated the domain of the cube root function to be \((\infty, \infty)\). We see by the graph that the range is also \((\infty, \infty)\). This graph is contained in quadrants I and III and passes through the origin. This function is symmetric about the origin.

In Section 1.7, you read about absolute value equations and inequalities. Now we shift our focus to the graph of the *absolute value function*.

**Absolute value function**

\[ f(x) = |x| \]

Some points that are on the graph of the absolute value function are \((-1, 1)\), \((0, 0)\), and \((1, 1)\). The domain of the absolute value function is the set of all real numbers \( \mathbb{R} \), yet the range is the set of nonnegative real numbers. The graph of this function is symmetric with respect to the y-axis and is contained in quadrants I and II.
A function whose output is the reciprocal of its input is called the \textit{reciprocal function}.

The only restriction on the domain of the reciprocal function is that $x \neq 0$. Therefore, we say the domain is the set of all real numbers excluding zero. The graph of the reciprocal function illustrates that its range is also the set of all real numbers except zero. Note that the reciprocal function is symmetric with respect to the origin and is contained in quadrants I and III.

**Even and Odd Functions**

Of the nine functions discussed above, several have similar properties of symmetry. The constant function, square function, and absolute value function are all symmetric with respect to the $y$-axis. The identity function, cube function, cube root function, and reciprocal function are all symmetric with respect to the origin. The term \textit{even} is used to describe functions that are symmetric with respect to the $y$-axis, or vertical axis, and the term \textit{odd} is used to describe functions that are symmetric with respect to the origin. Recall from Section 2.2 that symmetry can be determined both graphically and algebraically. The box below summarizes the graphic and algebraic characteristics of even and odd functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Symmetric with Respect to</th>
<th>On Replacing $x$ with $-x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>$y$-axis or vertical axis</td>
<td>$f(-x) = f(x)$</td>
</tr>
<tr>
<td>Odd</td>
<td>origin</td>
<td>$f(-x) = -f(x)$</td>
</tr>
</tbody>
</table>

The algebraic method for determining symmetry with respect to the $y$-axis, or vertical axis, is to substitute $-x$ for $x$. If the result is an equivalent equation, the function is symmetric with respect to the $y$-axis. Some examples of even functions are $f(x) = b$, $f(x) = x^2$, $f(x) = x^4$; and $f(x) = |x|$. In any of these equations, if $-x$ is substituted for $x$, the result is the same; that is, $f(-x) = f(x)$. Also note that, with the exception of the absolute value function, these examples are all even-degree polynomial equations. All constant functions are degree zero and are even functions.

The algebraic method for determining symmetry with respect to the origin is to substitute $-x$ for $x$. If the result is the negative of the original function, that is, if $f(-x) = -f(x)$, then the function is symmetric with respect to the origin and, hence, classified as an odd function. Examples of odd functions are $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$; and $f(x) = x^{10}$. In any of these functions, if $-x$ is substituted for $x$, the result is the negative of the original function. Note that with the exception of the cube root function, these equations are odd-degree polynomials.
Be careful, though, because functions that are combinations of even- and odd-degree polynomials can turn out to be neither even nor odd, as we will see in Example 1.

**EXAMPLE 1**  
**Determining Whether a Function Is Even, Odd, or Neither**

Determine whether the functions are even, odd, or neither.

a. \( f(x) = x^2 - 3 \)  
   b. \( g(x) = x^2 + x^3 \)  
   c. \( h(x) = x^2 - x \)

**Solution (a):**

Original function.

\[ f(x) = x^2 - 3 \]

Replace \( x \) with \( -x \).

\[ f(-x) = (-x)^2 - 3 \]

Simplify.

\[ f(-x) = x^2 - 3 = f(x) \]

Because \( f(-x) = f(x) \), we say that \( f(x) \) is an **even function**.

**Solution (b):**

Original function.

\[ g(x) = x^5 + x^3 \]

Replace \( x \) with \( -x \).

\[ g(-x) = (-x)^5 + (-x)^3 \]

Simplify.

\[ g(-x) = -x^5 - x^3 = -(x^5 + x^3) = -g(x) \]

Because \( g(-x) = -g(x) \), we say that \( g(x) \) is an **odd function**.

**Solution (c):**

Original function.

\[ h(x) = x^2 - x \]

Replace \( x \) with \( -x \).

\[ h(-x) = (-x)^2 - (-x) \]

Simplify.

\[ h(-x) = x^2 + x \]

\( h(-x) \) is **neither** \( -h(x) \) **nor** \( h(x) \); therefore the function \( h(x) \) is **neither even nor odd**.

In parts (a), (b), and (c), we classified these functions as either even, odd, or neither, using the algebraic test. Look back at them now and reflect on whether these classifications agree with your intuition. In part (a), we combined two functions: the square function and the constant function. Both of these functions are even, and adding even functions yields another even function. In part (b), we combined two odd functions: the fifth-power function and the cube function. Both of these functions are odd, and adding two odd functions yields another odd function. In part (c), we combined two functions: the square function and the identity function. The square function is even, and the identity function is odd. In this part, combining an even function with an odd function yields a function that is neither even nor odd and, hence, has no symmetry with respect to the vertical axis or the origin.

- **YOUR TURN**  
  Classify the functions as even, odd, or neither.

  a. \( f(x) = |x| + 4 \)  
  b. \( f(x) = x^3 - 1 \)
Increasing and Decreasing Functions

Look at the figure in the margin to the left. Graphs are read from left to right. If we start at the left side of the graph and trace the red curve with our pen, we see that the function values (values in the vertical direction) are decreasing until arriving at the point \((-2, -2)\). Then, the function values increase until arriving at the point \((-1, 1)\). The values then remain constant \((y = 1)\) between the points \((-1, 1)\) and \((0, 1)\). Proceeding beyond the point \((0, 1)\), the function values decrease again until the point \((2, -2)\). Beyond the point \((2, -2)\), the function values increase again until the point \((6, 4)\). Finally, the function values decrease and continue to do so.

When specifying a function as increasing, decreasing, or constant, the intervals are classified according to the \(x\)-coordinate. For instance, in this graph, we say the function is increasing when \(x\) is between \(-2\) and \(-1\) and again when \(x\) is between \(2\) and \(6\). The graph is classified as decreasing when \(x\) is less than \(-2\) and again when \(x\) is between \(0\) and \(2\) and again when \(x\) is greater than \(6\). The graph is classified as constant when \(x\) is between \(-1\) and \(0\). In interval notation, this is summarized as

\[
\text{Decreasing: } \left(\infty, -2\right) \cup \left(0, 2\right) \cup \left(6, \infty\right)
\]

\[
\text{Increasing: } \left(-2, -1\right) \cup (2, 6)
\]

\[
\text{Constant: } \left(-1, 0\right)
\]

An algebraic test for determining whether a function is increasing, decreasing, or constant is to compare the value \(f(x)\) of the function for particular points in the intervals.

**Increasing, decreasing, and constant functions**

1. A function \(f\) is **increasing** on an open interval \(I\) if for any \(x_1\) and \(x_2\) in \(I\), where \(x_1 < x_2\), then \(f(x_1) < f(x_2)\).
2. A function \(f\) is **decreasing** on an open interval \(I\) if for any \(x_1\) and \(x_2\) in \(I\), where \(x_1 < x_2\), then \(f(x_1) > f(x_2)\).
3. A function \(f\) is **constant** on an open interval \(I\) if for any \(x_1\) and \(x_2\) in \(I\), then \(f(x_1) = f(x_2)\).

In addition to classifying a function as increasing, decreasing, or constant, we can also determine the domain and range of a function by inspecting its graph from left to right:

- The domain is the set of all \(x\)-values (from left to right) where the function is defined.
- The range is the set of all \(y\)-values (from bottom to top) that the graph of the function corresponds to.
- A solid dot on the left or right end of a graph indicates that the graph terminates there and the point is included in the graph.
- An open dot indicates that the graph terminates there and the point is not included in the graph.
- Unless a dot is present, it is assumed that a graph continues indefinitely in the same direction. (An arrow is used in some books to indicate direction.)
3.2 Graphs of Functions

**Example 2** Finding Intervals When a Function Is Increasing or Decreasing

Given the graph of a function:

a. State the domain and range of the function.

b. Find the intervals when the function is increasing, decreasing, or constant.

**Solution (a):**

Domain: $[-5, \infty)$  
Range: $[0, \infty)$

**Solution (b):**

Reading the graph from left to right, we see that the graph

- decreases from the point $(-5, 7)$ to the point $(-2, 4)$.
- is constant from the point $(-2, 4)$ to the point $(0, 4)$.
- decreases from the point $(0, 4)$ to the point $(2, 0)$.
- increases from the point $(2, 0)$ on.

The intervals of increasing and decreasing correspond to the $x$-coordinates.

We say that this function is

- increasing on the interval $(2, \infty)$.
- decreasing on the interval $(-5, -2) \cup (0, 2)$.
- constant on the interval $(-2, 0)$.

**Note:** The intervals of increasing or decreasing are defined on open intervals. This should not be confused with the domain. For example, the point $x = -5$ is included in the domain of the function but not in the interval where the function is classified as decreasing.

**Average Rate of Change**

How do we know how much a function is increasing or decreasing? For example, is the price of a stock slightly increasing or is it doubling every week? One way we determine how much a function is increasing or decreasing is by calculating its average rate of change.

Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points that lie on the graph of a function $f$. Draw the line that passes through these two points $(x_1, y_1)$ and $(x_2, y_2)$. This line is called a secant line.

Note that the slope of the secant line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, and recall that the slope of a line is the rate of change of that line. The slope of the secant line is used to represent the average rate of change of the function.
**Average Rate of Change**

Let \((x_1, f(x_1))\) and \((x_2, f(x_2))\) be two distinct points, \((x_1 \neq x_2)\), on the graph of the function \(f\). The **average rate of change** of \(f\) between \(x_1\) and \(x_2\) is given by

\[
\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

**Example 3**  
Average Rate of Change

Find the average rate of change of \(f(x) = x^4\) from:

- a. \(x = -1\) to \(x = 0\)
- b. \(x = 0\) to \(x = 1\)
- c. \(x = 1\) to \(x = 2\)

**Solution (a):**

Write the average rate of change formula.

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Let \(x_1 = -1\) and \(x_2 = 0\).

Substitute \(f(-1) = (-1)^4 = 1\) and \(f(0) = 0^4 = 0\).

Simplify.

**Solution (b):**

Write the average rate of change formula.

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Let \(x_1 = 0\) and \(x_2 = 1\).

Substitute \(f(0) = 0^4 = 0\) and \(f(1) = 1^4 = 1\).

Simplify.

**Solution (c):**

Write the average rate of change formula.

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

Let \(x_1 = 1\) and \(x_2 = 2\).

Substitute \(f(1) = 1^4 = 1\) and \(f(2) = 2^4 = 16\).

Simplify.
Graphical Interpretation: Slope of the Secant Line

a. Average rate of change of \( f \) from \( x = -1 \) to \( x = 0 \):
   Decreasing at a rate of 1

b. Average rate of change of \( f \) from \( x = 0 \) to \( x = 1 \):
   Increasing at a rate of 1

c. Average rate of change of \( f \) from \( x = 1 \) to \( x = 2 \):
   Increasing at a rate of 15

**YOUR TURN** Find the average rate of change of \( f(x) = x^2 \) from:

a. \( x = -2 \) to \( x = 0 \)  
   b. \( x = 0 \) to \( x = 2 \)

The average rate of change can also be written in terms of the difference quotient.

**Words**
Let the difference between \( x_1 \) and \( x_2 \) be \( h \).
Solve for \( x_2 \).
Substitute \( x_2 - x_1 = h \) into the denominator and
\[ x_2 = x_1 + h \]

**Math**
Average rate of change = \[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]
\[ = \frac{f(x_1 + h) - f(x_1)}{h} \]
\[ = \frac{f(x + h) - f(x)}{h} \]
When written in this form, the average rate of change is called the **difference quotient**.

**Definition**  
**Difference Quotient**  
The expression \( \frac{f(x + h) - f(x)}{h} \), where \( h \neq 0 \), is called the **difference quotient**.

The difference quotient is more meaningful when \( h \) is small. In calculus the difference quotient is used to define a derivative.

### Example 4  Calculating the Difference Quotient

Calculate the difference quotient for the function \( f(x) = 2x^2 + 1 \).

**Solution:**

Find \( f(x + h) \).

\[
\begin{align*}
 f(x + h) &= 2(x + h)^2 + 1 \\
 &= 2(x^2 + 2xh + h^2) + 1 \\
 &= 2x^2 + 4xh + 2h^2 + 1
\end{align*}
\]

Find the difference quotient.

\[
\frac{f(x + h) - f(x)}{h} = \frac{2x^2 + 4xh + 2h^2 + 1 - (2x^2 + 1)}{h}
\]

Simplify.

\[
\begin{align*}
 f(x + h) - f(x) &= 2x^2 + 4xh + 2h^2 + 1 - 2x^2 - 1 \\
 &= 2x^2 + 4xh + 2h^2 \\
 &= 4xh + 2h^2
\end{align*}
\]

Factor the numerator.

\[
\begin{align*}
 f(x + h) - f(x) &= 4xh + 2h^2 \\
 &= h(4x + 2h)
\end{align*}
\]

Cancel (divide out) the common \( h \).

\[
\frac{f(x + h) - f(x)}{h} = \frac{4x + 2h}{h} \quad h \neq 0
\]

**Answer:**

\[
\frac{f(x + h) - f(x)}{h} = -2x - 3h
\]

**Your Turn** Calculate the difference quotient for the function \( f(x) = -x^2 + 2 \).

### Piecewise-Defined Functions

Most of the functions that we have seen in this text are functions defined by polynomials. Sometimes the need arises to define functions in terms of **pieces**. For example, most plumbers charge a flat fee for a house call and then an additional hourly rate for the job. For instance, if a particular plumber charges $100 to drive out to your house and work for 1 hour and then an additional $25 an hour for every additional hour he or she works on your job, we would define this function in pieces. If we let \( h \) be the number of hours worked, then the charge is defined as

\[
\text{Plumbing charge} = \begin{cases} 
 100 & h \leq 1 \\
 100 + 25(h - 1) & h > 1 
\end{cases}
\]
If we were to graph this function, we would see that there is 1 hour that is constant and after that the function continually increases.

Another piecewise-defined function is the absolute value function. The absolute value function can be thought of as two pieces: the line \( y = -x \) (when \( x \) is negative) and the line \( y = x \) (when \( x \) is nonnegative). We start by graphing these two lines on the same graph.

The absolute value function behaves like the line \( y = -x \) when \( x \) is negative (erase the blue graph in quadrant IV) and like the line \( y = x \) when \( x \) is positive (erase the red graph in quadrant III).

Absolute value function

\[
f(x) = |x| = \begin{cases} 
-x & x < 0 \\
 0 & x = 0 \\
 x & x > 0 
\end{cases}
\]

The next example is a piecewise-defined function given in terms of functions in our “library of functions.” Because the function is defined in terms of pieces of other functions, we draw the graph of each individual function, and then for each function, darken the piece corresponding to its part of the domain. This is like the procedure above for the absolute value function.

**EXAMPLE 5**  Graphing Piecewise-Defined Functions

Graph the piecewise-defined function, and state the domain, range, and intervals when the function is increasing, decreasing, or constant.

\[
G(x) = \begin{cases} 
x^2 & x < -1 \\
1 & -1 \leq x \leq 1 \\
x & x > 1 
\end{cases}
\]

**Solution:**

Graph each of the functions on the same plane.

**Square function:**
\( f(x) = x^2 \)

**Constant function:**
\( f(x) = 1 \)

**Identity function:**
\( f(x) = x \)

The points to focus on in particular are the \( x \)-values where the pieces change over—that is, \( x = -1 \) and \( x = 1 \).

Let’s now investigate each piece. When \( x < -1 \), this function is defined by the square function, \( f(x) = x^2 \), so darken that particular function to the left of \( x = -1 \). When \( -1 \leq x \leq 1 \), the function is defined by the constant function, \( f(x) = 1 \), so darken that particular function between the \( x \) values of \(-1 \) and \( 1 \). When \( x > 1 \), the function is defined by the identity function, \( f(x) = x \), so darken that function to the right of \( x = 1 \). Erase everything that is not darkened, and the resulting graph of the piecewise-defined function is given on the right.
This function is defined for all real values of \( x \), so the domain of this function is the set of all real numbers. The values that this function yields in the vertical direction are all real numbers greater than or equal to 1. Hence, the range of this function is \([1, \infty)\). The intervals of increasing, decreasing, and constant are as follows:

- Decreasing: \((-\infty, -1)\)
- Constant: \((-1, 1)\)
- Increasing: \((1, \infty)\)

The term **continuous** implies that there are no holes or jumps and that the graph can be drawn without picking up your pencil. A function that does have holes or jumps and cannot be drawn in one motion without picking up your pencil is classified as **discontinuous**, and the points where the holes or jumps occur are called **points of discontinuity**.

The previous example illustrates a **continuous** piecewise-defined function. At the \( x = -1 \) junction, the square function and constant function both pass through the point \((-1, 1)\). At the \( x = 1 \) junction, the constant function and the identity function both pass through the point \((1, 1)\). Since the graph of this piecewise-defined function has no holes or jumps, we classify it as a continuous function.

The next example illustrates a **discontinuous** piecewise-defined function.

**EXAMPLE 6** Graphing a Discontinuous Piecewise-Defined Function

Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

\[
f(x) = \begin{cases} 
1 - x & x < 0 \\
\quad x & 0 \leq x < 2 \\
-1 & x \geq 2 
\end{cases}
\]

**Solution:**

Graph these functions on the same plane.

- **Linear function:** 
  \( f(x) = 1 - x \)

- **Identity function:** 
  \( f(x) = x \)

- **Constant function:** 
  \( f(x) = -1 \)

To plot a piecewise-defined function using a graphing utility, use the **TEST** menu operations to define the inequalities in the function. Press:

- **Y=**
- **2nd MATH 1** (Less Than)
- **2nd MATH 6** (Greater Than or Equal To)
- **2nd MATH 2** (Less Than or Equal To)
- **2nd MATH 3** (Greater Than)

To avoid connecting graphs of the pieces, press **MODE** and select **Dot**. Set the viewing rectangle as \([-3, 4]\) by \([-2, 5]\); then press **GRAPH**.

Be sure to include the open circle and closed circle at the appropriate endpoints of each piece in the function.

The table of values supports the graph, except at \( x = 2 \). The function is not defined at \( x = 2 \).
At what intervals is the function increasing, decreasing, or constant? Remember that the intervals correspond to the \( x \)-values.

Decreasing: \((-\infty, 0)\)  \(\) Increasing: \((0, 2)\)  \(\) Constant: \((2, \infty)\)

The function is defined for all values of \( x \) except \( x = 2 \).

Domain: \((-\infty, 2) \cup (2, \infty)\)

The output of this function (vertical direction) takes on the \( y \)-values \( y \geq 0 \) and the additional single value \( y = -1 \).

Range: \([-1, -1]\) \(\] \(\cup [0, \infty)\) or \((-1) \cup [0, \infty)\)

We mentioned earlier that a discontinuous function has a graph that exhibits holes or jumps. In this example, the point \( x = 0 \) corresponds to a jump, because you would have to pick up your pencil to continue drawing the graph. The point \( x = 2 \) corresponds to both a hole and a jump. The hole indicates that the function is not defined at that point, and there is still a jump because the identity function and the constant function do not meet at the same \( y \)-value at \( x = 2 \).

**YOUR TURN** Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

\[
f(x) = \begin{cases} 
-x & x \leq -1 \\
2 & -1 < x < 1 \\
x & x \geq 1 
\end{cases}
\]

Piecewise-defined functions whose “pieces” are constants are called **step functions**. The reason for this name is that the graph of a step function looks like steps of a staircase. A common step function used in engineering is the **Heaviside step function** (also called the **unit step function**):

\[
H(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0 
\end{cases}
\]

This function is used in signal processing to represent a signal that turns on at some time and stays on indefinitely.

A common step function used in business applications is the **greatest integer function**.

**Greatest integer function**

\[
f(x) = \lfloor x \rfloor = \text{greatest integer less than or equal to } x.
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.0</th>
<th>1.3</th>
<th>1.5</th>
<th>1.7</th>
<th>1.9</th>
<th>2.0</th>
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</thead>
<tbody>
<tr>
<td>( f(x) = \lfloor x \rfloor )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
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### SECTION 3.2 SUMMARY

<table>
<thead>
<tr>
<th>Name</th>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Graph</th>
<th>Even/Odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$f(x) = mx + b$, $m \neq 0$</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\infty, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Neither (unless $y = x$)</td>
</tr>
<tr>
<td>Constant</td>
<td>$f(x) = c$</td>
<td>$(-\infty, \infty)$</td>
<td>$[c, c]$ or ${c}$</td>
<td><img src="image" alt="Graph" /></td>
<td>Even</td>
</tr>
<tr>
<td>Identity</td>
<td>$f(x) = x$</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\infty, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Odd</td>
</tr>
<tr>
<td>Square</td>
<td>$f(x) = x^2$</td>
<td>$(-\infty, \infty)$</td>
<td>$[0, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Even</td>
</tr>
<tr>
<td>Cube</td>
<td>$f(x) = x^3$</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\infty, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Odd</td>
</tr>
<tr>
<td>Square Root</td>
<td>$f(x) = \sqrt{x}$</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Neither</td>
</tr>
<tr>
<td>Cube Root</td>
<td>$f(x) = \sqrt[3]{x}$</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\infty, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Odd</td>
</tr>
<tr>
<td>Absolute Value</td>
<td>$f(x) =</td>
<td>x</td>
<td>$</td>
<td>$(-\infty, \infty)$</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>Reciprocal</td>
<td>$f(x) = \frac{1}{x}$</td>
<td>$(-\infty, 0) \cup (0, \infty)$</td>
<td>$(-\infty, 0) \cup (0, \infty)$</td>
<td><img src="image" alt="Graph" /></td>
<td>Odd</td>
</tr>
</tbody>
</table>

---

### Domain and Range of a Function

- **Implied domain:** Exclude any values that lead to the function being undefined (dividing by zero) or imaginary outputs (square root of a negative real number).
- **Graph:** Inspect the graph to determine the set of all inputs (domain) and the set of all outputs (range).
In Exercises 1–24, determine whether the function is even, odd, or neither.

1. \( G(x) = x + 4 \)
2. \( h(x) = 3 - x \)
3. \( f(x) = 3x^2 + 1 \)
4. \( F(x) = x^4 + 2x^2 \)
5. \( g(t) = 5t^3 - 3t \)
6. \( f(x) = 3x^5 + 4x^3 \)
7. \( h(x) = x^2 + 2x \)
8. \( G(x) = 2x^4 + 3x^3 \)
9. \( h(x) = x^{1/2} - x \)
10. \( g(x) = x^{-1} + x \)
11. \( f(x) = |x| + 5 \)
12. \( f(x) = |x| + x^2 \)
13. \( f(x) = |x| \)
14. \( f(x) = |x^3| \)
15. \( G(t) = |t - 3| \)
16. \( g(t) = |t + 2| \)
17. \( G(t) = \sqrt{t - 3} \)
18. \( f(x) = \sqrt{2 - x} \)
19. \( g(x) = \sqrt{x^2 + x} \)
20. \( f(x) = \sqrt{x^2 + 2} \)
21. \( h(x) = \frac{1}{x} + 3 \)
22. \( h(x) = \frac{1}{x} - 2x \)
23.
24.

In Exercises 25–36, state the (a) domain, (b) range, and (c) \( x \)-interval(s) where the function is increasing, decreasing, or constant. Find the values of (d) \( f(0) \), (e) \( f(-2) \), and (f) \( f(2) \).

25.
26.
27.
28.
In Exercises 37–44, find the difference quotient \( \frac{f(x + h) - f(x)}{h} \) for each function.

37. \( f(x) = x^2 - x \)
38. \( f(x) = x^2 + 2x \)
39. \( f(x) = 3x + x^2 \)
40. \( f(x) = 5x - x^2 \)
41. \( f(x) = x^2 - 3x + 2 \)
42. \( f(x) = x^2 - 2x + 5 \)
43. \( f(x) = -3x^2 + 5x - 4 \)
44. \( f(x) = -4x^2 + 2x - 3 \)

In Exercises 45–52, find the average rate of change of the function from \( x = 1 \) to \( x = 3 \).

45. \( f(x) = x^3 \)
46. \( f(x) = \frac{1}{x} \)
47. \( f(x) = |x| \)
48. \( f(x) = 2x \)
49. \( f(x) = 1 - 2x \)
50. \( f(x) = 9 - x^2 \)
51. \( f(x) = |5 - 2x| \)
52. \( f(x) = \sqrt{x^2 - 1} \)

In Exercises 53–78, graph the piecewise-defined functions. State the domain and range in interval notation. Determine the intervals where the function is increasing, decreasing, or constant.

53. \( f(x) = \begin{cases} x & x < 2 \\ 2 & x \geq 2 \end{cases} \)
54. \( f(x) = \begin{cases} -x & x < -1 \\ -1 & x \geq -1 \end{cases} \)
55. \( f(x) = \begin{cases} 1 & x < -1 \\ x^2 & x \geq -1 \end{cases} \)
56. \( f(x) = \begin{cases} x^2 & x < 2 \\ 4 & x \geq 2 \end{cases} \)
57. \( f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases} \)
58. \( f(x) = \begin{cases} -x & x \leq 0 \\ x^2 & x > 0 \end{cases} \)
59. \( f(x) = \begin{cases} -x + 2 & x < 1 \\ x^2 & x \geq 1 \end{cases} \)
60. \( f(x) = \begin{cases} 2 + x & x \leq -1 \\ x^2 & x > -1 \end{cases} \)
61. \( f(x) = \begin{cases} 5 - 2x & x < 2 \\ 3x - 2 & x \geq 2 \end{cases} \)
62. \( f(x) = \begin{cases} 3 - \frac{1}{2}x & x < -2 \\ 4 + \frac{3}{2}x & x \geq -2 \end{cases} \)
63. \( G(x) = \begin{cases} -1 & x < -1 \\ -1 \leq x \leq 3 \\ 3 & x > 3 \end{cases} \)
64. \( G(x) = \begin{cases} -1 & x < -1 \\ -1 < x < 3 \\ 3 & x > 3 \end{cases} \)
For Exercises 79 and 80, refer to the following:

A manufacturer determines that his profit and cost functions over one year are represented by the following graphs.

### Applications

#### 79. Business

Find the intervals on which profit is increasing, decreasing, and constant.

#### 80. Business

Find the intervals on which cost is increasing, decreasing, and constant.

81. **Budget: Costs.** The Kappa Kappa Gamma sorority decides to order custom-made T-shirts for its *Kappa Krush* mixer with the Sigma Alpha Epsilon fraternity. If the sorority orders 50 or fewer T-shirts, the cost is $10 per shirt. If it orders more than 50 but less than or equal to 100, the cost is $9 per shirt. If it orders more than 100, the cost is $8 per shirt. Find the cost function $C(x)$ as a function of the number of T-shirts $x$ ordered.

82. **Budget: Costs.** The marching band at a university is ordering some additional uniforms to replace existing uniforms that are worn out. If the band orders 50 or fewer, the cost is $176.12 per uniform. If it orders more than 50 but less than or equal to 100, the cost is $159.73 per uniform. If it orders more than 100, the cost is $152 per uniform. Find the cost function $C(x)$ as a function of the number of new uniforms $x$ ordered.

83. **Budget: Costs.** The Richmond rowing club is planning to enter the *Head of the Charles* race in Boston and is trying to figure out how much money to raise. The entry fee is $250 per boat for the first 10 boats and $175 for each additional boat. Find the cost function $C(x)$ as a function of the number of boats $x$ the club enters.
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84. Phone Cost: Long-Distance Calling. A phone company charges $0.39 per minute for the first 10 minutes of an international long-distance phone call and $0.12 per minute every minute after that. Find the cost function \( C(x) \) as a function of the length of the phone call \( x \) in minutes.

85. Event Planning. A young couple are planning their wedding reception at a yacht club. The yacht club charges a flat rate of $1000 to reserve the dining room for a private party. The cost of food is $35 per person for the first 100 people and $25 per person for every additional person beyond the first 100. Write the cost function \( C(x) \) as a function of the number of people \( x \) attending the reception.

86. Home Improvement. An irrigation company gives you an estimate for an eight-zone sprinkler system. The parts are $1400, and the labor is $25 per hour. Write a function \( C(x) \) that determines the cost of a new sprinkler system if you choose this irrigation company.

87. Sales. A famous author negotiates with her publisher the monies she will receive for her next suspense novel. She will receive $50,000 up front and a 15% royalty rate on the first 100,000 books sold, and 20% on any books sold beyond that. If the book sells for $20 and royalties are based on the selling price, write a royalties function \( R(x) \) as a function of total number \( x \) of books sold.

88. Sales. Rework Exercise 87 if the author receives $35,000 up front, 15% for the first 100,000 books sold, and 25% on any books sold beyond that.

89. Profit. Some artists are trying to decide whether they will make a profit if they set up a Web-based business to market and sell stained glass that they make. The costs associated with this business are $100 per month for the website and $700 per month for the studio they rent. The materials cost $35 for each work in stained glass, and the artists charge $100 for each unit they sell. Write the monthly profit as a function of the number of stained-glass units they sell.

90. Profit. Philip decides to host a shrimp boil at his house as a fundraiser for his daughter’s AAU basketball team. He orders gulf shrimp to be flown in from New Orleans. The shrimp costs $5 per pound. The shipping costs $30. If he charges $10 per person, write a function \( P(x) \) that represents either his loss or profit as a function of the number of people \( x \) that attend. Assume that each person will eat 1 pound of shrimp.

91. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing flat envelopes first class.

<table>
<thead>
<tr>
<th>Weight Less Than (ounces)</th>
<th>First-Class Rate (Flat Envelopes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.80</td>
</tr>
<tr>
<td>2</td>
<td>$0.97</td>
</tr>
<tr>
<td>3</td>
<td>$1.14</td>
</tr>
<tr>
<td>4</td>
<td>$1.31</td>
</tr>
<tr>
<td>5</td>
<td>$1.48</td>
</tr>
<tr>
<td>6</td>
<td>$1.65</td>
</tr>
<tr>
<td>7</td>
<td>$1.82</td>
</tr>
<tr>
<td>8</td>
<td>$1.99</td>
</tr>
<tr>
<td>9</td>
<td>$2.16</td>
</tr>
<tr>
<td>10</td>
<td>$2.33</td>
</tr>
<tr>
<td>11</td>
<td>$2.50</td>
</tr>
<tr>
<td>12</td>
<td>$2.67</td>
</tr>
<tr>
<td>13</td>
<td>$2.84</td>
</tr>
</tbody>
</table>

92. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing parcels first class.

<table>
<thead>
<tr>
<th>Weight Less Than (ounces)</th>
<th>First-Class Rate (Parcels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.13</td>
</tr>
<tr>
<td>2</td>
<td>$1.30</td>
</tr>
<tr>
<td>3</td>
<td>$1.47</td>
</tr>
<tr>
<td>4</td>
<td>$1.64</td>
</tr>
<tr>
<td>5</td>
<td>$1.81</td>
</tr>
<tr>
<td>6</td>
<td>$1.98</td>
</tr>
<tr>
<td>7</td>
<td>$2.15</td>
</tr>
<tr>
<td>8</td>
<td>$2.32</td>
</tr>
<tr>
<td>9</td>
<td>$2.49</td>
</tr>
<tr>
<td>10</td>
<td>$2.66</td>
</tr>
<tr>
<td>11</td>
<td>$2.83</td>
</tr>
<tr>
<td>12</td>
<td>$3.00</td>
</tr>
<tr>
<td>13</td>
<td>$3.17</td>
</tr>
</tbody>
</table>
A square wave is a waveform used in electronic circuit testing and signal processing. A square wave alternates regularly and instantaneously between two levels.

For Exercises 95 and 96, refer to the following table:

<table>
<thead>
<tr>
<th>Year</th>
<th>Millions of Tons of Carbon</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>500</td>
</tr>
<tr>
<td>1925</td>
<td>1000</td>
</tr>
<tr>
<td>1950</td>
<td>1500</td>
</tr>
<tr>
<td>1975</td>
<td>5000</td>
</tr>
<tr>
<td>2000</td>
<td>7000</td>
</tr>
</tbody>
</table>

95. Climate Change: Global Warming. What is the average rate of change in global carbon emissions from fossil fuel burning from
a. 1900 to 1950?
b. 1950 to 2000?

96. Climate Change: Global Warming. What is the average rate of change in global carbon emissions from fossil fuel burning from
a. 1950 to 1975?
b. 1975 to 2000?

For Exercises 97 and 98, use the following information:

The height (in feet) of a falling object with an initial velocity of 48 feet per second launched straight upward from the ground is given by

\[ h(t) = -16t^2 + 48t, \]

where \( t \) is time (in seconds).

97. Falling Objects. What is the average rate of change of the height as a function of time from \( t = 1 \) to \( t = 2 \)?

98. Falling Objects. What is the average rate of change of the height as a function of time from \( t = 1 \) to \( t = 3 \)?

For Exercises 99 and 100, refer to the following:

An analysis of sales indicates that demand for a product during a calendar year (no leap year) is modeled by

\[ d(t) = 3\sqrt{t^2 + 1} - 2.75t \]

where \( d \) is demand in thousands of units and \( t \) is the day of the year and \( t = 1 \) represents January 1.

99. Economics. Find the average rate of change of the demand of the product over the first quarter.

100. Economics. Find the average rate of change of the demand of the product over the fourth quarter.
In Exercises 101–104, explain the mistake that is made.

101. Graph the piecewise-defined function. State the domain and range.

\[ f(x) = \begin{cases} 
-x & x < 0 \\
 1 & x > 0 
\end{cases} \]

**Solution:**

Draw the graphs of \( f(x) = -x \) and \( f(x) = x \).

Darken the function \( f(x) = -x \) when \( x < 0 \) and the function \( f(x) = x \) when \( x > 0 \). This gives us the familiar absolute value graph.

Domain: \((-\infty, \infty)\) or \(\mathbb{R} \)
Range: \([-1, \infty)\)

This is incorrect. What mistake was made?

103. The cost of airport Internet access is $15 for the first 30 minutes and $1 per minute for each additional minute. Write a function describing the cost of the service as a function of minutes used online.

**Solution:**

\[ C(x) = \begin{cases} 
15 & x \leq 30 \\
15 + x & x > 30 
\end{cases} \]

This is incorrect. What mistake was made?

104. Most money market accounts pay a higher interest with a higher principal. If the credit union is offering 2% on accounts with less than or equal to $10,000 and 4% on the additional money over $10,000, write the interest function \( I(x) \) that represents the interest earned on an account as a function of dollars in the account.

**Solution:**

\[ I(x) = \begin{cases} 
0.02x & x \leq 10,000 \\
0.02(10,000) + 0.04x & x > 10,000 
\end{cases} \]

This is incorrect. What mistake was made?
In exercises 105–108, determine whether each statement is true or false.

105. The identity function is a special case of the linear function.
106. The constant function is a special case of the linear function.
107. If an odd function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.
108. If an even function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.

In exercises 109 and 110, for \( a \) and \( b \) real numbers, can the function given ever be a continuous function? If so, specify the value for \( a \) and \( b \) that would make it so.

109. \( f(x) = \begin{cases} \frac{ax}{x^2} & x \leq 2 \\ bx & x > 2 \end{cases} \)  
110. \( f(x) = \begin{cases} \frac{1}{x} & x < a \\ -\frac{1}{x} & x \geq a \end{cases} \)

In trigonometry you will learn about the tangent function, \( \tan x \). Plot the function \( f(x) = \tan x \), using a graphing utility. If you restrict the values of \( x \) so that \( -\frac{\pi}{2} < x < \frac{\pi}{2} \), the graph should resemble the graph below. Is the tangent function even, odd, or neither?

In trigonometry you will learn about the sine function, \( \sin x \). Plot the function \( f(x) = \sin x \), using a graphing utility. It should look like the graph on the right. Is the sine function even, odd, or neither?

In trigonometry you will learn about the cosine function, \( \cos x \). Plot the function \( f(x) = \cos x \), using a graphing utility. It should look like the graph on the right. Is the cosine function even, odd, or neither?

In trigonometry you will learn about the tangent function, \( \tan x \). Plot the function \( f(x) = \tan x \), using a graphing utility. If you restrict the values of \( x \) so that \( \frac{\pi}{2} < x < \frac{\pi}{2} \), the graph should resemble the graph below. Is the tangent function even, odd, or neither?

Plot the function \( f(x) = \frac{\sin x}{\cos x} \). What function is this?

Graph the function \( f(x) = [3x] \) using a graphing utility. State the domain and range.

Graph the function \( f(x) = \left[ \frac{1}{3x} \right] \) using a graphing utility. State the domain and range.
SECTION 3.3  GRAPHING TECHNIQUES: TRANSFORMATIONS

SKILLS OBJECTIVES
- Sketch the graph of a function using horizontal and vertical shifting of common functions.
- Sketch the graph of a function by reflecting a common function about the x-axis or y-axis.
- Sketch the graph of a function by stretching or compressing a common function.
- Sketch the graph of a function using a sequence of transformations.

CONCEPTUAL OBJECTIVES
- Identify the common functions by their graphs.
- Apply multiple transformations of common functions to obtain graphs of functions.
- Understand that domain and range also are transformed.

Horizontal and Vertical Shifts

The focus of the previous section was to learn the graphs that correspond to particular functions such as identity, square, cube, square root, cube root, absolute value, and reciprocal. Therefore, at this point, you should be able to recognize and generate the graphs of $y = x$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = |x|$, and $y = \frac{1}{x}$. In this section, we will discuss how to sketch the graphs of functions that are very simple modifications of these functions. For instance, a common function may be shifted (horizontally or vertically), reflected, or stretched (or compressed). Collectively, these techniques are called transformations.

Let’s take the absolute value function as an example. The graph of $f(x) = |x|$ was given in the last section. Now look at two examples that are much like this function: $g(x) = |x| + 2$ and $h(x) = |x - 1|$. Graphing these functions by point-plotting yields

\[
\begin{array}{c|c|c}
  x & f(x) & x & g(x) \\
  \hline
  -2 & 2 & -2 & 4 \\
  -1 & 1 & -1 & 3 \\
  0 & 0 & 0 & 2 \\
  1 & 1 & 1 & 3 \\
  2 & 2 & & \\
\end{array}
\]

Instead of point-plotting the function $g(x) = |x| + 2$, we could have started with the function $f(x) = |x|$ and shifted the entire graph up 2 units. Similarly, we could have generated the graph of the function $h(x) = |x - 1|$ by shifting the function $f(x) = |x|$ to the right 1 unit. In both cases, the base or starting function is $f(x) = |x|$. Why did we go up for $g(x)$ and to the right for $h(x)$?

Note that we could rewrite the functions $g(x)$ and $h(x)$ in terms of $f(x)$:

\[
g(x) = |x| + 2 = f(x) + 2 \\
h(x) = |x - 1| = f(x - 1)
\]
In the case of \( g(x) \), the shift \((+2)\) occurs “outside” the function—that is, outside the parentheses showing the argument. Therefore, the output for \( g(x) \) is two more than the typical output for \( f(x) \). Because the output corresponds to the vertical axis, this results in a shift \textit{upward} of two units. In general, shifts that occur \textit{outside} the function correspond to a \textit{vertical} shift corresponding to the sign of the shift. For instance, had the function been \( G(x) = |x| - 2 \), this graph would have started with the graph of the function \( f(x) \) and shifted down two units.

In the case of \( h(x) \), the shift occurs “inside” the function—that is, inside the parentheses showing the argument. Note that the point \((0, 0)\) that lies on the graph of \( f(x) \) was shifted to the point \((1, 0)\) on the graph of the function \( h(x) \). The \( y \)-value remained the same, but the \( x \)-value shifted to the right one unit. Similarly, the points \((-1, 1)\) and \((1, 1)\) were shifted to the points \((0, 1)\) and \((2, 1)\), respectively. In general, shifts that occur \textit{inside} the function correspond to a \textit{horizontal} shift opposite the sign. In this case, the graph of the function \( h(x) = |x - 1| \) shifted the graph of the function \( f(x) \) to the right one unit. If, instead, we had the function \( H(x) = |x + 1| \), this graph would have started with the graph of the function \( f(x) \) and shifted to the left one unit.

### Vertical shifts

**Assuming that \( c \) is a positive constant,**

<table>
<thead>
<tr>
<th>To Graph</th>
<th>Shift the Graph of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) + c )</td>
<td>( c ) units upward</td>
</tr>
<tr>
<td>( f(x) - c )</td>
<td>( c ) units downward</td>
</tr>
</tbody>
</table>

Adding or subtracting a constant \textit{outside} the function corresponds to a \textit{vertical} shift that goes \textit{with the sign}.

### Horizontal shifts

**Assuming that \( c \) is a positive constant,**

<table>
<thead>
<tr>
<th>To Graph</th>
<th>Shift the Graph of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x + c) )</td>
<td>( c ) units to the left</td>
</tr>
<tr>
<td>( f(x - c) )</td>
<td>( c ) units to the right</td>
</tr>
</tbody>
</table>

Adding or subtracting a constant \textit{inside} the function corresponds to a \textit{horizontal} shift that goes \textit{opposite the sign}.  

---

**Study Tip**

Shifts outside the function are \textit{vertical} shifts with the sign.

- Up (+)
- Down (−)

**Study Tip**

Shifts inside the function are \textit{horizontal} shifts opposite the sign.

- Left (+)
- Right (−)
EXAMPLE 1  Horizontal and Vertical Shifts

Sketch the graphs of the given functions using horizontal and vertical shifts:

a. \( g(x) = x^2 - 1 \)

b. \( H(x) = (x + 1)^2 \)

Solution:

In both cases, the function to start with is \( f(x) = x^2 \).

a. \( g(x) = x^2 - 1 \) can be rewritten as \( g(x) = f(x) - 1 \).

1. The shift (one unit) occurs outside of the function. Therefore, we expect a vertical shift that goes with the sign.
2. Since the sign is negative, this corresponds to a downward shift.
3. Shifting the graph of the function \( f(x) = x^2 \) down one unit yields the graph of \( g(x) = x^2 - 1 \).

b. \( H(x) = (x + 1)^2 \) can be rewritten as \( H(x) = f(x + 1) \).

1. The shift (one unit) occurs inside of the function. Therefore, we expect a horizontal shift that goes opposite the sign.
2. Since the sign is positive, this corresponds to a shift to the left.
3. Shifting the graph of the function \( f(x) = x^2 \) to the left one unit yields the graph of \( H(x) = (x + 1)^2 \).

Answer:

a. Graphs of \( y_1 = x^2 \) and \( y_2 = g(x) = x^2 - 1 \) are shown.

b. Graphs of \( y_1 = x^2 \) and \( y_2 = H(x) = (x + 1)^2 \) are shown.

YOUR TURN  Sketch the graphs of the given functions using horizontal and vertical shifts.

a. \( g(x) = x^2 + 1 \)

b. \( H(x) = (x - 1)^2 \)

It is important to note that the domain and range of the resulting function can be thought of as also being shifted. Shifts in the domain correspond to horizontal shifts, and shifts in the range correspond to vertical shifts.
EXAMPLE 2  Horizontal and Vertical Shifts and Changes in the Domain and Range

Graph the functions using translations and state the domain and range of each function.

a. \( g(x) = \sqrt{x + 1} \)  

b. \( G(x) = \sqrt{x} - 2 \)

Solution:

In both cases the function to start with is \( f(x) = \sqrt{x} \).

\[ \text{Domain: } [0, \infty) \]
\[ \text{Range: } [0, \infty) \]

a. \( g(x) = \sqrt{x + 1} \) can be rewritten as 
\( g(x) = f(x + 1) \).
1. The shift (one unit) is inside the function, which corresponds to a horizontal shift opposite the sign.
2. Shifting the graph of \( f(x) = \sqrt{x} \) to the left one unit yields the graph of \( g(x) = \sqrt{x + 1} \). Notice that the point \((0, 0)\), which lies on the graph of \( f(x) \), gets shifted to the point \((-1, 0)\) on the graph of \( g(x) \).

Although the original function \( f(x) = \sqrt{x} \) had an implicit restriction on the domain: \([0, \infty)\), the function \( g(x) = \sqrt{x + 1} \) has the implicit restriction that \( x \geq -1 \). We see that the output or range of \( g(x) \) is the same as the output of the original function \( f(x) \).

\[ \text{Domain: } [-1, \infty) \]
\[ \text{Range: } [0, \infty) \]

b. \( G(x) = \sqrt{x} - 2 \) can be rewritten as 
\( G(x) = f(x) - 2 \).
1. The shift (two units) is outside the function, which corresponds to a vertical shift with the sign.
2. The graph of \( G(x) = \sqrt{x} - 2 \) is found by shifting \( f(x) = \sqrt{x} \) down two units. Note that the point \((0, 0)\), which lies on the graph of \( f(x) \), gets shifted to the point \((0, -2)\) on the graph of \( G(x) \).

The original function \( f(x) = \sqrt{x} \) has an implicit restriction on the domain: \([0, \infty)\). The function \( G(x) = \sqrt{x} - 2 \) also has the implicit restriction that \( x \geq 0 \). The output or range of \( G(x) \) is always two units less than the output of the original function \( f(x) \).

\[ \text{Domain: } [0, \infty) \]
\[ \text{Range: } [-2, \infty) \]

YOUR TURN Sketch the graph of the functions using shifts and state the domain and range.

a. \( G(x) = \sqrt{x - 2} \)  
b. \( h(x) = |x| + 1 \)
The previous examples have involved graphing functions by shifting a known function either in the horizontal or vertical direction. Let us now look at combinations of horizontal and vertical shifts.

**EXAMPLE 3  Combining Horizontal and Vertical Shifts**

Sketch the graph of the function \( F(x) = (x + 1)^2 - 2 \). State the domain and range of \( F \).

**Solution:**

The base function is \( y = x^2 \).

1. The shift (one unit) is inside the function, so it represents a **horizontal shift** opposite the sign.
2. The \(-2\) shift is outside the function, which represents a **vertical shift** with the sign.
3. Therefore, we shift the graph of \( y = x^2 \) to the left one unit and down two units. For instance, the point \((0, 0)\) on the graph of \( y = x^2 \) shifts to the point \((-1, -2)\) on the graph of \( F(x) = (x + 1)^2 - 2 \).

**Domain:** \((-\infty, \infty)\)  **Range:** \([-2, \infty)\)

**YOUR TURN** Sketch the graph of the function \( f(x) = |x - 2| + 1 \). State the domain and range of \( f \).

All of the previous transformation examples involve starting with a common function and shifting the function in either the horizontal or the vertical direction (or a combination of both). Now, let’s investigate **reflections** of functions about the \( x \)-axis or \( y \)-axis.

**Reflection about the Axes**

To sketch the graphs of \( f(x) = x^2 \) and \( g(x) = -x^2 \) start by first listing points that are on each of the graphs and then connecting the points with smooth curves.
3.3 Graphing Techniques: Transformations

Note that if the graph of \( f(x) = x^2 \) is reflected about the \( x \)-axis, the result is the graph of \( g(x) = -x^2 \). Also note that the function \( g(x) \) can be written as the negative of the function \( f(x) \); that is, \( g(x) = -f(x) \). In general, \textit{reflection about the \( x \)-axis} is produced by multiplying a function by \(-1\).

Let’s now investigate reflection about the \( y \)-axis. To sketch the graphs of \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{-x} \) start by listing points that are on each of the graphs and then connecting the points with smooth curves.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that if the graph of \( f(x) = \sqrt{x} \) is reflected about the \( y \)-axis, the result is the graph of \( g(x) = \sqrt{-x} \). Also note that the function \( g(x) \) can be written as \( g(x) = f(-x) \). In general, \textit{reflection about the \( y \)-axis} is produced by replacing \( x \) with \(-x\) in the function. Notice that the domain of \( f \) is \([0, \infty)\), whereas the domain of \( g \) is \((-\infty, 0]\).

**Reflection about the axes**

The graph of \(-f(x)\) is obtained by reflecting the graph of \( f(x) \) about the \( x \)-axis. The graph of \( f(-x) \) is obtained by reflecting the graph of \( f(x) \) about the \( y \)-axis.

**Example 4**  Sketching the Graph of a Function Using Both Shifts and Reflections

Sketch the graph of the function \( G(x) = -\sqrt{x+1} \).

**Solution:**

Start with the square root function.

Shift the graph of \( f(x) \) to the left one unit to arrive at the graph of \( f(x + 1) \).

Reflect the graph of \( f(x + 1) \) about the \( x \)-axis to arrive at the graph of \(-f(x + 1)\).
Technology Tip

a. Graphs of \( y_1 = \sqrt{x}, y_2 = \sqrt{x + 2}, \)
\( y_3 = -\sqrt{x + 2}, \) and
\( y_4 = f(x) = \sqrt{2 - x} + 1 \) are shown.

\[ \text{Plot 1 Plot 2 Plot 3} \]
\[ \text{\( y_1 = \sqrt{x} \)} \]
\[ \text{\( y_2 = \sqrt{x + 2} \)} \]
\[ \text{\( y_3 = -\sqrt{x + 2} \)} \]
\[ \text{\( y_4 = f(x) = \sqrt{2 - x} + 1 \)} \]

- **Answer:**
  - Domain: \([1, \infty)\)
  - Range: \((-\infty, 2]\)

EXAMPLE 5  Sketching the Graph of a Function Using Both Shifts and Reflections

Sketch the graph of the function \( f(x) = \sqrt{2 - x} + 1 \).

**Solution:**

Start with the square root function.

- Shift the graph of \( g(x) \) to the left two units to arrive at the graph of \( g(x + 2) \).
- Reflect the graph of \( g(x + 2) \) about the \( y \)-axis to arrive at the graph of \( g(-x + 2) \).
- Shift the graph \( g(-x + 2) \) up one unit to arrive at the graph of \( g(-x + 2) + 1 \).

\[ g(x) = \sqrt{x} \]
\[ g(x + 2) = \sqrt{x + 2} \]
\[ g(-x + 2) = \sqrt{-x + 2} \]
\[ g(-x + 2) + 1 = \sqrt{2 - x} + 1 \]

**YOUR TURN** Use shifts and reflections to sketch the graph of the function \( f(x) = -\sqrt{x - 1} + 2 \). State the domain and range of \( f(x) \).

Look back at the order in which transformations were performed in Example 5: horizontal shift, reflection, and then vertical shift. Let us consider an alternate order of transformations.

**Words**

Start with the square root function.

- Shift the graph of \( g(x) \) up one unit to arrive at the graph of \( g(x + 1) \).
- Reflect the graph of \( g(x) + 1 \) about the \( y \)-axis to arrive at the graph of \( g(-x) + 1 \).
- Replace \( x \) with \( x - 2 \), which corresponds to a shift of the graph of \( g(-x) + 1 \) to the right two units to arrive at the graph of \( g(-x + 2) + 1 \).

In the last step we replaced \( x \) with \( x - 2 \), which required us to think ahead knowing the desired result was \( 2 - x \) inside the radical.

To avoid any possible confusion, follow this order of transformations:

1. Horizontal shifts: \( f(x \pm c) \)
2. Reflection: \( f(-x) \) and/or \(-f(x)\)
3. Vertical shifts: \( f(x) \pm c \)
Stretching and Compressing

Horizontal shifts, vertical shifts, and reflections change only the position of the graph in the Cartesian plane, leaving the basic shape of the graph unchanged. These transformations (shifts and reflections) are called **rigid transformations** because they alter only the position. **Nonrigid transformations**, on the other hand, distort the shape of the original graph. We now consider stretching and compressing of graphs in both the vertical and the horizontal direction.

A vertical stretch or compression of a graph occurs when the function is multiplied by a positive constant. For example, the graphs of the functions $f(x) = x^2$, $g(x) = 2f(x) = 2x^2$, and $h(x) = \frac{1}{2}f(x) = \frac{1}{2}x^2$ are illustrated below. Depending on if the constant is larger than 1 or smaller than 1 will determine whether it corresponds to a stretch (expansion) or compression (contraction) in the vertical direction.

Note that when the function $f(x) = x^2$ is multiplied by 2, so that $g(x) = 2f(x) = 2x^2$, the result is a graph stretched in the vertical direction. When the function $f(x) = x^2$ is multiplied by $\frac{1}{2}$, so that $h(x) = \frac{1}{2}f(x) = \frac{1}{2}x^2$, the result is a graph that is compressed in the vertical direction.

### Vertical Stretching and Vertical Compressing of Graphs

The graph of $cf(x)$ is found by:
- **Vertically stretching** the graph of $f(x)$ if $c > 1$
- **Vertically compressing** the graph of $f(x)$ if $0 < c < 1$

*Note: $c$ is any positive real number.*

#### EXAMPLE 6 Vertically Stretching and Compressing Graphs

Graph the function $h(x) = \frac{1}{4}x^3$.

**Solution:**
1. Start with the cube function.
2. Vertical compression is expected because $\frac{1}{4}$ is less than 1.

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
-2 & 4 \\
-1 & 1 \\
0 & 0 \\
1 & 1 \\
2 & 4 \\
\end{array}
\quad
\begin{array}{c|c}
 x & g(x) \\
\hline
-2 & 8 \\
-1 & 2 \\
0 & 0 \\
1 & 2 \\
2 & 8 \\
\end{array}
\quad
\begin{array}{c|c}
 x & h(x) \\
\hline
-2 & 2 \\
-1 & 1 \\
0 & 0 \\
1 & 1 \\
2 & 2 \\
\end{array}
\]
3. Determine a few points that lie on the graph of $h$.

$$(0, 0) \quad (2, 2) \quad (-2, -2)$$

Conversely, if the argument $x$ of a function $f$ is multiplied by a positive real number $c$, then the result is a horizontal stretch of the graph of $f$ if $0 < c < 1$. If $c > 1$, then the result is a horizontal compression of the graph of $f$.

**HORIZONTAL STRETCHING AND HORIZONTAL COMPRRESSING OF GRAPHS**

The graph of $f(cx)$ is found by:
- **Horizontally stretching** the graph of $f(x)$ if $0 < c < 1$
- **Horizontally compressing** the graph of $f(x)$ if $c > 1$

*Note: $c$ is any positive real number.*

---

**EXAMPLE 7** Vertically Stretching and Horizontally Compressing Graphs

Given the graph of $f(x)$, graph:

a. $2f(x)$  

b. $f(2x)$

**Solution (a):**

Since the function is multiplied (on the outside) by 2, the result is that each $y$-value of $f(x)$ is **multiplied by 2**, which corresponds to vertical stretching.
EXAMPLE 8 Sketching the Graph of a Function Using Multiple Transformations

Sketch the graph of the function \( H(x) = -2(x - 3)^2 \).

Solution:

Start with the square function, \( f(x) = x^2 \)

Shift the graph of \( f(x) \) to the right three units to arrive at the graph of \( f(x - 3) \).

Vertically stretch the graph of \( f(x - 3) \) by a factor of 2 to arrive at the graph of \( 2f(x - 3) \).

Reflect the graph \( 2f(x - 3) \) about the \( x \)-axis to arrive at the graph of \( -2f(x - 3) \).

In Example 8 we followed the same “inside out” approach with the functions to determine the order for the transformations: horizontal shift, vertical stretch, and reflection.

\[ \text{Answer: Stretching of the graph } f(x) = x^2. \]
In Exercises 1–12, match the function to the graph.

1. \( f(x) = x^2 + 1 \)
2. \( f(x) = (x - 1)^2 \)
3. \( f(x) = -(1 - x)^2 \)
4. \( f(x) = -x^2 - 1 \)
5. \( f(x) = -(x + 1)^2 \)
6. \( f(x) = -(1 - x)^2 + 1 \)
7. \( f(x) = \sqrt{x - 1} + 1 \)
8. \( f(x) = -\sqrt{x - 1} \)
9. \( f(x) = \sqrt{1 - x} - 1 \)
10. \( f(x) = \sqrt{-x} + 1 \)
11. \( f(x) = -\sqrt{-x} + 1 \)
12. \( f(x) = -\sqrt{1 - x} - 1 \)

**Transparency**

<table>
<thead>
<tr>
<th>Transformation</th>
<th>To graph the function...</th>
<th>Draw the graph of ( f ) and then...</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal shifts</td>
<td>( f(x + c) )</td>
<td>Shift the graph of ( f ) to the left ( c ) units.</td>
<td>Replace ( x ) by ( x + c ).</td>
</tr>
<tr>
<td></td>
<td>( f(x - c) )</td>
<td>Shift the graph of ( f ) to the right ( c ) units.</td>
<td>Replace ( x ) by ( x - c ).</td>
</tr>
<tr>
<td>Vertical shifts</td>
<td>( f(x) + c )</td>
<td>Shift the graph of ( f ) up ( c ) units.</td>
<td>Add ( c ) to ( f(x) ).</td>
</tr>
<tr>
<td></td>
<td>( f(x) - c )</td>
<td>Shift the graph of ( f ) down ( c ) units.</td>
<td>Subtract ( c ) from ( f(x) ).</td>
</tr>
<tr>
<td>Reflection about the x-axis</td>
<td>( -f(x) )</td>
<td>Reflect the graph of ( f ) about the x-axis.</td>
<td>Multiply ( f(x) ) by (-1).</td>
</tr>
<tr>
<td>Reflection about the y-axis</td>
<td>( f(-x) )</td>
<td>Reflect the graph of ( f ) about the y-axis.</td>
<td>Replace ( x ) by (-x).</td>
</tr>
<tr>
<td>Vertical stretch</td>
<td>( cf(x) ), where ( c &gt; 1 )</td>
<td>Vertically stretch the graph of ( f ).</td>
<td>Multiply ( f(x) ) by ( c ).</td>
</tr>
<tr>
<td>Vertical compression</td>
<td>( cf(x) ), where ( 0 &lt; c &lt; 1 )</td>
<td>Vertically compress the graph of ( f ).</td>
<td>Multiply ( f(x) ) by ( c ).</td>
</tr>
<tr>
<td>Horizontal stretch</td>
<td>( f(cx) ), where ( 0 &lt; c &lt; 1 )</td>
<td>Horizontally stretch the graph of ( f ).</td>
<td>Replace ( x ) by ( cx ).</td>
</tr>
<tr>
<td>Horizontal compression</td>
<td>( f(cx) ), where ( c &gt; 1 )</td>
<td>Horizontally compress the graph of ( f ).</td>
<td>Replace ( x ) by ( cx ).</td>
</tr>
</tbody>
</table>
In Exercises 13–18, write the function whose graph is the graph of \( y = |x| \), but is transformed accordingly.

13. Shifted up three units
14. Shifted to the left four units
15. Reflected about the \( y \)-axis
16. Reflected about the \( x \)-axis
17. Vertically stretched by a factor of 3
18. Vertically compressed by a factor of 3

In Exercises 19–24, write the function whose graph is the graph of \( y = x^3 \), but is transformed accordingly.

19. Shifted down four units
20. Shifted to the right three units
21. Shifted up three units and to the left one unit
22. Reflected about the \( x \)-axis
23. Reflected about the \( y \)-axis
24. Reflected about both the \( x \)-axis and the \( y \)-axis

In Exercises 25–48, use the given graph to sketch the graph of the indicated functions.

25.  
26.  
27.  
28.  

a. \( y = f(x - 2) \)  
  b. \( y = f(x) - 2 \)  
  a. \( y = f(x + 2) \)  
  b. \( y = f(x) + 2 \)  
  a. \( y = f(x) - 3 \)  
  b. \( y = f(x - 3) \)  
  a. \( y = f(x) + 3 \)  
  b. \( y = f(x + 3) \)
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29.  
30.  
31.  
32.  

\[
\begin{align*}
\text{a. } y &= -f(x) \\
\text{b. } y &= f(-x)
\end{align*}
\]

\[
\begin{align*}
\text{a. } y &= -f(x) \\
\text{b. } y &= f(-x)
\end{align*}
\]

\[
\begin{align*}
\text{a. } y &= 2f(x) \\
\text{b. } y &= f(2x)
\end{align*}
\]

\[
\begin{align*}
\text{a. } y &= 2f(x) \\
\text{b. } y &= f(2x)
\end{align*}
\]

33.  
34.  
35.  
36.  

\[
\begin{align*}
y &= f(x - 2) - 3 \\
y &= f(x + 1) - 2 \\
y &= -f(x - 1) + 2 \\
y &= -2f(x) + 1
\end{align*}
\]

37.  
38.  
39.  
40.  

\[
\begin{align*}
y &= -\frac{1}{2}g(x) \\
y &= \frac{1}{3}g(-x) \\
y &= -g(2x) \\
y &= g(\frac{1}{2}x)
\end{align*}
\]

41.  
42.  
43.  
44.  

\[
\begin{align*}
y &= \frac{1}{2}F(x - 1) + 2 \\
y &= \frac{1}{2}F(-x) \\
y &= -F(1 - x) \\
y &= -F(x - 2) - 1
\end{align*}
\]

45.  
46.  
47.  
48.  

\[
\begin{align*}
y &= 2G(x + 1) - 4 \\
y &= 2G(-x) + 1 \\
y &= -2G(x - 1) + 3 \\
y &= -G(x - 2) - 1
\end{align*}
\]

In Exercises 49–74, graph the function using transformations.

49.  
50.  
51.  
52.  

\[
\begin{align*}
y &= x^2 - 2 \\
y &= x^2 + 3 \\
y &= (x + 1)^2 \\
y &= (x - 2)^2
\end{align*}
\]

53.  
54.  
55.  
56.  

\[
\begin{align*}
y &= (x - 3)^2 + 2 \\
y &= (x + 2)^2 + 1 \\
y &= -(1 - x)^2 \\
y &= -(x + 2)^2
\end{align*}
\]

57.  
58.  
59.  
60.  

\[
\begin{align*}
y &= |x| \\
y &= -|x| \\
y &= -|x + 2| - 1 \\
y &= |1 - x| + 2
\end{align*}
\]

61.  
62.  
63.  
64.  

\[
\begin{align*}
y &= 2x^2 + 1 \\
y &= 2|x| + 1 \\
y &= -\sqrt{x - 2} \\
y &= \sqrt{2 - x}
\end{align*}
\]

65.  
66.  
67.  
68.  

\[
\begin{align*}
y &= -\sqrt{2 + x - 1} \\
y &= \sqrt{2 - x + 3} \\
y &= \sqrt{x - 1} + 2 \\
y &= \sqrt{x + 2} - 1
\end{align*}
\]

69.  
70.  
71.  
72.  

\[
\begin{align*}
y &= \frac{1}{x + 2} + 2 \\
y &= \frac{1}{3 - x} \\
y &= 2 - \frac{1}{x + 2} \\
y &= 2 - \frac{1}{1 - x}
\end{align*}
\]
In Exercises 75–80, transform the function into the form \( f(x) = c(x - h)^2 + k \), where \( c, k \), and \( h \) are constants, by completing the square. Use graph-shifting techniques to graph the function.

75. \( y = x^2 - 6x + 11 \)  
76. \( f(x) = x^2 + 2x - 2 \)  
77. \( f(x) = -x^2 - 2x \)  
78. \( f(x) = -x^2 + 6x - 7 \)  
79. \( f(x) = 2x^2 - 8x + 3 \)  
80. \( f(x) = 3x^2 - 6x + 5 \)

### Applications

81. **Salary.** A manager hires an employee at a rate of $10 per hour. Write the function that describes the current salary of the employee as a function of the number of hours worked per week, \( x \). After a year, the manager decides to award the employee a raise equivalent to paying him for an additional 5 hours per week. Write a function that describes the salary of the employee after the raise.

82. **Profit.** The profit associated with St. Augustine sod in Florida is typically \( P(x) = -x^2 + 14,000 - 48,700,000 \), where \( x \) is the number of pallets sold per year in a normal year. In rainy years Sod King gives away 10 free pallets per year. Write the function that describes the profit of \( x \) pallets of sod in rainy years.

83. **Taxes.** Every year in the United States each working American typically pays in taxes a percentage of his or her earnings (minus the standard deduction). Karen’s 2011 taxes were calculated based on the formula \( T(x) = 0.22(x - 6500) \). That year the standard deduction was $6500 and her tax bracket paid 22% in taxes. Write the function that will determine her 2012 taxes, assuming she receives the raise that places her in the 33% bracket.

84. **Medication.** The amount of medication that an infant requires is typically a function of the baby’s weight. The number of milliliters of an antiseizure medication \( A \) is given by \( A(x) = \sqrt{x} + 2 \), where \( x \) is the weight of the infant in ounces. In emergencies there is often not enough time to weigh the infant, so nurses have to estimate the baby’s weight. What is the function that represents the actual amount of medication the infant is given if his weight is overestimated by 3 ounces?

### Catch the Mistake

In Exercises 87–90, explain the mistake that is made.

87. Describe a procedure for graphing the function \( f(x) = \sqrt{x} - 3 + 2 \).

**Solution:**

a. Start with the function \( f(x) = \sqrt{x} \).
b. Shift the function to the left three units.
c. Shift the function up two units.

d. This is incorrect. What mistake was made?

88. Describe a procedure for graphing the function \( f(x) = -\sqrt{x} + 2 - 3 \).

**Solution:**

a. Start with the function \( f(x) = \sqrt{x} \).
b. Shift the function to the left two units.
c. Reflect the function about the y-axis.
d. Shift the function down three units.

d. This is incorrect. What mistake was made?
CHAPTER 3 Functions and Their Graphs

89. Describe a procedure for graphing the function 
\[ f(x) = |3 - x| + 1. \]
Solution:
  a. Start with the function \( f(x) = |x| \).
  b. Reflect the function about the y-axis.
  c. Shift the function to the left three units.
  d. Shift the function up one unit.
This is incorrect. What mistake was made?

90. Describe a procedure for graphing the function 
\[ f(x) = -2x^2 + 1. \]
Solution:
  a. Start with the function \( f(x) = x^2 \).
  b. Reflect the function about the y-axis.
  c. Shift the function up one unit.
  d. Expand in the vertical direction by a factor of 2.
This is incorrect. What mistake was made?

**CONCEPTUAL**

In Exercises 91–94, determine whether each statement is true or false.

91. The graph of \( y = |−x| \) is the same as the graph of \( y = |x| \).
92. The graph of \( y = \sqrt{-x} \) is the same as the graph of \( y = \sqrt{x} \).
93. If the graph of an odd function is reflected about the x-axis and then the y-axis, the result is the graph of the original odd function.
94. If the graph of \( y = \frac{1}{x} \) is reflected about the x-axis, it produces the same graph as if it had been reflected about the y-axis.

**CHALLENGE**

95. The point \((a, b)\) lies on the graph of the function \( y = f(x) \). What point is guaranteed to lie on the graph of \( f(x - 3) + 2 \)?
96. The point \((a, b)\) lies on the graph of the function \( y = f(x) \). What point is guaranteed to lie on the graph of \(-f(-x) + 1\)?

**TECHNOLOGY**

97. Use a graphing utility to graph:
  a. \( y = x^2 - 2 \) and \( y = |x^2 - 2| \)
  b. \( y = x^3 - 1 \) and \( y = |x^3 + 1| \)
What is the relationship between \( f(x) \) and \( |f(x)| \)?
98. Use a graphing utility to graph:
  a. \( y = x^2 - 2 \) and \( y = |x|^2 - 2 \)
  b. \( y = x^3 + 1 \) and \( y = |x|^3 + 1 \)
What is the relationship between \( f(x) \) and \( f(|x|) \)?
99. Use a graphing utility to graph:
  a. \( y = \sqrt{x} \) and \( y = \sqrt{0.1x} \)
  b. \( y = \sqrt{x} \) and \( y = \sqrt{10x} \)
What is the relationship between \( f(x) \) and \( f(ax) \), assuming that \( a \) is positive?
100. Use a graphing utility to graph:
  a. \( y = \sqrt{x} \) and \( y = 0.1\sqrt{x} \)
  b. \( y = \sqrt{x} \) and \( y = 10\sqrt{x} \)
What is the relationship between \( f(x) \) and \( af(x) \), assuming that \( a \) is positive?
101. Use a graphing utility to graph \( y = f(x) = [[0.5x]] + 1 \).
  Use transforms to describe the relationship between \( f(x) \) and \( y = [[x]] \).
102. Use a graphing utility to graph \( y = g(x) = 0.5[[x]] + 1 \).
  Use transforms to describe the relationship between \( g(x) \) and \( y = [[x]] \).
Two different functions can be combined using mathematical operations such as addition, subtraction, multiplication, and division. Also, there is an operation on functions called composition, which can be thought of as a function of a function. When we combine functions, we do so algebraically. Special attention must be paid to the domain and range of the combined functions.

Adding, Subtracting, Multiplying, and Dividing Functions

Consider the two functions \( f(x) = x^2 + 2x - 3 \) and \( g(x) = x + 1 \). The domain of both of these functions is the set of all real numbers. Therefore, we can add, subtract, or multiply these functions for any real number \( x \).

Addition: \( f(x) + g(x) = x^2 + 2x - 3 + x + 1 = x^2 + 3x - 2 \)

The result is in fact a new function, which we denote:

\( (f + g)(x) = x^2 + 3x - 2 \)

This is the sum function.

Subtraction: \( f(x) - g(x) = x^2 + 2x - 3 - (x + 1) = x^2 + x - 4 \)

The result is in fact a new function, which we denote:

\( (f - g)(x) = x^2 + x - 4 \)

This is the difference function.

Multiplication: \( f(x) \cdot g(x) = (x^2 + 2x - 3)(x + 1) = x^3 + 3x^2 - x - 3 \)

The result is in fact a new function, which we denote:

\( (f \cdot g)(x) = x^3 + 3x^2 - x - 3 \)

This is the product function.

Although both \( f \) and \( g \) are defined for all real numbers \( x \), we must restrict \( x \) so that \( x \neq -1 \) to form the quotient \( \frac{f}{g} \).

Division: \( \frac{f(x)}{g(x)} = \frac{x^2 + 2x - 3}{x + 1}, \quad x \neq -1 \)

The result is in fact a new function, which we denote:

\( \left( \frac{f}{g} \right)(x) = \frac{x^2 + 2x - 3}{x + 1}, \quad x \neq -1 \)

This is called the quotient function.

Two functions can be added, subtracted, and multiplied. The resulting function domain is therefore the intersection of the domains of the two functions. However, for division, any value of \( x \) (input) that makes the denominator equal to zero must be eliminated from the domain.
The previous examples involved polynomials. The domain of any polynomial is the set of all real numbers. Adding, subtracting, and multiplying polynomials result in other polynomials, which have domains of all real numbers. Let’s now investigate operations applied to functions that have a restricted domain.

The domain of the sum function, difference function, or product function is the intersection of the individual domains of the two functions. The quotient function has a similar domain in that it is the intersection of the two domains. However, any values that make the numerator zero must also be eliminated.

\[
\text{Function} \quad \text{Notation} \quad \text{Domain}
\]

- **Sum** \( (f + g)(x) = f(x) + g(x) \) \{domain of \( f \) \( \cap \) domain of \( g \) \}
- **Difference** \( (f - g)(x) = f(x) - g(x) \) \{domain of \( f \) \( \cap \) domain of \( g \) \}
- **Product** \( (f \cdot g)(x) = f(x) \cdot g(x) \) \{domain of \( f \) \( \cap \) domain of \( g \) \}
- **Quotient** \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \) \{domain of \( f \) \( \cap \) \{domain of \( g \) \} \( \cap \) \{\( g(x) \neq 0 \)\} \}

We can think of this in the following way: Any number that is in the domain of both the functions is in the domain of the combined function. The exception to this is the quotient function, which also eliminates values that make the denominator equal to zero.

**EXAMPLE 1  Operations on Functions: Determining Domains of New Functions**

For the functions \( f(x) = \sqrt{x - 1} \) and \( g(x) = \sqrt{4 - x} \), determine the sum function, difference function, product function, and quotient function. State the domain of these four new functions.

**Solution:**

- **Sum function:** \( f(x) + g(x) = \sqrt{x - 1} + \sqrt{4 - x} \)
- **Difference function:** \( f(x) - g(x) = \sqrt{x - 1} - \sqrt{4 - x} \)
- **Product function:** \( f(x) \cdot g(x) = \sqrt{x - 1} \cdot \sqrt{4 - x} = \sqrt{(x - 1)(4 - x)} = \sqrt{-x^2 + 5x - 4} \)
- **Quotient function:**

\[
\frac{f(x)}{g(x)} = \frac{\sqrt{x - 1}}{\sqrt{4 - x}} = \sqrt{\frac{x - 1}{4 - x}}
\]

The domain of the square root function is determined by setting the argument under the radical greater than or equal to zero.

- Domain of \( f(x) \): \([1, \infty)\)
- Domain of \( g(x) \): \((-\infty, 4]\)

The domain of the sum, difference, and product functions is

\[ [1, \infty) \cap (-\infty, 4] = [1, 4] \]

The quotient function has the additional constraint that the denominator cannot be zero. This implies that \( x \neq 4 \), so the domain of the quotient function is \([1, 4)\).

**YOUR TURN** Given the function \( f(x) = \sqrt{x + 3} \) and \( g(x) = \sqrt{1 - x} \), find \( (f + g)(x) \) and state its domain.
### Example 2  Quotient Function and Domain Restrictions

Given the functions \( F(x) = \sqrt{x} \) and \( G(x) = |x - 3| \), find the quotient function, \( \left( \frac{F}{G} \right)(x) \), and state its domain.

**Solution:**

The quotient function is written as
\[
\left( \frac{F}{G} \right)(x) = \frac{F(x)}{G(x)} = \frac{\sqrt{x}}{|x - 3|}
\]

Domain of \( F(x) \): \([0, \infty)\)  
Domain of \( G(x) \): \((-\infty, \infty)\)

The real numbers that are in both the domain for \( F(x) \) and the domain for \( G(x) \) are represented by the intersection \([0, \infty) \cap (-\infty, \infty) = [0, \infty)\). Also, the denominator of the quotient function is equal to zero when \( x = 3 \), so we must eliminate this value from the domain.

\[
\text{Domain of } \left( \frac{F}{G} \right)(x) : [0, 3) \cup (3, \infty)
\]

**Your Turn** For the functions given in Example 2, determine the quotient function \( \left( \frac{G}{F} \right)(x) \), and state its domain.

### Composition of Functions

Recall that a function maps every element in the domain to exactly one corresponding element in the range as shown in the figure on the right.

Suppose there is a sales rack of clothes in a department store. Let \( x \) correspond to the original price of each item on the rack. These clothes have recently been marked down 20%. Therefore, the function \( g(x) = 0.80x \) represents the current sale price of each item.

You have been invited to a special sale that lets you take 10% off the current sale price and an additional $5 off every item at checkout. The function \( f(g(x)) = 0.90g(x) - 5 \) determines the checkout price. Note that the output of the function \( g \) is the input of the function \( f \) as shown in the figure below.

![Composition of Functions Diagram](image)

This is an example of a composition of functions, when the output of one function is the input of another function. It is commonly referred to as a function of a function.

An algebraic example of this is the function \( y = \sqrt{x^2 - 2} \). Suppose we let \( g(x) = x^2 - 2 \) and \( f(x) = \sqrt{x} \). Recall that the independent variable in function notation is a placeholder. Since \( f(x) = \sqrt{x} \), then \( f(g(x)) = \sqrt{g(x)} \). Substituting the expression for \( g(x) \), we find \( f(g(x)) = \sqrt{x^2 - 2} \). The function \( y = \sqrt{x^2 - 2} \) is said to be a composite function, \( y = f(g(x)) \).
Note that the domain of \( g(x) \) is the set of all real numbers, and the domain of \( f(x) \) is the set of all nonnegative numbers. The domain of a composite function is the set of all \( x \) such that \( g(x) \) is in the domain of \( f \). For instance, in the composite function \( y = f(g(x)) \), we know that the allowable inputs into \( f \) are all numbers greater than or equal to zero. Therefore, we restrict the outputs of \( g(x) \geq 0 \) and find the corresponding \( x \)-values. Those \( x \)-values are the only allowable inputs and constitute the domain of the composite function \( y = f(g(x)) \).

The symbol that represents composition of functions is a small open circle; thus \((f \circ g)(x) = f(g(x))\) and is read aloud as “\( f \) of \( g \).” It is important not to confuse this with the multiplication sign: \((f \cdot g)(x) = f(x)g(x)\).

### Composition of functions

Given two functions \( f \) and \( g \), there are two composite functions that can be formed.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Words</th>
<th>Definition</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \circ g )</td>
<td>( f ) composed with ( g )</td>
<td>( f(g(x)) )</td>
<td>The set of all real numbers ( x ) in the domain of ( g ) such that ( g(x) ) is also in the domain of ( f ).</td>
</tr>
<tr>
<td>( g \circ f )</td>
<td>( g ) composed with ( f )</td>
<td>( g(f(x)) )</td>
<td>The set of all real numbers ( x ) in the domain of ( f ) such that ( f(x) ) is also in the domain of ( g ).</td>
</tr>
</tbody>
</table>

It is important to realize that there are two “filters” that allow certain values of \( x \) into the domain. The first filter is \( g(x) \). If \( x \) is not in the domain of \( g(x) \), it cannot be in the domain of \((f \circ g)(x) = f(g(x))\). Of those values for \( x \) that are in the domain of \( g(x) \), only some pass through, because we restrict the output of \( g(x) \) to values that are allowable as input into \( f \). This adds an additional filter.

The domain of \( f \circ g \) is always a subset of the domain of \( g \), and the range of \( f \circ g \) is always a subset of the range of \( f \).

### Example 3  Finding a Composite Function

Given the functions \( f(x) = x^2 + 1 \) and \( g(x) = x - 3 \), find \((f \circ g)(x)\).

**Solution:**

Write \( f(x) \) using placeholder notation.

\[ f(x) = (\Box)^2 + 1 \]

Express the composite function \( f \circ g \).

\[ f(g(x)) = (g(x))^2 + 1 \]

Substitute \( g(x) = x - 3 \) into \( f \).

\[ f(g(x)) = (x - 3)^2 + 1 \]

Eliminate the parentheses on the right side.

\[ f(g(x)) = x^2 - 6x + 10 \]

\[ (f \circ g)(x) = f(g(x)) = x^2 - 6x + 10 \]

**Your Turn**  Given the functions in Example 3, find \((g \circ f)(x)\).
3.4 Operations on Functions and Composition of Functions

**EXAMPLE 4**  Determining the Domain of a Composite Function

Given the functions \( f(x) = \frac{1}{x-1} \) and \( g(x) = \frac{1}{x} \), determine \( f \circ g \), and state its domain.

**Solution:**

Write \( f(x) \) using placeholder notation.

\[
\begin{align*}
\text{State:} & \quad f(x) = \frac{1}{(\square) - 1} \\
\text{Express the composite function} & \quad f(g(x)) = \frac{1}{g(x) - 1} \\
\text{Substitute } g(x) & \quad f(g(x)) = \frac{1}{x - 1} \\
\text{Multiply the right side by } x & \quad f(g(x)) = \frac{1 \cdot x}{x - 1} = \frac{x}{1 - x} \\
\end{align*}
\]

What is the domain of \((f \circ g)(x) = f(g(x))\)? By inspecting the final result of \(f(g(x))\), we see that the denominator is zero when \(x = 1\). Therefore, \(x \neq 1\). Are there any other values for \(x\) that are not allowed? The function \(g(x)\) has the domain \(x \neq 0\); therefore we must also exclude zero.

The domain of \((f \circ g)(x) = f(g(x))\) excludes \(x = 0\) and \(x = 1\) or, in interval notation, \([(-\infty, 0) \cup (1, \infty)]\).

**YOUR TURN**  For the functions \(f\) and \(g\) given in Example 4, determine the composite function \(g \circ f\) and state its domain.

The domain of the composite function cannot always be determined by examining the final form of \(f \circ g\).

**EXAMPLE 5**  Determining the Domain of a Composite Function (Without Finding the Composite Function)

Let \(f(x) = \frac{1}{x-2}\) and \(g(x) = \sqrt{x+3}\). Find the domain of \(f(g(x))\). Do not find the composite function.

**Solution:**

Find the domain of \(g\).  \([-3, \infty)\)

Find the range of \(g\).  \([0, \infty)\)

In \(f(g(x))\), the output of \(g\) becomes the input for \(f\). Since the domain of \(f\) is the set of all real numbers except 2, we eliminate any values of \(x\) in the domain of \(g\) that correspond to \(g(x) = 2\).

Let \(g(x) = 2\).

\[
\sqrt{x + 3} = 2
\]

Square both sides.

\[
x + 3 = 4
\]

Solve for \(x\).

\[
x = 1
\]

Eliminate \(x = 1\) from the domain of \(g\), \([-3, \infty)\).

State the domain of \(f(g(x))\).  \([-3, 1) \cup (1, \infty)\)

**CAUTION**

The domain of the composite function cannot always be determined by examining the final form of \(f \circ g\).

**Technology Tip**

The graphs of \(y_1 = f(x) = \frac{1}{x - 1}\), \(y_2 = g(x) = \frac{1}{x}\), and \(y_3 = (f \circ g)(x) = \frac{x}{1 - x}\) are shown.

**Classroom Example 3.4.4**

State the domains for \(f \circ g\) and \(g \circ f\) in Classroom Example 3.4.3.

**Answer:**

Domain \(f \circ g = [-2, 2]\)

Domain \(g \circ f = [0, \infty)\)

**Classroom Example 3.4.5**

Let \(f(x) = \sqrt{1+2x}\) and \(g(x) = \frac{1}{2}x - \frac{1}{4}\). What is the domain of \(f \circ g\)?

**Answer:** \(\mathbb{R}\)
CHAPTER 3  Functions and Their Graphs

Example 6  Evaluating a Composite Function

Given the functions \( f(x) = x^2 - 7 \) and \( g(x) = 5 - x^2 \), evaluate:

a. \( f(g(1)) \)  

b. \( f(g(-2)) \)  

c. \( g(f(3)) \)  

d. \( g(f(-4)) \)

Solution:

One way of evaluating these composite functions is to calculate the two individual composites in terms of \( x \): \( f(g(x)) \) and \( g(f(x)) \). Once those functions are known, the values can be substituted for \( x \) and evaluated.

Another way of proceeding is as follows:

a. Write the desired quantity.

   Find the value of the inner function \( g \).

   Substitute \( g(1) = 4 \) into \( f \).

   Evaluate \( f(4) \).

   \[ f(g(1)) = 9 \]

b. Write the desired quantity.

   Find the value of the inner function \( g \).

   Substitute \( g(-2) = 1 \) into \( f \).

   Evaluate \( f(1) \).

   \[ f(g(-2)) = -6 \]

c. Write the desired quantity.

   Find the value of the inner function \( f \).

   Substitute \( f(3) = 2 \) into \( g \).

   Evaluate \( g(2) \).

   \[ g(f(3)) = 1 \]

d. Write the desired quantity.

   Find the value of the inner function \( f \).

   Substitute \( f(-4) = 9 \) into \( g \).

   Evaluate \( g(9) \).

   \[ g(f(-4)) = -76 \]

Your Turn

Given the functions \( f(x) = x^3 - 3 \) and \( g(x) = 1 + x^3 \), evaluate \( f(g(1)) \) and \( g(f(1)) \).

Application Problems

Recall the example at the beginning of this chapter regarding the clothes that are on sale. Often, real-world applications are modeled with composite functions. In the clothes example, \( x \) is the original price of each item. The first function maps its input (original price) to an output (sale price). The second function maps its input (sale price) to an output (checkout price). Example 7 is another real-world application of composite functions.

Three temperature scales are commonly used:

- The degree Celsius (°C) scale

  - This scale was devised by dividing the range between the freezing (0°C) and boiling (100°C) points of pure water at sea level into 100 equal parts. This scale is used in science and is one of the standards of the “metric” (SI) system of measurements.
The Kelvin (K) temperature scale
- This scale shifts the Celsius scale down so that the zero point is equal to absolute zero (about \(-273.15^\circ C\)), a hypothetical temperature at which there is a complete absence of heat energy.
- Temperatures on this scale are called kelvins, not degrees kelvin, and kelvin is not capitalized. The symbol for the kelvin is K.

The degree Fahrenheit (°F) scale
- This scale evolved over time and is still widely used mainly in the United States, although Celsius is the preferred “metric” scale.
- With respect to pure water at sea level, the degrees Fahrenheit are gauged by the spread from 32°F (freezing) to 212°F (boiling).

The equations that relate these temperature scales are

\[
F = \frac{9}{5}C + 32 \quad C = K - 273.15
\]

**EXAMPLE 7** Applications Involving Composite Functions

Determine degrees Fahrenheit as a function of kelvins.

**Solution:**

Degrees Fahrenheit is a function of degrees Celsius.

Now substitute \( C = K - 273.15 \) into the equation for \( F \).

Simplify.

\[
F = \frac{9}{5} (K - 273.15) + 32
\]

\[
F = \frac{9}{5}K - 491.67 + 32
\]

\[
F = \frac{9}{5}K - 459.67
\]

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**SECTION 3.4 SUMMARY**

**Operations on Functions**

<table>
<thead>
<tr>
<th>Function</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>((f + g)(x) = f(x) + g(x))</td>
</tr>
<tr>
<td>Difference</td>
<td>((f - g)(x) = f(x) - g(x))</td>
</tr>
<tr>
<td>Product</td>
<td>((f \cdot g)(x) = f(x) \cdot g(x))</td>
</tr>
<tr>
<td>Quotient</td>
<td>(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0)</td>
</tr>
</tbody>
</table>

The domain of the sum, difference, and product functions is the intersection of the domains, or common domain shared by both \( f \) and \( g \). The domain of the quotient function is also the intersection of the domain shared by both \( f \) and \( g \) with an additional restriction that \( g(x) \neq 0 \).

**Composition of Functions**

\((f \circ g)(x) = f(g(x))\)

The domain restrictions cannot always be determined simply by inspecting the final form of \( f(g(x)) \). Rather, the domain of the composite function is a subset of the domain of \( g(x) \). Values of \( x \) must be eliminated if their corresponding values of \( g(x) \) are not in the domain of \( f \).
In Exercises 1–10, given the functions \(f\) and \(g\), find \(f + g\), \(f - g\), \(f \cdot g\), and \(\frac{f}{g}\), and state the domain of each.

1. \(f(x) = 2x + 1\)  
   \(g(x) = 1 - x\)
2. \(f(x) = 3x + 2\)  
   \(g(x) = 2x - 4\)
3. \(f(x) = 2x^2 - x\)  
   \(g(x) = x^2 - 4\)
4. \(f(x) = 3x + 2\)  
   \(g(x) = x^2 - 25\)
5. \(f(x) = \frac{1}{x}\)
6. \(f(x) = \frac{2x + 3}{x - 4}\)  
   \(g(x) = \frac{x - 4}{3x + 2}\)
7. \(f(x) = \sqrt{x}\)
8. \(f(x) = \sqrt{x - 1}\)
9. \(f(x) = \sqrt{4 - x}\)
10. \(f(x) = \sqrt{1 - 2x}\)

In Exercises 11–20, for the given functions \(f\) and \(g\), find the composite functions \(f \circ g\) and \(g \circ f\), and state their domains.

11. \(f(x) = 2x + 1\)  
   \(g(x) = x^2 - 3\)
12. \(f(x) = x^2 - 1\)  
   \(g(x) = 2 - x\)
13. \(f(x) = \frac{1}{x - 1}\)  
   \(g(x) = x + 2\)
14. \(f(x) = \frac{2}{x - 3}\)  
   \(g(x) = 2 + x\)
15. \(f(x) = |x|\)  
   \(g(x) = \frac{1}{x - 1}\)
16. \(f(x) = |x - 1|\)
17. \(f(x) = \sqrt{x - 1}\)
18. \(f(x) = \sqrt{2 - x}\)
19. \(f(x) = x^3 + 4\)  
   \(g(x) = (x - 4)^{\frac{1}{3}}\)
20. \(f(x) = \sqrt{x^2 - 1}\)

In Exercises 21–38, evaluate the functions for the specified values, if possible.

\(f(x) = x^2 + 10\)  
\(g(x) = \sqrt{x - 1}\)

21. \((f + g)(2)\)
22. \((f + g)(10)\)
23. \((f - g)(2)\)
24. \((f - g)(5)\)
25. \((f \cdot g)(4)\)
26. \((f \cdot g)(5)\)
27. \(\left(\frac{f}{g}\right)(10)\)
28. \(\left(\frac{f}{g}\right)(2)\)
29. \(f(g(2))\)
30. \(f(g(1))\)
31. \(g(f(-3))\)
32. \(g(f(4))\)
33. \(f(g(0))\)
34. \(g(f(0))\)
35. \(f(g(-3))\)
36. \(g(f(\sqrt{7}))\)
37. \((f \circ g)(4)\)
38. \((g \circ f)(-3)\)

In Exercises 39–50, evaluate \(f(g(1))\) and \(g(f(2))\), if possible.

39. \(f(x) = \frac{1}{x}\)  
   \(g(x) = 2x + 1\)
40. \(f(x) = x^2 + 1\)  
   \(g(x) = \frac{1}{2 - x}\)
41. \(f(x) = \sqrt{1 - x}\)  
   \(g(x) = x^2 + 2\)
42. \(f(x) = \sqrt{3 - x}\)  
   \(g(x) = x^2 + 1\)
43. \(f(x) = \frac{1}{|x - 1|}\)  
   \(g(x) = x + 3\)
44. \(f(x) = \frac{1}{x}\)  
   \(g(x) = |2x - 3|\)
45. \(f(x) = \sqrt{x - 1}\)  
   \(g(x) = x^2 + 5\)
46. \(f(x) = \sqrt{x - 3}\)  
   \(g(x) = \frac{1}{x - 3}\)
47. \(f(x) = \frac{1}{x^2 - 3}\)  
   \(g(x) = \sqrt{x - 3}\)
48. \(f(x) = \frac{x}{2 - x}\)  
   \(g(x) = 4 - x^2\)
49. \(f(x) = (x - 1)^{\frac{1}{3}}\)  
   \(g(x) = x^2 + 2x + 1\)
50. \(f(x) = (1 - x)^{\frac{1}{2}}\)  
   \(g(x) = (x - 3)^{\frac{1}{3}}\)
In Exercises 51–60, show that \( f(g(x)) = x \) and \( g(f(x)) = x \).

51. \( f(x) = 2x + 1, \quad g(x) = \frac{x - 1}{2} \)

52. \( f(x) = \frac{x - 2}{3}, \quad g(x) = 3x + 2 \)

53. \( f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 1 \) \( x \geq 1 \)

54. \( f(x) = 2 - x^2, \quad g(x) = \sqrt{2 - x} \) \( x \leq 2 \)

55. \( f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x} \) \( x \neq 0 \)

56. \( f(x) = (5 - x)^{1/3}, \quad g(x) = 5 - x^3 \)

57. \( f(x) = 4x^2 - 9, \quad g(x) = \frac{\sqrt{x + 9}}{2} \) \( x \geq 0 \)

58. \( f(x) = \sqrt{8x - 1}, \quad g(x) = \frac{x^3 + 1}{8} \)

59. \( f(x) = \frac{1}{x - 1}, \quad g(x) = \frac{x + 1}{x} \) \( x \neq 0, \ x \neq 1 \)

60. \( f(x) = \sqrt{25 - x^2}, \quad g(x) = \sqrt{25 - x^2} \) \( 0 \leq x \leq 5 \)

In Exercises 61–66, write the function as a composite of two functions \( f \) and \( g \). (More than one answer is correct.)

61. \( f(g(x)) = 2(3x - 1)^2 + 5(3x - 1) \)

62. \( f(g(x)) = \frac{1}{1 + x^2} \)

63. \( f(g(x)) = \frac{2}{|x - 3|} \)

64. \( f(g(x)) = \sqrt{1 - x^2} \)

65. \( f(g(x)) = \frac{3}{\sqrt{x + 1} - 2} \)

66. \( f(g(x)) = \frac{\sqrt{x}}{3\sqrt{x} + 2} \)

### Applications

Exercises 67 and 68 depend on the relationship between degrees Fahrenheit, degrees Celsius, and kelvins:

\[
F = \frac{9}{5} C + 32 \quad C = K - 273.15
\]

67. **Temperature.** Write a composite function that converts kelvins into degrees Fahrenheit.

68. **Temperature.** Convert the following degrees Fahrenheit to kelvins: 32°F and 212°F.

69. **Dog Run.** Suppose that you want to build a square fenced-in area for your dog. Fencing is purchased in linear feet.
   a. Write a composite function that determines the area of your dog pen as a function of how many linear feet are purchased.
   b. If you purchase 100 linear feet, what is the area of your dog pen?
   c. If you purchase 200 linear feet, what is the area of your dog pen?

70. **Dog Run.** Suppose that you want to build a circular fenced-in area for your dog. Fencing is purchased in linear feet.
   a. Write a composite function that determines the area of your dog pen as a function of how many linear feet are purchased.
   b. If you purchase 100 linear feet, what is the area of your dog pen?
   c. If you purchase 200 linear feet, what is the area of your dog pen?

71. **Market Price.** Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price \( p \) and the number of units for sale \( x \) are related by the demand equation:

\[
p = 3000 - \frac{1}{2}x
\]

Assume that the cost \( C(x) \) of producing \( x \) items is governed by the equation

\[
C(x) = 2000 + 10x
\]

and the revenue \( R(x) \) generated by selling \( x \) units is governed by

\[
R(x) = 100x
\]

   a. Write the cost as a function of price \( p \).
   b. Write the revenue as a function of price \( p \).
   c. Write the profit as a function of price \( p \).

72. **Market Price.** Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price \( p \) and the number of units for sale \( x \) are related by the demand equation:

\[
p = 10,000 - \frac{1}{4}x
\]

Assume that the cost \( C(x) \) of producing \( x \) items is governed by the equation

\[
C(x) = 30,000 + 5x
\]

and the revenue \( R(x) \) generated by selling \( x \) units is governed by

\[
R(x) = 1000x
\]

   a. Write the cost as a function of price \( p \).
   b. Write the revenue as a function of price \( p \).
   c. Write the profit as a function of price \( p \).
In Exercises 73 and 74, refer to the following:

The cost of manufacturing a product is a function of the number of hours $t$ the assembly line is running per day. The number of products manufactured $n$ is a function of the number of hours $t$ the assembly line is operating and is given by the function $n(t)$. The cost of manufacturing the product $C$ measured in thousands of dollars is a function of the quantity manufactured, that is, the function $C(n)$.

73. Business. If the quantity of a product manufactured during a day is given by
\[ n(t) = 50t - t^2 \]
and the cost of manufacturing the product is given by
\[ C(n) = 10n + 1375 \]

a. Find a function that gives the cost of manufacturing the product in terms of the number of hours $t$ the assembly line was functioning, $C(n(t))$.

b. Find the cost of production on a day when the assembly line was running for 16 hours. Interpret your answer.

74. Business. If the quantity of a product manufactured during a day is given by
\[ n(t) = 100t - 4t^2 \]
and the cost of manufacturing the product is given by
\[ C(n) = 5n + 2375 \]

a. Find a function that gives the cost of manufacturing the product in terms of the number of hours $t$ the assembly line was functioning, $C(n(t))$.

b. Find the cost of production on a day when the assembly line was running for 24 hours. Interpret your answer.

In Exercises 75 and 76, refer to the following:

Surveys performed immediately following an accidental oil spill at sea indicate the oil moved outward from the source of the spill in a nearly circular pattern. The radius of the oil spill $r$ measured in miles is a function of time $t$ measured in days from the start of the spill, while the area of the oil spill is a function of radius, that is, the function $A(r)$.

75. Environment: Oil Spill. If the radius of the oil spill is given by
\[ r(t) = 10t - 0.2t^2 \]
and the area of the oil spill is given by
\[ A(r) = \pi r^2 \]

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.

b. Find the area of the oil spill to the nearest square mile 7 days after the start of the spill.

76. Environment: Oil Spill. If the radius of the oil spill is given by
\[ r(t) = 8t - 0.1t^2 \]
and the area of the oil spill is given by
\[ A(r) = \pi r^2 \]

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.

b. Find the area of the oil spill to the nearest square mile 5 days after the start of the spill.

77. Environment: Oil Spill. An oil spill makes a circular pattern around a ship such that the radius in feet grows as a function of time in hours $r(t) = 150\sqrt{t}$. Find the area of the spill as a function of time.

78. Pool Volume. A 20 foot by 10 foot rectangular pool has been built. If 50 cubic feet of water is pumped into the pool per hour, write the water-level height (feet) as a function of time (hours).

79. Fireworks. A family is watching a fireworks display. If the family is 2 miles from where the fireworks are being launched and the fireworks travel vertically, what is the distance between the family and the fireworks as a function of height above ground?

80. Real Estate. A couple are about to put their house up for sale. They bought the house for $172,000 a few years ago, and when they list it with a realtor they will pay a 6% commission. Write a function that represents the amount of money they will make on their home as a function of the asking price $p$.

*catch the mistake*

In Exercises 81–86, for the functions $f(x) = x + 2$ and $g(x) = x^2 - 4$, find the indicated function and state its domain. Explain the mistake that is made in each problem.

81. \[ \frac{g}{f} \]
Solution:
\[ \frac{g(x)}{f(x)} = \frac{x^2 - 4}{x + 2} = \frac{(x - 2)(x + 2)}{x + 2} = x - 2 \]
Domain: $(-\infty, \infty)$
This is incorrect. What mistake was made?

82. \[ \frac{f}{g} \]
Solution:
\[ \frac{f(x)}{g(x)} = \frac{x + 2}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)} = \frac{1}{x - 2} \]
Domain: $(-\infty, 2) \cup (2, \infty)$
This is incorrect. What mistake was made?
83. \( f \circ g \)

Solution:
\[
(f \circ g)(x) = f(x)g(x) = (x + 2)(x^2 - 4) = x^3 + 2x^2 - 4x - 8
\]

Domain: \((-\infty, \infty)\)

This is incorrect. What mistake was made?

84. Given the function \( f(x) = x^2 + 7 \) and \( g(x) = \sqrt{x - 3} \), find \( f \circ g \) and state the domain.

Solution:
\[
(f \circ g)(x) = f(g(x)) = (\sqrt{x - 3})^2 + 7 = f(x) = x^2 + 7 = x - 4
\]

Domain: \((-\infty, \infty)\)

This is incorrect. What mistake was made?

**CONCEPTUAL**

In Exercises 87–90, determine whether each statement is true or false.

87. When adding, subtracting, multiplying, or dividing two functions, the domain of the resulting function is the union of the domains of the individual functions.

88. For any functions \( f \) and \( g \), \( f(g(x)) = g(f(x)) \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).

89. For any functions \( f \) and \( g \), \((f \circ g)(x) = x \) exists for all values of \( x \) that are in the domain of \( g(x) \), provided the range of \( g \) is a subset of the domain of \( f \).

90. The domain of a composite function can be found by inspection, without knowledge of the domain of the individual functions.

**CHALLENGE**

91. For the functions \( f(x) = x + a \) and \( g(x) = \frac{1}{x - a} \), find \( g \circ f \) and state its domain.

92. For the functions \( f(x) = ax^2 + bx + c \) and \( g(x) = \frac{1}{x - c} \), find \( g \circ f \) and state its domain.

93. For the functions \( f(x) = \sqrt{x + a} \) and \( g(x) = x^2 - a \), find \( g \circ f \) and state its domain.

94. For the functions \( f(x) = \frac{1}{x^2} \) and \( g(x) = \frac{1}{x^3} \), find \( g \circ f \) and state its domain. Assume \( a > 1 \) and \( b > 1 \).

**TECHNOLOGY**

95. Using a graphing utility, plot \( y_1 = \sqrt{x + 7} \) and \( y_2 = \sqrt{9 - x} \).

96. Using a graphing utility, plot \( y_1 = \sqrt{x + 3} \), \( y_2 = \frac{1}{\sqrt{3 - x}} \), and \( y_3 = \frac{y_1}{y_2} \). What is the domain of \( y_3 \)?

97. Using a graphing utility, plot \( y_1 = \sqrt{x^2 - 3x - 4} \), \( y_2 = \frac{1}{x^2 - 14} \), and \( y_3 = \frac{1}{y_1^2 - 14} \). If \( y_1 \) represents a function \( f \) and \( y_2 \) represents a function \( g \), then \( y_3 \) represents the composite function \( g \circ f \). The graph of \( y_1 \) is only defined for the domain of \( g \circ f \). State the domain of \( g \circ f \).

98. Using a graphing utility, plot \( y_1 = \sqrt{1 - x} \), \( y_2 = x^2 + 2 \), and \( y_3 = y_1^2 + 2 \). If \( y_1 \) represents a function \( f \) and \( y_2 \) represents a function \( g \), then \( y_3 \) represents the composite function \( g \circ f \). The graph of \( y_1 \) is only defined for the domain of \( g \circ f \). State the domain of \( g \circ f \).
Every human being has a blood type, and every human being has a DNA sequence. These are examples of functions, where a person is the input and the output is blood type or DNA sequence. These relationships are classified as functions because each person can have one and only one blood type or DNA strand. The difference between these functions is that many people have the same blood type, but DNA is unique to each individual. Can we map backwards? For instance, if you know the blood type, do you know specifically which person it came from? No, but, if you know the DNA sequence, you know exactly to which person it corresponds. When a function has a one-to-one correspondence, like the DNA example, then mapping backwards is possible. The map back is called the inverse function.

**Determine Whether a Function Is One-to-One**

In Section 3.1, we defined a function as a relationship that maps an input (contained in the domain) to exactly one output (found in the range). Algebraically, each value for \( x \) can correspond to only a single value for \( y \). Recall the square, identity, absolute value, and reciprocal functions from our library of functions in Section 3.3.

All of the graphs of these functions satisfy the vertical line test. Although the square function and the absolute value function map each value of \( x \) to exactly one value for \( y \), these two functions map two values of \( x \) to the same value for \( y \). For example, \((-1, 1)\) and \((1, 1)\) lie on both graphs. The identity and reciprocal functions, on the other hand, map each \( x \) to a single value for \( y \), and no two \( x \)-values map to the same \( y \)-value. These two functions are examples of one-to-one functions.

**Definition**

A function \( f(x) \) is **one-to-one** if no two elements in the domain correspond to the same element in the range; that is,

\[
\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).
\]

In other words, it is one-to-one if no two inputs map to the same output.
3.5 One-to-One Functions and Inverse Functions

**EXAMPLE 1** Determining Whether a Function Defined as a Set of Points Is a One-to-One Function

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function.

\[ f = \{ (0, 0), (1, 1), (1, -1) \} \]
\[ g = \{ (-1, 1), (0, 0), (1, 1) \} \]
\[ h = \{ (-1, -1), (0, 0), (1, 1) \} \]

**Solution:**

\[
\begin{array}{c|c}
\text{Domain} & \text{Range} \\
0 & 0 \\
1 & -1 \\
1 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
\text{Domain} & \text{Range} \\
0 & 0 \\
-1 & 1 \\
1 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
\text{Domain} & \text{Range} \\
-1 & -1 \\
0 & 0 \\
1 & 1 \\
\end{array}
\]

\( f \) is not a function. \( g \) is a function, but not one-to-one. \( h \) is a one-to-one function.

Just as there is a graphical test for functions, the vertical line test, there is a graphical test for one-to-one functions, the *horizontal line test*. Note that a horizontal line can be drawn on the square and absolute value functions so that it intersects the graph of each function at two points. The identity and reciprocal functions, however, will intersect a horizontal line in at most only one point. This leads us to the horizontal line test for one-to-one functions.

**Definition**

**Horizontal Line Test**

If every horizontal line intersects the graph of a function in at most one point, then the function is classified as a one-to-one function.

**EXAMPLE 2** Using the Horizontal Line Test to Determine Whether a Function Is One-to-One

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function. Assume that \( x \) is the independent variable and \( y \) is the dependent variable.

\[ x = y^2 \quad y = x^2 \quad y = x^3 \]

Classroom Example 3.5.1

Determine whether these functions are one-to-one functions.

a. \( f = \{ (-2, -2), (-1, -1), (0, 0), (1, -1) \} \)

b. \( g = \{ (3, 1), (2, -1), (0, 2) \} \)

**Answer:** a. no  b. yes
Solution:

\[ y = x^2 \]

Not a function
(fails vertical line test)

\[ y = x^3 \]

One-to-one function
(passes both horizontal and vertical line tests)

\[ x = y^2 \]

Function, but not one-to-one
(passes vertical line test but fails horizontal line test)

**YOUR TURN** Determine whether each of the functions is a one-to-one function.

- **a.** \( f(x) = x + 2 \)
- **b.** \( f(x) = x^2 + 1 \)

Another way of writing the definition of a one-to-one function is:

If \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

In the Your Turn following Example 2, we found (using the horizontal line test) that \( f(x) = x + 2 \) is a one-to-one function, but that \( f(x) = x^2 + 1 \) is not a one-to-one function. We can also use this alternative definition to determine algebraically whether a function is one-to-one.

**Words**

State the function.

Let there be two real numbers, \( x_1 \) and \( x_2 \), such that \( f(x_1) = f(x_2) \).

Subtract 2 from both sides of the equation.

\[ f(x) = x + 2 \text{ is a one-to-one function.} \]

**Math**

\[ f(x) = x + 2 \]

\[ x_1 + 2 = x_2 + 2 \]

\[ x_1 = x_2 \]

**Words**

State the function.

Let there be two real numbers, \( x_1 \) and \( x_2 \), such that \( f(x_1) = f(x_2) \).

Subtract 1 from both sides of the equation.

Solve for \( x_1 \).

\[ f(x) = x^2 + 2 \text{ is not a one-to-one function.} \]

**Math**

\[ f(x) = x^2 + 1 \]

\[ x_1^2 + 1 = x_2^2 + 1 \]

\[ x_1^2 = x_2^2 \]

\[ x_1 = \pm x_2 \]
### 3.5 One-to-One Functions and Inverse Functions

#### Example 3

**Determining Algebraically Whether a Function Is One-to-One**

Determine algebraically whether the following functions are one-to-one:

a. $f(x) = 5x^3 - 2$

b. $f(x) = |x + 1|$

g. $g(x) = (x - 2)^3 - 2$

**Solution (a):**

Find $f(x_1)$ and $f(x_2)$.

Let $f(x_1) = f(x_2)$.

Add 2 to both sides of the equation.

Divide both sides of the equation by 5.

Take the cube root of both sides of the equation.

Simplify.

$f(x) = 5x^3 - 2$ is a one-to-one function.

**Solution (b):**

Find $f(x_1)$ and $f(x_2)$.

Let $f(x_1) = f(x_2)$.

Solve the absolute value equation.

$x = x_2$ or $x = -x_2 - 2$

$f(x) = |x + 1|$ is not a one-to-one function.

#### Inverse Functions

If a function is one-to-one, then the function maps each $x$ to exactly one $y$, and no two $x$-values map to the same $y$-value. This implies that there is a one-to-one correspondence between the inputs (domain) and outputs (range) of a one-to-one function $f(x)$. In the special case of a one-to-one function, it would be possible to map from the output (range of $f$) back to the input (domain of $f$), and this mapping would also be a function. The function that maps the output back to the input of a function $f$ is called the **inverse function** and is denoted $f^{-1}(x)$.

A one-to-one function $f$ maps every $x$ in the domain to a unique and distinct corresponding $y$ in the range. Therefore, the inverse function $f^{-1}$ maps every $y$ back to a unique and distinct $x$.

The function notations $f(x) = y$ and $f^{-1}(y) = x$ indicate that if the point $(x, y)$ satisfies the function, then the point $(y, x)$ satisfies the inverse function.

For example, let the function $h(x) = \{(-1, 0), (1, 2), (3, 4)\}$.

$h = \{(-1, 0), (1, 2), (3, 4)\}$

**Domain**

-1 \[\rightarrow\] 0

1 \[\rightarrow\] 2

3 \[\rightarrow\] 4

**Range**

- \[\leftarrow\] 0

2 \[\leftarrow\] 4

$h^{-1} = \{(0, -1), (2, 1), (4, 3)\}$
The inverse function undoes whatever the function does. For example, if \( f(x) = 5x \), then the function \( f \) maps any value \( x \) in the domain to a value \( 5x \) in the range. If we want to map backwards or undo the \( 5x \), we develop a function called the inverse function that takes \( 5x \) as input and maps back to \( x \) as output. The inverse function is \( f^{-1}(x) = \frac{1}{5}x \). Note that if we input \( 5x \) into the inverse function, the output is \( x: f^{-1}(5x) = \frac{1}{5}(5x) = x \).

**Inverse Function**

If \( f \) and \( g \) denote two one-to-one functions such that

\[
\begin{align*}
    f(g(x)) &= x \text{ for every } x \text{ in the domain of } g \\
    g(f(x)) &= x \text{ for every } x \text{ in the domain of } f,
\end{align*}
\]

then \( g \) is the inverse of the function \( f \). The function \( g \) is denoted by \( f^{-1} \) (read “\( f \)-inverse”).

**Note:** \( f^{-1} \) is used to denote the inverse of \( f \). The \( -1 \) is not used as an exponent and, therefore, does not represent the reciprocal of \( f: \frac{1}{f} \).

Two properties hold true relating one-to-one functions to their inverses: (1) the range of the function is the domain of the inverse, and the range of the inverse is the domain of the function, and (2) the composite function that results with a function and its inverse (and vice versa) is the identity function \( x \).

\[
\text{Domain of } f = \text{range of } f^{-1} \quad \text{and} \quad \text{range of } f = \text{domain of } f^{-1}
\]

\[f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x\]

**Example 4  Verifying Inverse Functions**

Verify that \( f^{-1}(x) = \frac{1}{2}x - 2 \) is the inverse of \( f(x) = 2x + 4 \).

**Solution:**

Show that \( f^{-1}(f(x)) = x \) and \( f(f^{-1}(x)) = x \).

Write \( f^{-1} \) using placeholder notation.

\[
f^{-1}(\square) = \frac{1}{2} (\square) - 2
\]

Substitute \( f(x) = 2x + 4 \) into \( f^{-1} \).

\[
f^{-1}(f(x)) = \frac{1}{2} (2x + 4) - 2 = x
\]

Simplify.

\[
f^{-1}(f(x)) = x
\]

Write \( f \) using placeholder notation.

\[
f(\square) = 2(\square) + 4
\]

Substitute \( f^{-1}(x) = \frac{1}{2}x - 2 \) into \( f \).

\[
f(f^{-1}(x)) = 2\left( \frac{1}{2}x - 2 \right) + 4
\]

Simplify.

\[
f(f^{-1}(x)) = x - 4 + 4 = x
\]

\[
f(f^{-1}(x)) = x
\]
Note the relationship between the domain and range of \( f \) and \( f^{-1} \).

<table>
<thead>
<tr>
<th>( f(x) = 2x + 4 )</th>
<th>( f^{-1}(x) = \frac{1}{2}x - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
</tbody>
</table>

**Example 5**  
**Verifying Inverse Functions with Domain Restrictions**

Verify that \( f^{-1}(x) = x^2 \), for \( x \geq 0 \), is the inverse of \( f(x) = \sqrt{x} \).

**Solution:**

Show that \( f^{-1}(f(x)) = x \) and \( f(f^{-1}(x)) = x \).

Write \( f^{-1} \) using placeholder notation.

Substitute \( f(x) = \sqrt{x} \) into \( f^{-1} \).

Write \( f \) using placeholder notation.

Substitute \( f^{-1}(x) = x^2, x \geq 0 \) into \( f \).

<table>
<thead>
<tr>
<th>( f(x) = \sqrt{x} )</th>
<th>( f^{-1}(x) = x^2, x \geq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, \infty))</td>
<td>([0, \infty))</td>
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</tbody>
</table>

**Graphical Interpretation of Inverse Functions**

In Example 4, we showed that \( f^{-1}(x) = \frac{1}{2}x - 2 \) is the inverse of \( f(x) = 2x + 4 \). Let’s now investigate the graphs that correspond to the function \( f \) and its inverse \( f^{-1} \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f^{-1}(x) )</th>
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<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>(-3)</td>
<td>(-2)</td>
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<tr>
<td>(-2)</td>
<td>(0)</td>
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<tr>
<td>(-1)</td>
<td>(2)</td>
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<tr>
<td>(2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(4)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

Note that the point \((-3, -2)\) lies on the function and the point \((-2, -3)\) lies on the inverse. In fact, every point \((a, b)\) that lies on the function corresponds to a point \((b, a)\) that lies on the inverse.

Draw the line \( y = x \) on the graph. In general, the point \((b, a)\) on the inverse \( f^{-1}(x) \) is the reflection (about \( y = x \)) of the point \((a, b)\) on the function \( f(x) \).

In general, if the point \((a, b)\) is on the graph of a function, then the point \((b, a)\) is on the graph of its inverse.

**Study Tip**

If the point \((a, b)\) is on the function, then the point \((b, a)\) is on the inverse. Notice the interchanging of the \( x \)- and \( y \)-coordinates.
We have developed the definition of an inverse function and described properties of inverses. At this point, you should be able to determine whether two functions are inverses of one another. Let's turn our attention to another problem: How do you find the inverse of a function?

### Finding the Inverse Function

If the point \((a, b)\) lies on the graph of a function, then the point \((b, a)\) lies on the graph of the inverse function. The symmetry about the line \(y = x\) tells us that the roles of \(x\) and \(y\) interchange. Therefore, if we start with every point \((x, y)\) that lies on the graph of a function, then every point \((y, x)\) lies on the graph of its inverse. Algebraically, this corresponds to interchanging \(x\) and \(y\). Finding the inverse of a finite set of ordered pairs is easy: simply interchange the \(x\)- and \(y\)-coordinates. Earlier, we found that if \(h(x) = \{(−1, 0), (1, 2), (3, 4)\}\), then \(h^{-1}(x) = \{(0, −1), (2, 1), (4, 3)\}\). But how do we find the inverse of a function defined by an equation?

Recall the mapping relationship if \(f\) is a one-to-one function. This relationship implies that \(f(x) = y\) and \(f^{-1}(y) = x\). Let’s use these two identities to find the inverse. Now consider the
function defined by \( f(x) = 3x - 1 \). To find \( f^{-1} \), we let \( f(x) = y \), which yields \( y = 3x - 1 \). Solve for the variable \( x: x = \frac{y + 1}{3} \).

Recall that \( f^{-1}(y) = x \), so we have found the inverse to be \( f^{-1}(y) = \frac{1}{3}y + \frac{1}{3} \). It is customary to write the independent variable as \( x \), so we write the inverse as \( f^{-1}(x) = \frac{1}{3}x + \frac{1}{3} \). Now that we have found the inverse, let’s confirm that the properties \( f^{-1}(f(x)) = x \) and \( f(f^{-1}(x)) = x \) hold.

\[
f(f^{-1}(x)) = 3\left(\frac{1}{3}x + \frac{1}{3}\right) - 1 = x + 1 - 1 = x
\]

\[
f^{-1}(f(x)) = \frac{1}{3}(3x - 1) + \frac{1}{3} = x - \frac{1}{3} + \frac{1}{3} = x
\]

**Finding the inverse of a function**

Let \( f \) be a one-to-one function. Then the following procedure can be used to find the inverse function \( f^{-1} \) if the inverse exists.

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
<th>Example</th>
</tr>
</thead>
</table>
| 1    | Let \( y = f(x) \). | \( f(x) = -3x + 5 \)  
\( y = -3x + 5 \) |
| 2    | Solve the resulting equation for \( x \) in terms of \( y \) (if possible). | \( 3x = -y + 5 \)  
\( x = \frac{-1}{3}y + \frac{5}{3} \) |
| 3    | Let \( x = f^{-1}(y) \). | \( f^{-1}(y) = -\frac{1}{3}y + \frac{5}{3} \) |
| 4    | Let \( y = x \) (interchange \( x \) and \( y \)). | \( f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3} \) |

The same result is found if we first interchange \( x \) and \( y \) and then solve for \( y \) in terms of \( x \).

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
<th>Example</th>
</tr>
</thead>
</table>
| 1    | Let \( y = f(x) \). | \( f(x) = -3x + 5 \)  
\( y = -3x + 5 \) |
| 2    | Interchange \( x \) and \( y \). | \( x = -3y + 5 \) |
| 3    | Solve for \( y \) in terms of \( x \). | \( 3y = -x + 5 \)  
\( y = \frac{1}{3}x + \frac{5}{3} \) |
| 4    | Let \( y = f^{-1}(x) \) | \( f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3} \) |

Note the following:

- Verify first that a function is one-to-one prior to finding an inverse (if it is not one-to-one, then the inverse does not exist).
- State the domain restrictions on the inverse function. The domain of \( f \) is the range of \( f^{-1} \) and vice versa.
- To verify that you have found the inverse, show that \( f(f^{-1}(x)) = x \) for all \( x \) in the domain of \( f^{-1} \) and \( f^{-1}(f(x)) = x \) for all \( x \) in the domain of \( f \).

Classroom Example 3.5.6*

Given the graph of \( f(x) \) below, plot the graph of the inverse \( f^{-1}(x) \).

(a) Find the inverse of \( f(x) \).

(b) Plot the graph of \( f^{-1}(x) \).

(Answer: There is a closed hole at (2, 0) and an open hole at (4, 0).)
CHAPTER 3 Functions and Their Graphs

Technology Tip
Using a graphing utility, plot
\[ y_1 = f(x) = \sqrt{x + 2}, \]
\[ y_2 = f^{-1}(x) = x^2 - 2, \]
and \( y_3 = x. \)

Note that the function \( f(x) \) and its inverse \( f^{-1}(x) \) are symmetric about the line \( y = x. \)

Study Tip
Had we ignored the domain and range in Example 7, we would have found the inverse function to be the square function \( f(x) = x^2 - 2, \) which is not a one-to-one function. It is only when we restrict the domain of the square function that we get a one-to-one function.

Classroom Example 3.5.7
Determine the inverse of the following functions and state the domain and range.

a. \( f(x) = \sqrt{1 - 3x} \)
b. \( g(x) = -\sqrt{2 + x} + 1 \)

Answer:

a. \[ f^{-1}(x) = \frac{1 - x^2}{3} \]
   Domain: \( (-\infty, \infty) \]
   Range: \( (-\infty, \frac{1}{3}] \]
b. \[ g^{-1}(x) = (1 - x)^2 - 2 \]
   Domain: \( (-\infty, 0] \]
   Range: \( [-2, \infty) \]

EXAMPLE 7 The Inverse of a Square Root Function

Find the inverse of the function \( f(x) = \sqrt{x + 2} \). State the domain and range of both \( f \) and \( f^{-1}. \)

Solution:

\( f(x) \) is a one-to-one function because it passes the horizontal line test.

**Step 1** Let \( y = f(x) \).
\[ y = \sqrt{x + 2} \]

**Step 2** Interchange \( x \) and \( y \).
\[ x = \sqrt{y + 2} \]

**Step 3** Solve for \( y \).
\[ x^2 = y + 2 \]
\[ y = x^2 - 2 \]

**Step 4** Let \( y = f^{-1}(x) \).
\[ f^{-1}(x) = x^2 - 2 \]

Note any domain restrictions. (State the domain and range of both \( f \) and \( f^{-1} \).)

\[ f: \quad \text{Domain: } [-2, \infty) \quad \text{Range: } [0, \infty) \]
\[ f^{-1}: \quad \text{Domain: } [0, \infty) \quad \text{Range: } [-2, \infty) \]

The inverse of \( f(x) = \sqrt{x + 2} \) is \( f^{-1}(x) = x^2 - 2 \) for \( x \geq 0 \).

Check.

\[ f^{-1}(f(x)) = x \text{ for all } x \text{ in the domain of } f. \]
\[ f^{-1}(f(x)) = (\sqrt{x + 2})^2 - 2 = x + 2 - 2 \text{ for } x \geq -2 = x \]

\[ f(f^{-1}(x)) = x \text{ for all } x \text{ in the domain of } f^{-1}. \]
\[ f(f^{-1}(x)) = \sqrt{(x^2 - 2) + 2} = \sqrt{x^2} \text{ for } x \geq 0 = x \]

Note that the function \( f(x) = \sqrt{x + 2} \) and its inverse \( f^{-1}(x) = x^2 - 2 \) for \( x \geq 0 \) are symmetric about the line \( y = x. \)

YOUR TURN Find the inverse of the given function. State the domain and range of the inverse function.

a. \( f(x) = 7x - 3 \)  
   b. \( g(x) = \sqrt{x - 1} \)
**EXAMPLE 8**  A Function That Does Not Have an Inverse Function

Find the inverse of the function \( f(x) = |x| \) if it exists.

**Solution:**
The function \( f(x) = |x| \) fails the horizontal line test and therefore is not a one-to-one function. Because \( f \) is not a one-to-one function, its inverse function does not exist.

**Study Tip**
The range of the function is equal to the domain of its inverse function.

**EXAMPLE 9**  Finding the Inverse Function

The function \( f(x) = \frac{2}{x + 3}, x \neq -3 \), is a one-to-one function. Find its inverse.

**Solution:**

**STEP 1**  Let \( y = f(x) \).

\[ y = \frac{2}{x + 3} \]

**STEP 2**  Interchange \( x \) and \( y \).

\[ x = \frac{2}{y + 3} \]

**STEP 3**  Solve for \( y \).

Multiply the equation by \( (y + 3) \).

\[ x(y + 3) = 2 \]

Eliminate the parentheses.

\[ xy + 3x = 2 \]

Subtract \( 3x \) from both sides.

\[ xy = -3x + 2 \]

Divide the equation by \( x \).

\[ y = \frac{-3x + 2}{x} = -3 + \frac{2}{x} \]

**STEP 4**  Let \( y = f^{-1}(x) \).

\[ f^{-1}(x) = -3 + \frac{2}{x}, x \neq 0 \]

**Note any domain restrictions on \( f^{-1}(x) \).**

The inverse of the function \( f(x) = \frac{2}{x + 3}, x \neq -3 \), is

\[ f^{-1}(x) = -3 + \frac{2}{x}, x \neq 0 \]

**Check.**

\[ f^{-1}(f(x)) = -3 + \frac{2}{\left( \frac{2}{x + 3} \right)} = -3 + (x + 3) = x, x \neq -3 \]

\[ f(f^{-1}(x)) = \frac{2}{\left( -3 + \frac{2}{x} \right) + 3} = \frac{2}{\frac{2}{x}} = x, x \neq 0 \]

**YOUR TURN**  The function \( f(x) = \frac{4}{x - 1}, x \neq 1 \), is a one-to-one function.

Find its inverse.

Note in Example 9 that the domain of \( f \) is \((-\infty, -3) \cup (-3, \infty)\) and the domain of \( f^{-1} \) is \((-\infty, 0) \cup (0, \infty)\). Therefore, we know that the range of \( f \) is \((-\infty, 0) \cup (0, \infty)\), and the range of \( f^{-1} \) is \((-\infty, -3) \cup (-3, \infty)\).
EXAMPLE 10  Finding the Inverse of a Piecewise-Defined Function

The function \( f(x) = \begin{cases} 3x & x < 0 \\ x^2 & x \geq 0 \end{cases} \) is a one-to-one function. Find its inverse.

Solution:
From the graph of \( f \) we can make a table with corresponding domain and range values.

<table>
<thead>
<tr>
<th>DOMAIN OF ( f )</th>
<th>RANGE OF ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( -\infty, 0 ))</td>
<td>(( -\infty, 0 ))</td>
</tr>
<tr>
<td>([ 0, \infty ))</td>
<td>([ 0, \infty ))</td>
</tr>
</tbody>
</table>

From this information we can also list domain and range values for \( f^{-1} \).

<table>
<thead>
<tr>
<th>DOMAIN OF ( f^{-1} )</th>
<th>RANGE OF ( f^{-1} )</th>
<th>RANGE OF ( f )</th>
<th>DOMAIN OF ( f^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( -\infty, 0 ))</td>
<td>(( -\infty, 0 ))</td>
<td>([ 0, \infty ))</td>
<td>([ 0, \infty ))</td>
</tr>
</tbody>
</table>

\( f(x) = 3x \) on \(( -\infty, 0 )\); find \( f^{-1}(x) \) on \(( -\infty, 0 )\).

**STEP 1** Let \( y = f(x) \).

**STEP 2** Solve for \( x \) in terms of \( y \).

**STEP 3** Solve for \( y \).

**STEP 4** Let \( y = f^{-1}(x) \).

\( f(x) = x^2 \) on \([ 0, \infty )\); find \( f^{-1}(x) \) on \([ 0, \infty )\).

**STEP 1** Let \( y = f(x) \).

**STEP 2** Solve for \( x \) in terms of \( y \).

**STEP 3** Solve for \( y \).

**STEP 4** Let \( y = f^{-1}(x) \).

**STEP 5** The range of \( f^{-1} \) is \([ 0, \infty )\)

Combining the two pieces yields a piecewise-defined inverse function.

\[
 f^{-1}(x) = \begin{cases} 
 \frac{1}{3}x & x < 0 \\
 \sqrt{x} & x \geq 0 
\end{cases}
\]
### Section 3.5 Summary

#### One-to-One Functions
Each input in the domain corresponds to exactly one output in the range, and no two inputs map to the same output. There are three ways to test a function to determine whether it is a one-to-one function.

1. **Discrete points**: For the set of all points \((a, b)\) verify that no \(y\)-values are repeated.
2. **Algebraic equations**: Let \(f(x_1) = f(x_2)\); if it can be shown that \(x_1 = x_2\), then the function is one-to-one.
3. **Graphs**: Use the horizontal line test; if any horizontal line intersects the graph of the function in more than one point, then the function is not one-to-one.

#### Properties of Inverse Functions
1. If \(f\) is a one-to-one function, then \(f^{-1}\) exists.
2. **Domain and range**
   - Domain of \(f = \text{range of } f^{-1}\)
   - Domain of \(f^{-1} = \text{range of } f\)
3. **Composition of inverse functions**
   - \(f^{-1}(f(x)) = x\) for all \(x\) in the domain of \(f\).
   - \(f(f^{-1}(x)) = x\) for all \(x\) in the domain of \(f^{-1}\).
4. The graphs of \(f\) and \(f^{-1}\) are symmetric with respect to the line \(y = x\).

#### Procedure for Finding the Inverse of a Function
1. Let \(y = f(x)\).
2. Interchange \(x\) and \(y\).
3. Solve for \(y\).
4. Let \(y = f^{-1}(x)\).

### Section 3.5 Exercises

**Skills**

In Exercises 1–16, determine whether the given relation is a function. If it is a function, determine whether it is a one-to-one function.

1. ![Table of MONTH and AVERAGE TEMPERATURE](Image)

2. ![Table of PERSON and 10-DIGIT PHONE #](Image)

3. ![Table of PERSON and SPouse](Image)

4. ![Table of PERSON and COURSE GRADE](Image)
5. \{(0, 1), (1, 2), (2, 3), (3, 4)\}  
6. \{(0, -2), (2, 0), (5, 3), (-5, -7)\}  
7. \{(0, 0), (9, \frac{3}{2}), (4, \frac{2}{3}), (2, 0), (5, 3), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}  
8. \{(0, 1), (1, 1), (2, 1), (3, 1)\}  
9. \{(0, 1), (1, 0), (2, 1), (\frac{1}{2}, 1), (5, 4), (\frac{3}{2}, 4)\}  
10. \{(0, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{8}), (1, 1), (2, 8)\}  

11. In Exercises 17–24, determine algebraically and graphically whether the function is one-to-one.  
12. \(f(x) = |x - 3|\)  
13. \(f(x) = (x - 2)^3 + 1\)  
14. \(f(x) = \frac{1}{x - 1}\)  
15. \(f(x) = \frac{1}{x + 2}\)  

16. In Exercises 25–34, verify that the function \(f^{-1}(x)\) is the inverse of \(f(x)\) by showing that \(f(f^{-1}(x)) = x\) and \(f^{-1}(f(x)) = x\). Graph \(f(x)\) and \(f^{-1}(x)\) on the same axes to show the symmetry about the line \(y = x\).  
17. \(f(x) = 2x + 1; f^{-1}(x) = \frac{x - 1}{2}\)  
18. \(f(x) = \sqrt{x - 1}, x \geq 1; f^{-1}(x) = x^2 + 1, x \geq 0\)  
19. \(f(x) = \frac{1}{x}; f^{-1}(x) = \frac{1}{x}, x \neq 0\)  
20. \(f(x) = \sqrt[x]{x}\)  
21. \(f(x) = x^2 - 4\)  
22. \(f(x) = \sqrt{x + 1}\)  
23. \(f(x) = x^5 - 1\)  
24. \(f(x) = \frac{1}{x + 2}\)  
25. \(f(x) = 2x + 1; f^{-1}(x) = \frac{x - 1}{2}\)  
26. \(f(x) = \frac{x - 2}{3}; f^{-1}(x) = 3x + 2\)  
27. \(f(x) = \sqrt{x - 1}, x \geq 1; f^{-1}(x) = x^2 + 1, x \geq 0\)  
28. \(f(x) = 2 - x^2, x \geq 0; f^{-1}(x) = \sqrt{2 - x}, x \leq 2\)  
29. \(f(x) = \frac{1}{x}; f^{-1}(x) = \frac{1}{x}, x \neq 0\)  
30. \(f(x) = (5 - x)^2; f^{-1}(x) = 5 - x^2\)  
31. \(f(x) = \frac{1}{2x + 6}, x \neq -3; f^{-1}(x) = \frac{1}{2x} - 3, x \neq 0\)  
32. \(f(x) = \frac{3}{4 - x}, x \neq 4; f^{-1}(x) = 4 - \frac{3}{x}, x \neq 0\)  
33. \(f(x) = \frac{x + 3}{x + 4}, x \neq -4; f^{-1}(x) = \frac{3 - 4x}{x - 1}, x \neq 1\)  
34. \(f(x) = \frac{x - 5}{3 - x}, x \neq 3; f^{-1}(x) = \frac{3x + 5}{x + 1}, x \neq -1\)
In Exercises 35–42, graph the inverse of the one-to-one function that is given.

35. 

36. 

37. 

38. 

39. 

40. 

41. 

42. 

In Exercises 43–60, the function $f$ is one-to-one. Find its inverse, and check your answer. State the domain and range of both $f$ and $f^{-1}$.

43. $f(x) = x - 1$
44. $f(x) = 7x$
45. $f(x) = -3x + 2$
46. $f(x) = 2x + 3$
47. $f(x) = x^3 + 1$
48. $f(x) = x^3 - 1$
49. $f(x) = \sqrt{x - 3}$
50. $f(x) = \sqrt{3 - x}$
51. $f(x) = x^2 - 1, x \geq 0$
52. $f(x) = 2x^2 + 1, x \geq 0$
53. $f(x) = (x + 2)^2 - 3, x \geq -2$
54. $f(x) = (x - 3)^2 - 2, x \geq 3$
55. $f(x) = \frac{2}{x}$
56. $f(x) = -\frac{3}{x}$
57. $f(x) = \frac{2}{3 - x}$
58. $f(x) = \frac{7}{x + 2}$
59. $f(x) = \frac{7x + 1}{5 - x}$
60. $f(x) = \frac{2x + 5}{7 + x}$

In Exercises 61–64, graph the piecewise-defined function to determine whether it is a one-to-one function. If it is a one-to-one function, find its inverse.

61. $G(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{x} & x \geq 0 \end{cases}$
62. $G(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \sqrt{\frac{1}{x}} & x \geq 0 \end{cases}$
63. $f(x) = \begin{cases} x & x \leq -1 \\ x^3 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$
64. $f(x) = \begin{cases} x + 3 & x \leq -2 \\ |x| & -2 < x < 2 \\ x^2 & x \geq 2 \end{cases}$
CHAPTER 3 Functions and Their Graphs

\section*{Applications}

65. \textbf{Temperature.} The equation used to convert from degrees Celsius to degrees Fahrenheit is \( f(x) = \frac{9}{5}x + 32 \).
Determine the inverse function \( f^{-1}(x) \). What does the inverse function represent?

66. \textbf{Temperature.} The equation used to convert from degrees Fahrenheit to degrees Celsius is \( C(x) = \frac{5}{9}(x - 32) \).
Determine the inverse function \( C^{-1}(x) \). What does the inverse function represent?

67. \textbf{Budget.} The Richmond rowing club is planning to enter the Head of the Charles race in Boston and is trying to figure out how much money to raise. The entry fee is $250 per boat for the first 10 boats and $175 for each additional boat. Find the cost function \( C(x) \) as a function of the number of boats the club enters \( x \). Find the inverse function that will yield how many boats the club can enter as a function of how much money it will raise.

68. \textbf{Long-Distance Calling Plans.} A phone company charges $.39 per minute for the first 10 minutes of a long-distance phone call and $.12 per minute every minute after that. Find the cost function \( C(x) \) as a function of the length of the phone call in minutes \( x \). Suppose you buy a “prepaid” phone card that is planned for a single call. Find the inverse function that determines how many minutes you can talk as a function of how much you prepaid.

69. \textbf{Salary.} A student works at Target making $10 per hour and the weekly number of hours worked per week \( x \) varies. If Target withholds 25\% of his earnings for taxes and Social Security, write a function \( E(x) \) that expresses the student’s take-home pay each week. Find the inverse function \( E^{-1}(x) \). What does the inverse function tell you?

70. \textbf{Salary.} A grocery store pays you $8 per hour for the first 40 hours per week and time and a half for overtime. Write a piecewise-defined function that represents your weekly earnings \( E(x) \) as a function of the number of hours worked \( x \). Find the inverse function \( E^{-1}(x) \). What does the inverse function tell you?

In Exercises 71–74, refer to the following:

By analyzing available empirical data it was determined that during an illness a patient’s body temperature fluctuated during one 24-hour period according to the function
\[ T(t) = 0.0003(t - 24)^3 + 101.70 \]
where \( T \) represents the patient’s temperature in degrees Fahrenheit and \( t \) represents the time of day in hours measured from 12:00 A.M. (midnight).

71. \textbf{Health/Medicine.} Find the domain and range of the function \( T(t) \).

72. \textbf{Health/Medicine.} Find time as a function of temperature, that is, the inverse function \( t(T) \).

73. \textbf{Health/Medicine.} Find the domain and range of the function \( t(T) \) found in Exercise 72.

74. \textbf{Health/Medicine.} At what time, to the nearest hour, was the patient’s temperature 99.5°F?

\section*{Catch the Mistake}

In Exercises 75–78, explain the mistake that is made.

75. Is \( x = y^2 \) a one-to-one function?

\textbf{Solution:}
Yes, this graph represents a one-to-one function because it passes the horizontal line test.

This is incorrect. What mistake was made?

76. A linear one-to-one function is graphed below. Draw its inverse.

\textbf{Solution:}
Note that the points \((3, 3)\) and \((0, -4)\) lie on the graph of the function.

By symmetry, the points \((-3, 3)\) and \((0, -4)\) lie on the graph of the inverse.

This is incorrect. What mistake was made?
77. Given the function \( f(x) = x^2 \), find the inverse function \( f^{-1}(x) \).

**Solution:**

**Step 1:** Let \( y = f(x) \).

\[ y = x^2 \]

**Step 2:** Solve for \( x \).

\[ x = \sqrt{y} \]

**Step 3:** Interchange \( x \) and \( y \).

\[ y = \sqrt{x} \]

**Step 4:** Let \( y = f^{-1}(x) \).

\[ f^{-1}(x) = \sqrt{x} \]

Check: \( f(f^{-1}(x)) = (\sqrt{x})^2 = x \) and \( f^{-1}(f(x)) = \sqrt{x^2} = x \).

The inverse of \( f(x) = x^2 \) is \( f^{-1}(x) = \sqrt{x} \).

This is incorrect. What mistake was made?

---

**3.5 One-to-One Functions and Inverse Functions**

77. Given the function \( f(x) = \sqrt{x - \frac{2}{9}} \), find the inverse function \( f^{-1}(x) \), and state the domain restrictions on \( f^{-1}(x) \).

**Solution:**

**Step 1:** Let \( y = f(x) \).

\[ y = \sqrt{x - \frac{2}{9}} \]

**Step 2:** Interchange \( x \) and \( y \).

\[ x = \sqrt{y - \frac{2}{9}} \]

**Step 3:** Solve for \( y \).

\[ y = x^2 + 2 \]

**Step 4:** Let \( y = f^{-1}(x) \).

\[ f^{-1}(x) = x^2 + 2 \]

**Step 5:** Domain restrictions: \( f(x) = \sqrt{x - \frac{2}{9}} \) has the domain restriction that \( x \geq 2 \).

The inverse of \( f(x) = \sqrt{x - \frac{2}{9}} \) is \( f^{-1}(x) = x^2 + 2 \).

The domain of \( f^{-1}(x) \) is \( x \geq 2 \).

This is incorrect. What mistake was made?

---

**CONCEPTUAL**

In Exercises 79–82, determine whether each statement is true or false.

79. Every even function is a one-to-one function.

80. Every odd function is a one-to-one function.

81. It is not possible that \( f = f^{-1} \).

82. A function \( f \) has an inverse. If the function lies in quadrant II, then its inverse lies in quadrant IV.

83. If \( (0, \, b) \) is the \( y \)-intercept of a one-to-one function \( f \), what is the \( x \)-intercept of the inverse \( f^{-1} \)?

84. If \( (a, \, 0) \) is the \( x \)-intercept of a one-to-one function \( f \), what is the \( y \)-intercept of the inverse \( f^{-1} \)?

---

**CHALLENGE**

85. The unit circle is not a function. If we restrict ourselves to the semicircle that lies in quadrants I and II, the graph represents a function, but it is not a one-to-one function. If we further restrict ourselves to the quarter circle lying in quadrant I, the graph does represent a one-to-one function. Determine the equations of both the one-to-one function and its inverse. State the domain and range of both.

86. Find the inverse of \( f(x) = \frac{c}{x} \), \( c \neq 0 \).

87. Under what conditions is the linear function \( f(x) = mx + b \) a one-to-one function?

88. Assuming that the conditions found in Exercise 87 are met, determine the inverse of the linear function.

---

**TECHNOLOGY**

In Exercises 89–92, graph the following functions and determine whether they are one-to-one.

89. \( f(x) = |4 - x^3| \)  
90. \( f(x) = \frac{3}{x^3 + 2} \)

91. \( f(x) = x^{1/3} - x^5 \)  
92. \( f(x) = \frac{1}{x^{1/2}} \)

---

In Exercises 93–96, graph the functions \( f \) and \( g \) and the line \( y = x \) in the same screen. Do the two functions appear to be inverses of each other?

93. \( f(x) = \sqrt{3x - 5} \); \( g(x) = \frac{x^2}{3} + \frac{5}{3} \)

94. \( f(x) = \sqrt{4 - 3x} \); \( g(x) = \frac{4}{3} - \frac{x^2}{3}, x \geq 0 \)

95. \( f(x) = (x - 7)^{1/3} + 2 \); \( g(x) = x^3 - 6x^2 + 12x - 1 \)

96. \( f(x) = \sqrt{x + 3} - 2 \); \( g(x) = x^3 + 6x^2 + 12x + 6 \)
In 2005, the national average cost of residential electricity was 9.53 ¢/kWh (cents per kilowatt-hour). For example, if a residence used 3400 kWh, then the bill would be $324.02, and if a residence used 2500 kWh, then the bill would be $238.25.

In this section we discuss mathematical models for different applications. Two quantities in the real world often vary with respect to one another. Sometimes, they vary directly. For example, the more money we make, the more total dollars of federal income tax we expect to pay. Sometimes, quantities vary inversely. For example, when interest rates on mortgages decrease, we expect the number of homes purchased to increase because a buyer can afford “more house” with the same mortgage payment when rates are lower. In this section we discuss quantities varying directly, inversely, and jointly.

Direct Variation

When one quantity is a constant multiple of another quantity, we say that the quantities are directly proportional to one another.

**Direct variation**

Let \( x \) and \( y \) represent two quantities. The following are equivalent statements:

- \( y = kx \), where \( k \) is a nonzero constant.
- \( y \) varies directly with \( x \).
- \( y \) is directly proportional to \( x \).

The constant \( k \) is called the constant of variation or the constant of proportionality.

In 2005, the national average cost of residential electricity was 9.53 ¢/kWh (cents per kilowatt-hour). For example, if a residence used 3400 kWh, then the bill would be $324.02, and if a residence used 2500 kWh, then the bill would be $238.25.
**EXAMPLE 1  Finding the Constant of Variation**

In the United States, the cost of electricity is directly proportional to the number of kilowatt-hours (kWh) used. If a household in Tennessee on average used 3098 kWh per month and had an average monthly electric bill of $179.99, find a mathematical model that gives the cost of electricity in Tennessee in terms of the number of kilowatt-hours used.

**Solution:**

Write the direct variation model.

\[ y = kx \]

Label the variables and constant.

\[ x = \text{number of kWh} \]
\[ y = \text{cost (dollars)} \]
\[ k = \text{cost per kWh} \]

Substitute the given data \( x = 3098 \) kWh and \( y = \$179.99 \) into \( y = kx \).

\[ 179.99 = 3098k \]

Solve for \( k \).

\[ k = \frac{179.99}{3098} = 0.05810 \]

\[ y = 0.0581x \]

In Tennessee the cost of electricity is \( \$0.581/\text{kWh} \).

**YOUR TURN** Find a mathematical model that describes the cost of electricity in California if the cost is directly proportional to the number of kWh used and a residence that consumes 4000 kWh is billed $480.

Not all variation we see in nature is direct variation. Isometric growth, where the various parts of an organism grow in direct proportion to each other, is rare in living organisms. If organisms grew isometrically, young children would look just like adults, only smaller. In contrast, most organisms grow nonisometrically; the various parts of organisms do not increase in size in a one-to-one ratio. The relative proportions of a human body change dramatically as the human grows. Children have proportionately larger heads and shorter legs than adults. Allometric growth is the pattern of growth whereby different parts of the body grow at different rates with respect to each other. Some human body characteristics vary directly, and others can be mathematically modeled by direct variation with powers.

**Direct variation with powers**

Let \( x \) and \( y \) represent two quantities. The following are equivalent statements:

- \( y = kx^n \), where \( k \) is a nonzero constant.
- \( y \) varies directly with the \( n \)th power of \( x \).
- \( y \) is directly proportional to the \( n \)th power of \( x \).

One example of direct variation with powers is height and weight of humans. Weight (in pounds) is directly proportional to the cube of height (feet).

\[ W = kH^3 \]
EXAMPLE 2  Direct Variation with Powers

The following is a personal ad:

*Single professional male (6 ft/194 lbs) seeks single professional female for long-term relationship. Must be athletic, smart, like the movies and dogs, and have height and weight similarly proportioned to mine.*

Find a mathematical equation that describes the height and weight of the male who wrote the ad. How much would a 5’6” woman weigh who has the same proportionality as the male?

**Solution:**

Write the direct variation (cube) model for height versus weight.

\[ W = kH^3 \]

Substitute the given data \( W = 194 \) and \( H = 6 \) into \( W = kH^3 \).

\[ 194 = k(6)^3 \]

Solve for \( k \).

\[ k = \frac{194}{216} = 0.898148 \approx 0.90 \]

\[ W = 0.9H^3 \]

Let \( H = 5.5 \) ft.

\[ W = 0.9(5.5)^3 = 149.73 \]

A woman 5’6” tall with the same height and weight proportionality as the male would weigh \( 150 \) lb.

**YOUR TURN**  A brother and sister both have weight (pounds) that varies as the cube of height (feet) and they share the same proportionality constant. The sister is 6 feet tall and weighs 170 pounds. Her brother is 6 feet 4 inches. How much does he weigh?

---

**Inverse Variation**

Two fundamental topics covered in economics are supply and demand. Supply is the quantity that producers are willing to sell at a given price. For example, an artist may be willing to paint and sell 5 portraits if each sells for $50, but that same artist may be willing to sell 100 portraits if each sells for $10,000. Demand is the quantity of a good that consumers are not only willing to purchase but also have the capacity to buy at a given price. For example, consumers may purchase 1 billion Big Macs from McDonald’s every year, but perhaps only 1 million filet mignons are sold at Outback. There may be 1 billion people who want to buy the filet mignon but don’t have the financial means to do so. Economists study the equilibrium between supply and demand.

Demand can be modeled with an inverse variation of price: when the price increases, demand decreases, and vice versa.

**Inverse variation**

Let \( x \) and \( y \) represent two quantities. The following are equivalent statements:

- \( y = \frac{k}{x} \), where \( k \) is a nonzero constant.
- \( y \) varies inversely with \( x \).
- \( y \) is inversely proportional to \( x \).

The constant \( k \) is called the constant of variation or the constant of proportionality.
EXAMPLE 3  Inverse Variation

The number of potential buyers of a house decreases as the price of the house increases (see graph on the right). If the number of potential buyers of a house in a particular city is inversely proportional to the price of the house, find a mathematical equation that describes the demand for houses as it relates to price. How many potential buyers will there be for a $2 million house?

Solution:

Write the inverse variation model.

\[ y = \frac{k}{x} \]

Label the variables and constant.

\( x \) = price of house in thousands of dollars

\( y \) = number of buyers

Select any point that lies on the curve. \((200, 500)\)

Substitute the given data \( x = 200 \) and \( y = 500 \) into \( y = \frac{k}{x} \).

\[ 500 = \frac{k}{200} \]

Solve for \( k \).

\[ k = 200 \cdot 500 = 100,000 \]

\[ y = \frac{100,000}{x} \]

Let \( x = 2000 \).

\[ y = \frac{100,000}{2000} = 50 \]

There are only 50 potential buyers for a $2 million house in this city.

Answer: 5000

\[ \text{Answer: 5000} \]

\[ \text{Answer: 5000} \]

Two quantities can vary inversely with the \( n \)th power of \( x \).

If \( x \) and \( y \) are related by the equation \( y = \frac{k}{x^n} \), then we say that \( y \) varies inversely with the \( n \)th power of \( x \), or \( y \) is inversely proportional to the \( n \)th power of \( x \).

Joint Variation and Combined Variation

We now discuss combinations of variations. When one quantity is proportional to the product of two or more other quantities, the variation is called joint variation. When direct variation and inverse variation occur at the same time, the variation is called combined variation.
An example of a joint variation is simple interest (Section 1.2), which is defined as

\[ I = Prt \]

where
- \( I \) is the interest in dollars
- \( P \) is the principal (initial) dollars
- \( r \) is the interest rate (expressed in decimal form)
- \( t \) is time in years

The interest earned is proportional to the product of three quantities (principal, interest rate, and time). Note that if the interest rate increases, then the interest earned also increases. Similarly, if either the initial investment (principal) or the time the money is invested increases, then the interest earned also increases.

An example of combined variation is the combined gas law in chemistry,

\[ P = \frac{kT}{V} \]

where
- \( P \) is pressure
- \( T \) is temperature (kelvins)
- \( V \) is volume
- \( k \) is a gas constant

This relation states that the pressure of a gas is directly proportional to the temperature and inversely proportional to the volume containing the gas. For example, as the temperature increases, the pressure increases, but when the volume decreases, pressure increases.

As an example, the gas in the headspace of a soda bottle has a fixed volume. Therefore, as temperature increases, the pressure increases. Compare the different pressures of opening a twist-off cap on a bottle of soda that is cold versus one that is hot. The hot one feels as though it “releases more pressure.”

**EXAMPLE 4 Combined Variation**

The gas in the headspace of a soda bottle has a volume of 9.0 ml, pressure of 2 atm (atmospheres), and a temperature of 298 K (standard room temperature of 77°F). If the soda bottle is stored in a refrigerator, the temperature drops to approximately 279 K (42°F). What is the pressure of the gas in the headspace once the bottle is chilled?

**Solution:**

Write the combined gas law.

\[ P = \frac{kT}{V} \]

Let \( P = 2 \) atm, \( T = 298 \) K, and \( V = 9.0 \) ml.

\[ 2 = \frac{k \cdot 298}{9} \]

Solve for \( k \).

\[ k = \frac{18}{298} \]

Let \( k = \frac{18}{298}, \ T = 279, \) and \( V = 9.0 \) in \( P = \frac{kT}{V} \).

\[ P = \frac{18}{298} \cdot \frac{279}{9} \approx 1.87 \]

Since we used the same physical units for both the chilled and room-temperature soda bottles, the pressure is in atmospheres. \( P = 1.87 \) atm
In Exercises 1–16, write an equation that describes each variation. Use $k$ as the constant of variation.

1. $y$ varies directly with $x$.
2. $s$ varies directly with $t$.
3. $V$ varies directly with $x^3$.
4. $A$ varies directly with $x^2$.
5. $z$ varies directly with $m$.
6. $h$ varies directly with $\sqrt{r}$.
7. $f$ varies inversely with $\lambda$.
8. $P$ varies inversely with $r^2$.
9. $F$ varies directly with $w$ and inversely with $L$.
10. $V$ varies directly with $T$ and inversely with $P$.
11. $v$ varies directly with both $g$ and $t$.
12. $S$ varies directly with both $t$ and $d$.
13. $R$ varies inversely with both $P$ and $T$.
14. $y$ varies inversely with both $x$ and $z$.
15. $y$ is directly proportional to the square root of $x$.
16. $y$ is inversely proportional to the cube of $t$.

In Exercises 17–36, write an equation that describes each variation.

17. $d$ is directly proportional to $t$. $d = r$ when $t = 1$.
18. $F$ is directly proportional to $m$. $F = a$ when $m = 1$.
19. $V$ is directly proportional to both $l$ and $w$. $V = 6h$ when $w = 3$ and $l = 2$.
20. $A$ is directly proportional to both $b$ and $h$. $A = 10$ when $b = 5$ and $h = 4$.
21. $A$ varies directly with the square of $r$. $A = 9\pi$ when $r = 3$.
22. $V$ varies directly with the cube of $r$. $V = 36\pi$ when $r = 3$.
23. $V$ varies directly with both $h$ and $r^2$. $V = 1$ when $r = 2$ and $h = \frac{4}{\pi}$.
24. $W$ is directly proportional to both $R$ and the square of $I$. $W = 4$ when $R = 100$ and $I = 0.25$.
25. $V$ varies inversely with $P$. $V = \frac{1000}{P}$ when $P = 400$.
26. $I$ varies inversely with the square of $d$. $I = 42$ when $d = 16$.
27. $F$ varies inversely with both $\lambda$ and $L$. $F = \frac{20\pi}{m^2}$ when $\lambda = 1$ $\mu$m and $L = 100$ kilometers.
28. \( y \) varies inversely with both \( x \) and \( z \). \( y = 32 \) when \( x = 4 \) and \( z = 0.05 \).

29. \( t \) varies inversely with \( s \). \( t = 2.4 \) when \( s = 8 \).

30. \( W \) varies inversely with the square of \( d \). \( W = 180 \) when \( d = 0.2 \).

31. \( R \) varies inversely with the square of \( I \). \( R = 0.4 \) when \( I = 3.5 \).

32. \( y \) varies inversely with both \( x \) and the square root of \( z \). \( y = 12 \) when \( x = 0.2 \) and \( z = 4 \).

33. \( R \) varies directly with \( L \) and inversely with \( A \). \( R = 0.5 \) when \( L = 20 \) and \( A = 0.4 \).

34. \( F \) varies directly with \( m \) and inversely with \( d \). \( F = 32 \) when \( m = 20 \) and \( d = 8 \).

35. \( F \) varies directly with both \( m_1 \) and \( m_2 \) and inversely with the square of \( d \). \( F = 20 \) when \( m_1 = 8 \), \( m_2 = 16 \), and \( d = 0.4 \).

36. \( w \) varies directly with the square root of \( g \) and inversely with the square of \( t \). \( w = 20 \) when \( g = 16 \) and \( t = 0.5 \).

### Applications

37. **Wages.** Jason and Valerie both work at Panera Bread and have the following paycheck information for a certain week. Find an equation that shows their wages \( W \) varying directly with the number of hours worked \( H \).

<table>
<thead>
<tr>
<th>Employee</th>
<th>Hours Worked</th>
<th>Wages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jason</td>
<td>23</td>
<td>$172.50</td>
</tr>
<tr>
<td>Valerie</td>
<td>32</td>
<td>$240.00</td>
</tr>
</tbody>
</table>

38. **Sales Tax.** The sales tax in Orange and Seminole counties in Florida differs by only 0.5%. A new resident knows this but doesn’t know which of the counties has the higher tax. The resident lives near the border of the counties and is in the market for a new plasma television and wants to purchase it in the county with the lower tax. If the tax on a pair of $40 sneakers is $2.60 in Orange County and the tax on a $12 T-shirt is $0.84 in Seminole County, write two equations: one for each county that describes the tax \( T \), which is directly proportional to the purchase price \( P \).

For Exercises 39 and 40, refer to the following:

The ratio of the speed of an object to the speed of sound determines the Mach number. Aircraft traveling at a subsonic speed (less than the speed of sound) have a Mach number less than 1. In other words, the speed of an aircraft is directly proportional to its Mach number. Aircraft traveling at a supersonic speed (greater than the speed of sound) have a Mach number greater than 1. The speed of sound at sea level is approximately 760 miles per hour.

39. **Military.** The U.S. Navy Blue Angels fly F-18 Hornets that are capable of Mach 1.7. How fast can F-18 Hornets fly at sea level?

40. **Military.** The U.S. Air Force’s newest fighter aircraft is the F-35, which is capable of Mach 1.9. How fast can an F-35 fly at sea level?

Exercises 41 and 42 are examples of the golden ratio, or phi, a proportionality constant that appears in nature. The numerical approximate value of phi is 1.618. From www.goldenratio.net.

41. **Human Anatomy.** The length of your forearm \( F \) (wrist to elbow) is directly proportional to the length of your hand \( H \) (length from wrist to tip of middle finger). Write the equation that describes this relationship if the length of your forearm is 11 inches and the length of your hand is 6.8 inches.

42. **Human Anatomy.** Each section of your index finger, from the tip to the base of the wrist, is larger than the preceding one by about the golden (Fibonacci) ratio. Find an equation that represents the ratio of each section of your finger related to the previous one if one section is eight units long and the next section is five units long.
For Exercises 43 and 44, refer to the following:

Hooke’s law in physics states that if a spring at rest (equilibrium position) has a weight attached to it, then the distance the spring stretches is directly proportional to the force (weight), according to the formula:

\[ F = kx \]

where \( F \) is the force in Newtons (N), \( x \) is the distance stretched in meters (m), and \( k \) is the spring constant (N/m).

For Exercises 49 and 50, refer to the following:

In physics, the inverse square law states that any physical quantity or strength is inversely proportional to the square of the distance from the source of that physical quantity. In particular, the intensity of light radiating from a point source is inversely proportional to the square of the distance from the source. Below is a table of average distances from the Sun:

<table>
<thead>
<tr>
<th>Planet</th>
<th>Distance to the Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>58,000 km</td>
</tr>
<tr>
<td>Earth</td>
<td>150,000 km</td>
</tr>
<tr>
<td>Mars</td>
<td>228,000 km</td>
</tr>
</tbody>
</table>

43. **Physics.** A force of 30 N will stretch the spring 10 centimeters. How far will a force of 72 N stretch the spring?

44. **Physics.** A force of 30 N will stretch the spring 10 centimeters. How much force is required to stretch the spring 18 centimeters?

45. **Business.** A cell phone company develops a pay-as-you-go cell phone plan in which the monthly cost varies directly as the number of minutes used. If the company charges $17.70 in a month when 236 minutes are used, what should the company charge for a month in which 500 minutes are used?

46. **Economics.** Demand for a product varies inversely with the price per unit of the product. Demand for the product is 10,000 units when the price is $5.75 per unit. Find the demand for the product (to the nearest hundred units) when the price is $6.50.

47. **Sales.** Levi’s makes jeans in a variety of price ranges for juniors. The Flare 519 jeans sell for about $20, whereas the 646 Vintage Flare jeans sell for $300. The demand for Levi’s jeans is inversely proportional to the price. If 300,000 pairs of the 519 jeans were bought, approximately how many of the Vintage Flare jeans were bought?

48. **Sales.** Levi’s makes jeans in a variety of price ranges for men. The Silver Tab Baggy jeans sell for about $30, whereas the Offender jeans sell for about $160. The demand for Levi’s jeans is inversely proportional to the price. If 400,000 pairs of the Silver Tab Baggy jeans were bought, approximately how many of the Offender jeans were bought?

49. **Solar Radiation.** The solar radiation on the Earth is approximately 1400 watts per square meter. How much solar radiation is there on Mars? Round to the nearest hundred watts per square meter.

50. **Solar Radiation.** The solar radiation on the Earth is approximately 1400 watts per square meter. How much solar radiation is there on Mercury? Round to the nearest hundred watts per square meter.

51. **Investments.** Marilyn receives a $25,000 bonus from her company and decides to put the money toward a new car that she will need in two years. Simple interest is directly proportional to the principal and the time invested. She compares two different banks’ rates on money market accounts. If she goes with Bank of America, she will earn $750 in interest, but if she goes with the Navy Federal Credit Union, she will earn $1500. What is the interest rate on money market accounts at both banks?

52. **Investments.** Connie and Alvaro sell their house and buy a fixer-upper house. They made $130,000 on the sale of their previous home. They know it will take 6 months before the general contractor will start their renovation, and they want to take advantage of a 6-month CD that pays simple interest. What is the rate of the 6-month CD if they will make $3250 in interest?

53. **Chemistry.** A gas contained in a 4 milliliter container at a temperature of 300 K has a pressure of 1 atmosphere. If the temperature decreases to 275 K, what is the resulting pressure?

54. **Chemistry.** A gas contained in a 4 milliliter container at a temperature of 300 K has a pressure of 1 atmosphere. If the container changes to a volume of 3 milliliters, what is the resulting pressure?
In Exercises 55 and 56, explain the mistake that is made.

55. $y$ varies directly with $t$ and indirectly with $x$. When $x = 4$ and $t = 2$, then $y = 1$. Find an equation that describes this variation.

Solution:
Write the variation equation: $y = ktx$
Let $x = 4$, $t = 2$, and $y = 1$. $1 = k(2)(4)$
Solve for $k$. $k = \frac{1}{8}$
Substitute $k = \frac{1}{8}$ into $y = ktx$. $y = \frac{1}{8}tx$
This is incorrect. What mistake was made?

56. $y$ varies directly with $t$ and the square of $x$. When $x = 4$ and $t = 1$, then $y = 8$. Find an equation that describes this variation.

Solution:
Write the variation equation: $y = kt\sqrt{x}$
Let $x = 4$, $t = 1$, and $y = 8$. $8 = k(1)\sqrt{4}$
Solve for $k$. $k = 4$
Substitute $k = 4$ into $y = kt\sqrt{x}$. $y = 4t\sqrt{x}$
This is incorrect. What mistake was made?

In Exercises 57 and 58, determine whether each statement is true or false.

57. The area of a triangle is directly proportional to both the base and the height of the triangle (joint variation).

58. Average speed is directly proportional to both distance and time (joint variation).

In Exercises 59 and 60, match the variation with the graph.

59. Inverse variation

60. Direct variation

Exercises 61 and 62 involve the theory governing laser propagation through the Earth’s atmosphere.

The three parameters that help classify the strength of optical turbulence are:
- $C_n^2$, index of refraction structure parameter
- $k$, wave number of the laser, which is inversely proportional to the wavelength $\lambda$ of the laser:
  $$k = \frac{2\pi}{\lambda}$$
- $L$, propagation distance

The variance of the irradiance of a laser $\sigma^2$ is directly proportional to $C_n^2$, $k^{7/6}$, and $L^{11/6}$.

61. When $C_n^2 = 1.0 \times 10^{-13}$ m$^{-2/3}$, $L = 2$ km, and $\lambda = 1.55$ $\mu$m, the variance of irradiance for a plane wave $\sigma^2_{pl}$ is 7.1. Find the equation that describes this variation.

62. When $C_n^2 = 1.0 \times 10^{-13}$ m$^{-2/3}$, $L = 2$ km, and $\lambda = 1.55$ $\mu$m, the variance of irradiance for a spherical wave $\sigma^2_{sp}$ is 2.3. Find the equation that describes this variation.
For Exercises 63–66, refer to the following:

Data from 1995 to 2006 for oil prices in dollars per barrel, the U.S. Dow Jones Utilities Stock Index, New Privately Owned Housing, and 5-year Treasury Constant Maturity Rate are given in the table. (Data are from Forecast Center’s Historical Economic and Market Home Page at www.neatideas.com/djutil.htm.)

Use the calculator \( \text{STAT} \) \( \text{EDIT} \) commands to enter the table with \( L_1 \) as the oil price, \( L_2 \) as the utilities stock index, \( L_3 \) as number of housing units, and \( L_4 \) as the 5-year maturity rate.

<table>
<thead>
<tr>
<th>January of Each Year</th>
<th>Oil Price, $ per Barrel</th>
<th>U.S. Dow Jones Utilities Stock Index</th>
<th>New, Privately Owned Housing Units</th>
<th>5-year Treasury Constant Maturity Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>17.99</td>
<td>193.12</td>
<td>1407</td>
<td>7.76</td>
</tr>
<tr>
<td>1996</td>
<td>18.88</td>
<td>230.85</td>
<td>1467</td>
<td>5.36</td>
</tr>
<tr>
<td>1997</td>
<td>25.17</td>
<td>232.53</td>
<td>1355</td>
<td>6.33</td>
</tr>
<tr>
<td>1998</td>
<td>16.71</td>
<td>263.29</td>
<td>1525</td>
<td>5.42</td>
</tr>
<tr>
<td>1999</td>
<td>12.47</td>
<td>302.80</td>
<td>1748</td>
<td>4.60</td>
</tr>
<tr>
<td>2000</td>
<td>27.18</td>
<td>315.14</td>
<td>1636</td>
<td>6.58</td>
</tr>
<tr>
<td>2001</td>
<td>29.58</td>
<td>372.32</td>
<td>1600</td>
<td>4.86</td>
</tr>
<tr>
<td>2002</td>
<td>19.67</td>
<td>285.71</td>
<td>1698</td>
<td>4.34</td>
</tr>
<tr>
<td>2003</td>
<td>32.94</td>
<td>207.75</td>
<td>1853</td>
<td>3.05</td>
</tr>
<tr>
<td>2004</td>
<td>32.27</td>
<td>271.94</td>
<td>1911</td>
<td>3.12</td>
</tr>
<tr>
<td>2005</td>
<td>46.84</td>
<td>343.46</td>
<td>2137</td>
<td>3.71</td>
</tr>
<tr>
<td>2006</td>
<td>65.51</td>
<td>413.84</td>
<td>2265</td>
<td>4.35</td>
</tr>
</tbody>
</table>

63. An increase in oil price in dollars per barrel will drive the U.S. Dow Jones Utilities Stock Index to soar.
   a. Use the calculator commands \( \text{STAT} \) \( \text{linReg} \) \((ax + b)\), and \( \text{STATPLOT} \) to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the oil price in dollars per barrel.
   b. If the U.S. Dow Jones Utilities Stock Index varies directly as the oil price in dollars per barrel, then use the calculator commands \( \text{STAT} \) \( \text{PwrReg} \) and \( \text{STATPLOT} \) to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the oil price in dollars per barrel.
   c. Use the equations you found in (a) and (b) to predict the stock index when the oil price hits $72.70 per barrel in September 2006. Which answer is closer to the actual stock index of 417? Round all answers to the nearest whole number.

64. An increase in oil price in dollars per barrel will affect the interest rates across the board—in particular, the 5-year Treasury constant maturity rate.
   a. Use the calculator commands \( \text{STAT} \) \( \text{linReg} \) \((ax + b)\), and \( \text{STATPLOT} \) to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the oil price in dollars per barrel.
   b. If the 5-year Treasury constant maturity rate varies inversely as the oil price in dollars per barrel, then use the calculator commands \( \text{STAT} \) \( \text{PwrReg} \) and \( \text{STATPLOT} \) to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the oil price in dollars per barrel.
   c. Use the equations you found in (a) and (b) to predict the maturity rate when the oil price hits $72.70 per barrel in September 2006. Which answer is closer to the actual maturity rate at 5.02%? Round all answers to two decimal places.
65. An increase in interest rates—in particular, the 5-year Treasury constant maturity rate—will affect the number of new, privately owned housing units.

a. Use the calculator commands \texttt{STAT \{ LinReg \} \{ ax+b \}}, and \texttt{STATPLOT} to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the 5-year rate.

b. If the number of new privately owned housing units varies inversely as the 5-year Treasury constant maturity rate, then use the calculator commands \texttt{STAT \{ PwrReg \}} and \texttt{STATPLOT} to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the 5-year rate.

c. Use the equations you found in (a) and (b) to predict the number of housing units when the maturity rate is 5.02\% in September 2006. Which answer is closer to the actual number of new, privately owned housing units of 1861? Round all answers to the nearest unit.

66. An increase in the number of new, privately owned housing units will affect the U.S. Dow Jones Utilities Stock Index.

a. Use the calculator commands \texttt{STAT \{ LinReg \} \{ ax+b \}}, and \texttt{STATPLOT} to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the number of housing units.

b. If the U.S. Dow Jones Utilities Stock Index varies directly as the number of new, privately owned housing units, then use the calculator commands \texttt{STAT \{ PwrReg \}} and \texttt{STATPLOT} to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the number of housing units.

c. Use the equations you found in (a) and (b) to predict the utilities stock index if there are 1861 new, privately owned housing units in September 2006. Which answer is closer to the actual stock index of 417? Round all answers to the nearest whole number.

For Exercises 67 and 68, refer to the following:

Data for retail gasoline price in dollars per gallon for the period March 2000 to March 2008 are given in the following table. (Data are from Energy Information Administration, Official Energy Statistics from the U.S. government at http://tonto.eia.doe.gov/oeg/info/gdu/gaspump.html.) Use the calculator \texttt{STAT \{ EDIT \}} command to enter the table below with \( L_1 \) as the year (\( x = 1 \) for year 2000) and \( L_2 \) as the gasoline price in dollars per gallon.

<table>
<thead>
<tr>
<th>March of Each Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retail Gasoline Price $ per Gallon</td>
<td>1.517</td>
<td>1.409</td>
<td>1.249</td>
<td>1.693</td>
<td>1.736</td>
<td>2.079</td>
<td>2.425</td>
<td>2.563</td>
<td>3.244</td>
</tr>
</tbody>
</table>

67. a. Use the calculator commands \texttt{STAT \{ LinReg \}} to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the year (\( x = 1 \) for year 2000) and \( y \) as the gasoline price in dollars per gallon. Round all answers to three decimal places.

b. Use the equation to predict the gasoline price in March 2006. Round all answers to three decimal places. Is the answer close to the actual price?

c. Use the equation to predict the gasoline price in March 2009. Round all answers to three decimal places.

68. a. Use the calculator commands \texttt{STAT \{ PwrReg \}} to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the year (\( x = 1 \) for year 2000) and \( y \) as the gasoline price in dollars per gallon. Round all answers to three decimal places.

b. Use the equation to predict the gasoline price in March 2006. Round all answers to three decimal places. Is the answer close to the actual price?

c. Use the equation to predict the gasoline price in March 2009. Round all answers to three decimal places.
Transformations of Functions

Being a creature of habit, Dylan usually sets out each morning at 7 AM from his house for a jog. Figure 1 shows the graph of a function, \( y = d(t) \), that represents Dylan’s jog on Friday.

a. Use the graph in Figure 1 to fill in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y = d(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Describe a jogging scenario that fits the graph and table above.

b. The graph shown in Figure 2 represents Dylan’s jog on Saturday. It is a transformation of the function \( y = d(t) \) shown in Figure 1.

Complete the table of values below for this transformation. You may find it helpful to refer to the table in part (a).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
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<td></td>
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<td></td>
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</tr>
</tbody>
</table>

What is the real-world meaning of this transformation? How is Dylan’s jog on Saturday different from his usual jog? How is it the same?

The original function (in Figure 1) is represented by the equation \( y = d(t) \). Write an equation, in terms of \( d(t) \), that represents the function graphed in Figure 2. Explain.

c. The graph shown in Figure 3 represents Dylan’s jog on Sunday. It is a transformation of the function \( y = d(t) \) shown in Figure 1.

Complete the table of values below for this transformation.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
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<td></td>
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</tbody>
</table>

What is the real-world meaning of this transformation? How is Dylan’s jog on Sunday different from his usual jog?
The original function (in Figure 1) is represented by the equation \( y = d(t) \). Use function notation to represent the function graphed in Figure 3. Explain.

**d.** Suppose Dylan’s jog on Monday can be represented by the equation \( y = \frac{1}{2}d(t) \).

Complete the table of values below and sketch a graph at the right for this transformation.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
<th>( t )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
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</tr>
</tbody>
</table>

What is the real-world meaning of this transformation? How does Dylan’s jog on Monday differ from his usual jog? How is it the same?

**e.** Suppose Dylan has a goal of cutting his usual jogging time in half, while covering the same distance. Represent this scenario as a transformation of \( y = d(t) \) shown in Figure 1. Complete the table, sketch a graph, and write an equation in function notation. Explain why your equation makes sense. Finally, discuss whether you think Dylan’s goal is realistic.

\[ y = \quad \text{________________________} \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
<th>( t )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
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</tbody>
</table>
The U.S. National Oceanic and Atmospheric Association (NOAA) monitors temperature and carbon emissions at its observatory in Mauna Loa, Hawaii. NOAA's goal is to help foster an informed society that uses a comprehensive understanding of the role of the oceans, coasts, and atmosphere in the global ecosystem to make the best social and economic decisions. The data presented in this chapter is from the Mauna Loa Observatory, where historical atmospheric measurements have been recorded for the last 50 years. You will develop linear models based on this data to predict temperature and carbon emissions in the future.

The following table summarizes average yearly temperature in degrees Fahrenheit °F and carbon dioxide emissions in parts per million (ppm) for Mauna Loa, Hawaii.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp</td>
<td>44.45</td>
<td>43.29</td>
<td>43.61</td>
<td>43.35</td>
<td>46.66</td>
<td>45.71</td>
<td>45.53</td>
<td>47.53</td>
<td>45.86</td>
<td>46.23</td>
</tr>
<tr>
<td>CO₂</td>
<td>316.9</td>
<td>320.0</td>
<td>325.7</td>
<td>331.1</td>
<td>338.7</td>
<td>345.9</td>
<td>354.2</td>
<td>360.6</td>
<td>369.4</td>
<td>379.7</td>
</tr>
</tbody>
</table>

1. Plot the temperature data with time on the horizontal axis and temperature on the vertical axis. Let t = 0 correspond to 1960.

2. Find a linear function that models the temperature in Mauna Loa.
   c. Use linear regression and all data given.

3. Predict what the temperature will be in Mauna Loa in 2020.
   a. Apply the line found in Exercise 2(a).
   b. Apply the line found in Exercise 2(b).
   c. Apply the line found in Exercise 2(c).

4. Predict what the temperature will be in Mauna Loa in 2100.
   a. Apply the line found in Exercise 2(a).
   b. Apply the line found in Exercise 2(b).
   c. Apply the line found in Exercise 2(c).

5. Do you think your models support the claim of "global warming"? Explain.

6. Plot the carbon dioxide emissions data with time on the horizontal axis and carbon dioxide levels on the vertical axis. Let t = 0 correspond to 1960.

7. Find a linear function that models the CO₂ emissions (ppm) in Mauna Loa.
   c. Use linear regression and all data given.

8. Predict the expected CO₂ levels in Mauna Loa in 2020.
   a. Apply the line found in Exercise 7(a).
   b. Apply the line found in Exercise 7(b).
   c. Apply the line found in Exercise 7(c).

9. Predict the expected CO₂ levels in Mauna Loa in 2100.
   a. Apply the line found in Exercise 7(a).
   b. Apply the line found in Exercise 7(b).
   c. Apply the line found in Exercise 7(c).

10. Do you think your models support the claim of the "greenhouse effect"? Explain.
### Chapter 3 Review

<table>
<thead>
<tr>
<th>Section</th>
<th>Concept</th>
<th>Key Ideas/Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3.1</strong></td>
<td>Functions</td>
<td>All functions are relations, but not all relations are functions.</td>
</tr>
<tr>
<td>Relations and functions</td>
<td>A vertical line can intersect a function in at most one point.</td>
<td></td>
</tr>
<tr>
<td>Functions defined by equations</td>
<td>Placeholders notation: $f(x) = 3x^2 - 6x + 2$, $g(x) = 3(x)^2 - 6(x) + 2$</td>
<td></td>
</tr>
<tr>
<td>Function notation</td>
<td>Difference quotient: $f(x + h) - f(x) / h$, $h \neq 0$</td>
<td></td>
</tr>
<tr>
<td>Domain of a function</td>
<td>Are there any restrictions on $x$?</td>
<td></td>
</tr>
<tr>
<td><strong>3.2</strong></td>
<td>Graphs of functions; piecewise-defined functions; increasing and decreasing functions; average rate of change</td>
<td>Common functions</td>
</tr>
<tr>
<td>Recognizing and classifying functions</td>
<td>$f(x) = mx + b$, $f(x) = x$, $f(x) = x^2$,</td>
<td></td>
</tr>
<tr>
<td>&amp; $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) =</td>
<td>x</td>
<td>$, $f(x) = 1/x$ &amp;</td>
</tr>
<tr>
<td>Even and odd functions</td>
<td>Even: Symmetry about $y$-axis: $f(-x) = f(x)$</td>
<td></td>
</tr>
<tr>
<td>&amp; Odd: Symmetry about origin: $f(-x) = -f(x)$ &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Increasing and decreasing functions</td>
<td>• Increasing: rises (left to right)</td>
<td></td>
</tr>
<tr>
<td>&amp; • Decreasing: falls (left to right) &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average rate of change</td>
<td>$f(x_2) - f(x_1) / (x_2 - x_1)$, $x_1 \neq x_2$</td>
<td></td>
</tr>
<tr>
<td>Piecewise-defined functions</td>
<td>Points of discontinuity</td>
<td></td>
</tr>
<tr>
<td><strong>3.3</strong></td>
<td>Graphing techniques: Transformations</td>
<td>Shift the graph of $f(x)$.</td>
</tr>
<tr>
<td>Horizontal and vertical shifts</td>
<td>$f(x + c)$ $c$ units to the left, $c &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>&amp; $f(x - c)$ $c$ units to the right, $c &gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; $f(x) + c$ $c$ units upward, $c &gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; $f(x) - c$ $c$ units downward, $c &gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reflection about the axes</td>
<td>$-f(x)$ Reflection about the x-axis</td>
<td></td>
</tr>
<tr>
<td>&amp; $f(-x)$ Reflection about the y-axis &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stretching and compressing</td>
<td>$cf(x)$ if $c &gt; 1$; stretch vertically</td>
<td></td>
</tr>
<tr>
<td>&amp; $cf(x)$ if $0 &lt; c &lt; 1$; compress vertically</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; $f(cx)$ if $c &gt; 1$; compress horizontally</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; $f(cx)$ if $0 &lt; c &lt; 1$; stretch horizontally &amp;</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 3.4 Operations on functions and composition of functions

**Adding, subtracting, multiplying, and dividing functions**

- \((f + g)(x) = f(x) + g(x)\)
- \((f - g)(x) = f(x) - g(x)\)
- \((f \cdot g)(x) = f(x) \cdot g(x)\)

Domain of the resulting function is the intersection of the individual domains.

- \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\), \(g(x) \neq 0\)

Domain of the quotient is the intersection of the domains of \(f\) and \(g\), and any points when \(g(x) = 0\) must be eliminated.

**Composition of functions**

- \((f \circ g)(x) = f(g(x))\)

The domain of the composite function is a subset of the domain of \(g(x)\). Values for \(x\) must be eliminated if their corresponding values \(g(x)\) are not in the domain of \(f\).

### 3.5 One-to-one functions and inverse functions

**Determine whether a function is one-to-one**

- No two \(x\)-values map to the same \(y\)-value.
- If \(f(x_1) = f(x_2)\), then \(x_1 = x_2\).
- A horizontal line may intersect a one-to-one function in at most one point.

**Inverse functions**

- Only one-to-one functions have inverses.
- \(f^{-1}(f(x)) = x\) and \(f(f^{-1}(x)) = x\).
- Domain of \(f\) = range of \(f^{-1}\).
- Range of \(f\) = domain of \(f^{-1}\).

**Graphical interpretation of inverse functions**

- The graph of a function and its inverse are symmetric about the line \(y = x\).
- If the point \((a, b)\) lies on the graph of a function, then the point \((b, a)\) lies on the graph of its inverse.

**Finding the inverse function**

1. Let \(y = f(x)\).
2. Interchange \(x\) and \(y\).
3. Solve for \(y\).
4. Let \(y = f^{-1}(x)\).

### 3.6 Modeling functions using variation

**Direct variation**

\[ y = kx \]

**Inverse variation**

\[ y = \frac{k}{x} \]

**Joint variation and combined variation**

- **Joint**: One quantity is directly proportional to the product of two or more other quantities.
- **Combined**: Direct variation and inverse variation occur at the same time.
3.1 Functions

Determine whether each relation is a function.

1. \{ (1, 2), (3, 4), (2, 4), (3, 7) \}

2. \{ (−2, 3), (1, −3), (0, 4), (2, 6) \}

3. \{ (4, 7), (2, 6), (3, 8), (1, 7) \}

4. \{ (x^2, y^2) = (36) \}

5. \{ (x, y) = (4, 7), (2, 6), (3, 8), (1, 7) \}

6. \{ y = \sqrt{x} \}

7. \{ y = x + 2 \}

8. \{ y = \sqrt{x} \}

9. \{ y = \sqrt{x} \}

10. \{ y = \sqrt{x} \}

Use the graphs of the functions to find:

11. a. \( f(-2) \)  b. \( f(4) \)
    c. \( x \), where \( f(x) = 0 \)

12. a. \( f(-4) \)  b. \( f(0) \)
    c. \( x \), where \( f(x) = 0 \)

13. Evaluate the given quantities using the following three functions.

\[ f(x) = 4x - 7 \quad F(t) = t^2 + 4t - 3 \quad g(x) = |x^2 + 2x + 4| \]

14. a. \( f(-5) \)  b. \( f(0) \)
    c. \( x \), where \( f(x) = 0 \)

Find the domain of the given function. Express the domain in interval notation.

15. \( f(3) \)

16. \( F(4) \)

17. \( f(-7) \cdot g(3) \)

18. \( \frac{F(0)}{g(0)} \)

19. \( \frac{f(2) - F(2)}{g(0)} \)

20. \( f(3 + h) \)

21. \( \frac{f(3 + h) - f(3)}{h} \)

22. \( \frac{F(t + h) - F(t)}{h} \)

23. \( f(x) = -3x - 4 \)

24. \( g(x) = x^2 - 2x + 6 \)

25. \( h(x) = \frac{1}{x + 4} \)

26. \( F(x) = \frac{7}{x^2 + 3} \)

27. \( G(x) = \sqrt{x} - 4 \)

28. \( H(x) = \frac{1}{\sqrt{2x} - 6} \)

29. If \( f(x) = \frac{D}{x^2 - 16} \), \( f(4) \) and \( f(-4) \) are undefined, and \( f(5) = 2 \), find \( D \).

30. Construct a function that is undefined at \( x = -3 \) and \( x = 2 \) such that the point \((0, -4)\) lies on the graph of the function.

3.2 Graphs of Functions

Determine whether the function is even, odd, or neither.

31. \( f(x) = 2x - 7 \)

32. \( g(x) = 7x^2 + 4x^3 - 2x \)

33. \( h(x) = x^3 - 7x \)

34. \( f(x) = x^4 + 3x^2 \)

35. \( f(x) = x^{1/4} + x \)

36. \( f(x) = \sqrt{x} + 4 \)

37. \( f(x) = \frac{1}{x^3} + 3x \)

38. \( f(x) = \frac{1}{x^3} + 3x^4 + |x| \)
Use the graph of the functions to find:

a. Domain
b. Range
c. Intervals on which the function is increasing, decreasing, or constant.

39. 40.

41. Find the average rate of change of \( f(x) = 4 - x^2 \) from \( x = 0 \) to \( x = 2 \).

42. Find the average rate of change of \( f(x) = |2x - 1| \) from \( x = 1 \) to \( x = 5 \).

Graph the piecewise-defined function. State the domain and range in interval notation.

43. \( F(x) = \begin{cases} x^2 & x < 0 \\ 2 & x \geq 0 \end{cases} \)

44. \( f(x) = \begin{cases} 4 & 0 < x \leq 1 \\ x^2 + 4 & x > 1 \end{cases} \)

45. \( f(x) = \begin{cases} x^2 & x \leq 0 \\ -\sqrt{x} & 0 < x \leq 1 \\ |x + 2| & x > 1 \end{cases} \)

46. \( F(x) = \begin{cases} x^2 & x < 0 \\ x^3 & 0 < x < 1 \\ -|x| - 1 & x \geq 1 \end{cases} \)

Applications

47. Tutoring Costs. A tutoring company charges $25.00 for the first hour of tutoring and $10.50 for every 30-minute period after that. Find the cost function \( C(x) \) as a function of the length of the tutoring session. Let \( x \) = number of 30-minute periods.

48. Salary. An employee who makes $30.00 per hour also earns time and a half for overtime (any hours worked above the normal 40-hour work week). Write a function \( E(x) \) that describes her weekly earnings as a function of the number of hours worked \( x \).

3.3 Graphing Techniques: Transformations

Graph the following functions using graphing aids.

49. \( y = -(x - 2)^2 + 4 \)

50. \( y = |x - 5| - 7 \)

51. \( y = \sqrt{x - 3} + 2 \)

52. \( y = \frac{1}{x - 2} - 4 \)

53. \( y = -\frac{1}{2}x^3 \)

54. \( y = 2x^2 + 3 \)

Use the given graph to graph the following:

55. 56.

57. 58.

59. Shifted to the left three units

60. Shifted down four units

61. Shifted to the right two units and up three units

62. Reflected about the y-axis

63. Stretched by a factor of 5 and shifted down six units

64. Compressed by a factor of 2 and shifted up three units

Write the function whose graph is the graph of \( y = \sqrt{x} \), but is transformed accordingly, and state the domain of the resulting function.

65. \( y = x^2 + 4x - 8 \)

66. \( y = 2x^2 + 6x - 5 \)
Given the functions \( g \) and \( h \), find \( g + h, g - h, g \cdot h \), and \( \frac{g}{h} \) and state the domain.

67. \( g(x) = -3x - 4 \)
   \( h(x) = x - 3 \)

68. \( g(x) = 2x + 3 \)
   \( h(x) = x^2 + 6 \)

69. \( g(x) = \frac{1}{x^2} \)
   \( h(x) = \sqrt{x} \)

70. \( g(x) = \frac{x + 3}{2x - 4} \)
   \( h(x) = \frac{3x - 1}{x - 2} \)

71. \( g(x) = \sqrt{x - 4} \)
   \( h(x) = \sqrt{2x + 1} \)

72. \( g(x) = x^2 - 4 \)
   \( h(x) = x + 2 \)

For the given functions \( f \) and \( g \), find the composite functions \( f \circ g \) and \( g \circ f \), and state the domains.

73. \( f(x) = 3x - 4 \)
   \( g(x) = 2x + 1 \)

74. \( f(x) = x^3 + 2x - 1 \)
   \( g(x) = x + 3 \)

75. \( f(x) = \frac{2}{x + 3} \)
   \( g(x) = \frac{1}{4 - x} \)

76. \( f(x) = \sqrt{2x^2 - 5} \)
   \( g(x) = \sqrt{x + 6} \)

77. \( f(x) = \sqrt{x - 5} \)
   \( g(x) = x^2 - 4 \)

78. \( f(x) = \frac{1}{\sqrt{x}} \)
   \( g(x) = \frac{1}{x^2 - 4} \)

Evaluate \( f(g(3)) \) and \( g(f(-1)) \), if possible.

79. \( f(x) = 4x^2 - 3x + 2 \)
   \( g(x) = 6x - 3 \)

80. \( f(x) = \sqrt{4-x} \)
   \( g(x) = x^2 + 5 \)

81. \( f(x) = \frac{x}{2x - 3} \)
   \( g(x) = |5x + 2| \)

82. \( f(x) = \frac{1}{x - 1} \)
   \( g(x) = x^2 - 1 \)

83. \( f(x) = x^2 - x + 10 \)
   \( g(x) = \sqrt{x - 4} \)

84. \( f(x) = \frac{4}{x^2 - 2} \)
   \( g(x) = \frac{1}{x^2 - 9} \)

Write the function as a composite \( f(g(x)) \) of two functions \( f \) and \( g \).

85. \( h(x) = 3(x - 2)^2 + 4(x - 2) + 7 \)

86. \( h(x) = \frac{\sqrt{x}}{1 - \sqrt{x}} \)

87. \( h(x) = \frac{1}{\sqrt{x^2 + 7}} \)

88. \( h(x) = \sqrt{3x + 4} \)

Applications

89. Rain. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time, in minutes, \( r(t) = 25\sqrt{t + 2} \), find the area of the ripple as a function of time.

90. Geometry. Let the area of a rectangle be given by \( 42 = l \cdot w \), and let the perimeter be \( 36 = 2 \cdot l + 2 \cdot w \). Express the perimeter in terms of \( w \).

### 3.5 One-to-One Functions and Inverse Functions

Determine whether the given function is a one-to-one function.

91. \[
\begin{array}{c|c}
\text{Domain} & \text{Range} \\
\hline
\text{BROTHER} & \text{SISTER} \\
Chris & Paula \\
Harold & Vickie \\
Tom & Renee \\
Danny & Gabriel \\
\end{array}
\]
92. **Domain**\hspace{1cm} **Function**\hspace{1cm} **Range**

<table>
<thead>
<tr>
<th>STUDENTS IN PRECALC</th>
<th>GRADE IN PRECALCULUS COURSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bill</td>
<td>A</td>
</tr>
<tr>
<td>Tracey</td>
<td>C</td>
</tr>
<tr>
<td>Tonja</td>
<td>B</td>
</tr>
<tr>
<td>Troy</td>
<td>D</td>
</tr>
<tr>
<td>Maria</td>
<td>E</td>
</tr>
<tr>
<td>Martin</td>
<td>F</td>
</tr>
</tbody>
</table>

93. \{ (2, 3), (-1, 2), (3, 3), (-3, -4), (-2, 1) \}

94. \{ (-3, 9), (5, 25), (2, 4), (3, 9) \}

95. \{ (-2, 0), (4, 5), (3, 7) \}

96. \{ (-8, -6), (-4, 2), (0, 3), (2, -8), (7, 4) \}

97. \( y = \sqrt{x} \) 98. \( y = x^2 \) 99. \( f(x) = x^3 \) 100. \( f(x) = \frac{1}{x^2} \)

Verify that the function \( f^{-1}(x) \) is the inverse of \( f(x) \) by showing that \( f(f^{-1}(x)) = x \). Graph \( f(x) \) and \( f^{-1}(x) \) on the same graph and show the symmetry about the line \( y = x \).

101. \( f(x) = 3x + 4; f^{-1}(x) = \frac{x - 4}{3} \)

102. \( f(x) = \frac{1}{4x - 7}; f^{-1}(x) = \frac{1 + 7x}{4x} \)

103. \( f(x) = \sqrt{x + 4}; f^{-1}(x) = x^2 - 4; x \geq 0 \)

104. \( f(x) = \frac{x + 2}{x - 7}; f^{-1}(x) = \frac{7x + 2}{x - 1} \)

The function \( f \) is one-to-one. Find its inverse and check your answer. State the domain and range of both \( f \) and \( f^{-1} \).

105. \( f(x) = 2x + 1 \) 106. \( f(x) = x^3 + 2 \)

107. \( f(x) = \sqrt{x + 4} \) 108. \( f(x) = (x + 4)^2 + 3; x \geq -4 \)

109. \( f(x) = \frac{x + 6}{x + 3} \) 110. \( f(x) = 2\sqrt{x - 5} - 8 \)

Applications

111. **Salary.** A pharmaceutical salesperson makes $22,000 base salary a year plus 8% of the total products sold. Write a function \( S(x) \) that represents her yearly salary as a function of the total dollars worth of products sold \( x \). Find \( S^{-1}(x) \). What does this inverse function tell you?

112. **Volume.** Express the volume \( V \) of a rectangular box that has a square base of length \( x \) and is 3 feet high as a function of the square length. Find \( V^{-1} \). If a certain volume is desired, what does the inverse tell you?

3.6 Modeling Functions Using Variation

Write an equation that describes each variation.

113. \( C \) is directly proportional to \( r \). \( C = 2\pi \) when \( r = 1 \).

114. \( V \) is directly proportional to both \( l \) and \( w \). \( V = 12h \) when \( w = 6 \) and \( l = 2 \).

115. \( A \) varies directly with the square of \( r \). \( A = 25\pi \) when \( r = 5 \).

116. \( F \) varies inversely with both \( \lambda \) and \( L \). \( F = \frac{20\pi}{10 \mu m} \) when \( \lambda = 10 \mu m \) and \( L = 10 \text{ km} \).

Applications

117. **Wages.** Cole and Dickson both work at the same museum and have the following paycheck information for a certain week. Find an equation that shows their wages \( (W) \) varying directly with the number of hours \( (H) \) worked.

<table>
<thead>
<tr>
<th>EMPLOYEE</th>
<th>HOURS WORKED</th>
<th>WAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cole</td>
<td>27</td>
<td>$229.50</td>
</tr>
<tr>
<td>Dickson</td>
<td>30</td>
<td>$255.00</td>
</tr>
</tbody>
</table>

118. **Sales Tax.** The sales tax in two neighboring counties differs by 1%. A new resident knows the difference but doesn’t know which county has the higher tax rate. The resident lives near the border of the two counties and wants to buy a new car. If the tax on a $50.00 jacket is $3.50 in County A and the tax on a $20.00 calculator is $1.60 in County B, write two equations (one for each county) that describe the tax \( (T) \), which is directly proportional to the purchase price \( (P) \).

Technology Exercises

**Section 3.1**

119. Use a graphing utility to graph the function and find the domain. Express the domain in interval notation.

\[ f(x) = \frac{1}{\sqrt{x^2 - 2x - 3}} \]

120. Use a graphing utility to graph the function and find the domain. Express the domain in interval notation.

\[ f(x) = \frac{x^2 - 4x - 5}{x^2 - 9} \]
Section 3.2

121. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) x intervals where the function is increasing, decreasing, and constant.

\[ f(x) = \begin{cases} 
1 - x & x < -1 \\
\lfloor x \rfloor & -1 \leq x < 2 \\
x + 1 & x \geq 2 
\end{cases} \]

122. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) x intervals where the function is increasing, decreasing, and constant.

\[ f(x) = \begin{cases} 
x^2 - 1 & -2 < x < 2 \\
\sqrt{x^2 - 2} + 4 & x > 2 
\end{cases} \]

Section 3.3

123. Use a graphing utility to graph \( f(x) = x^2 - x - 6 \) and \( g(x) = x^2 - 5x \). Use transforms to describe the relationship between \( f(x) \) and \( g(x) \).

124. Use a graphing utility to graph \( f(x) = 2x^2 - 3x - 5 \) and \( g(x) = -2x^2 - x + 6 \). Use transforms to describe the relationship between \( f(x) \) and \( g(x) \).

Section 3.4

125. Using a graphing utility, plot \( y_1 = \sqrt{2x + 3} \), \( y_2 = \sqrt{4 - x} \), and \( y_3 = \frac{y_1}{y_2} \). What is the domain of \( y_3 \)?

126. Using a graphing utility, plot \( y_1 = \sqrt{x^2 - 4} \), \( y_2 = x^2 - 5 \), and \( y_3 = y_1^2 - 5 \). If \( y_1 \) represents a function \( f \) and \( y_2 \) represents a function \( g \), then \( y_3 \) represents the composite function \( g \circ f \). The graph of \( y_3 \) is only defined for the domain of \( g \circ f \). State the domain of \( g \circ f \).

Section 3.5

127. Use a graphing utility to graph the function and determine whether it is one-to-one.

\[ f(x) = \frac{6}{\sqrt{x^3 - 1}} \]

128. Use a graphing utility to graph the functions \( f \) and \( g \) and the line \( y = x \) in the same screen. Are the two functions inverses of each other?

\[ f(x) = \sqrt{x - 3} + 1 \], \( g(x) = x^4 - 4x^3 + 6x^2 - 4x + 3 \]

Section 3.6

From December 1999 to December 2007, data for gold price in dollars per ounce are given in the table below. (Data are from Finfacts Ireland Business & Finance Portal, Ireland’s Top Business website at www.finfacts.ie/Private/curency/goldmarketprice.htm.) Use the calculator commands \[ \text{STAT EDIT} \] to enter the table below with \( L_1 \) as the year \( (x = 1 \text{ for year 1999}) \) and \( L_2 \) as the gold price in dollars per ounce.

<table>
<thead>
<tr>
<th>Year</th>
<th>Gold Price in $ per Ounce</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td>287.90</td>
</tr>
<tr>
<td>2000</td>
<td>272.15</td>
</tr>
<tr>
<td>2001</td>
<td>278.70</td>
</tr>
<tr>
<td>2002</td>
<td>346.70</td>
</tr>
<tr>
<td>2003</td>
<td>414.80</td>
</tr>
<tr>
<td>2004</td>
<td>438.10</td>
</tr>
<tr>
<td>2005</td>
<td>517.20</td>
</tr>
<tr>
<td>2006</td>
<td>636.30</td>
</tr>
<tr>
<td>2007</td>
<td>833.20</td>
</tr>
</tbody>
</table>

129. a. Use the calculator commands \[ \text{STAT LinReg} \] to model the data using the least-squares regression. Find the equation of the least-squares regression line using \( x \) as the year \( (x = 1 \text{ for year 1999}) \) and \( y \) as the gold price in dollars per ounce. Round all answers to two decimal places.

b. Use the equation to predict the gold price in December 2005. Round all answers to two decimal places. Is the answer close to the actual price?

c. Use the equation to predict the gold price in December 2008. Round all answers to two decimal places.

130. a. Use the calculator commands \[ \text{STAT PwrReg} \] to model the data using the power function. Find the variation constant and equation of variation using \( x \) as the year \( (x = 1 \text{ for year 1999}) \) and \( y \) as the gold price in dollars per ounce. Round all answers to two decimal places.

b. Use the equation to predict the gold price in December 2005. Round all answers to two decimal places. Is the answer close to the actual price?

c. Use the equation to predict the gold price in December 2008. Round all answers to two decimal places.
CHAPTER 3 PRACTICE TEST

Assuming that \( x \) represents the independent variable and \( y \) represents the dependent variable, classify the relationships as:

a. not a function
b. a function, but not one-to-one
c. a one-to-one function

1. \( f(x) = |2x + 3| \)  
2. \( x = y^2 + 2 \)  
3. \( y = \sqrt[3]{x + 1} \)

Use \( f(x) = \sqrt{x - 2} \) and \( g(x) = x^2 + 11 \), and determine the desired quantity or expression. In the case of an expression, state the domain.

4. \( f(11) - 2g(-1) \)
5. \( \left( \frac{f}{g} \right)(x) \)
6. \( \left( \frac{g}{f} \right)(x) \)
7. \( g(f(x)) \)
8. \( f(g(6)) \)
9. \( f(g(\sqrt{7})) \)

Determine whether the function is odd, even, or neither.
10. \( f(x) = |x| - x^2 \)  
11. \( f(x) = 9x^3 + 5x - 3 \)  
12. \( f(x) = \frac{2}{x} \)

Graph the functions. State the domain and range of each function.
13. \( f(x) = \sqrt{-x^3 + 2} \)  
14. \( f(x) = -2(x - 1)^2 \)
15. \( f(x) = \begin{cases} -x & x < -1 \\ 1 & -1 < x < 2 \\ x^2 & x \geq 2 \end{cases} \)

Use the graphs of the function to find:
16. \( y = f(x) \)

a. \( f(3) \)  
b. \( f(0) \)  
c. \( f(-4) \)  
d. \( x, \text{ where } f(x) = 3 \)  
e. \( x, \text{ where } f(x) = 0 \)

17. \( y = g(x) \)

a. \( g(3) \)  
b. \( g(0) \)  
c. \( g(-4) \)  
d. \( x, \text{ where } g(x) = 0 \)

e. \( x, \text{ where } f(x) = 0 \)

18. \( \frac{f(x + h) - f(x)}{h} \) for:
19. \( f(x) = 3x^2 - 4x + 1 \)  
20. \( f(x) = 5 - 7x \)

Find the average rate of change of the given functions.
21. \( f(x) = 64 - 16x^2 \) for \( x = 0 \) to \( x = 2 \)
22. \( f(x) = \sqrt{x - 1} \) for \( x = 2 \) to \( x = 10 \)

Given the function \( f \), find the inverse if it exists. State the domain and range of both \( f \) and \( f^{-1} \).
23. \( f(x) = \sqrt{x - 5} \)  
24. \( f(x) = x^2 + 5 \)
25. \( f(x) = \frac{2x + 1}{5 - x} \)  
26. \( f(x) = \begin{cases} -x & x \leq 0 \\ -x^2 & x > 0 \end{cases} \)

27. What domain restriction can be made so that \( f(x) = x^2 \) has an inverse?

28. If the point \((-2, 5)\) lies on the graph of a function, what point lies on the graph of its inverse function?

29. Discount. Suppose a suit has been marked down 40% off the original price. An advertisement in the newspaper has an “additional 30% off the sale price” coupon. Write a function that determines the “checkout” price of the suit.

30. Temperature. Degrees Fahrenheit (\( ^\circ F \)), degrees Celsius (\( ^\circ C \)), and kelvins (K) are related by the two equations:
\( F = \frac{9}{5}C + 32 \) and \( K = C + 273.15 \). Write a function whose input is kelvins and output is degrees Fahrenheit.

31. Circles. If a quarter circle is drawn by tracing the unit circle in quadrant III, what does the inverse of that function look like? Where is it located?

32. Sprinkler. A sprinkler head malfunctions at midfield in an NFL football field. The puddle of water forms a circular pattern around the sprinkler head with a radius in yards that grows as a function of time, in hours: \( r(t) = 10\sqrt{t} \). When will the puddle reach the sidelines? (A football field is 30 yards from sideline to sideline.)
33. **Internet.** The cost of airport Internet access is $15 for the first 30 minutes and $1 per minute for each minute after that. Write a function describing the cost of the service as a function of minutes used.

Use variation to find a model for the given problem.

34. \( y \) varies directly with the square of \( x \). \( y = 8 \) when \( x = 5 \).

35. \( F \) varies directly with \( m \) and inversely with \( p \). \( F = 20 \) when \( m = 2 \) and \( p = 3 \).

36. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) \( x \) intervals where the function is increasing, decreasing, and constant.

\[
f(x) = \begin{cases} 
5 & -4 \leq x < -2 \\
|3 + 2x - x^2| & -2 \leq x \leq 4
\end{cases}
\]

37. Use a graphing utility to graph the function and determine whether it is one-to-one.

\[
y = x^3 - 12x^2 + 48x - 65
\]
CHAPTERS 1—3 CUMULATIVE TEST

1. Simplify \( \frac{2}{3 - \sqrt{5}} \)
2. Factor completely: \( 10x^2 - 29x - 21 \)
3. Simplify and state the domain: \( \frac{x^3 - 4x}{x + 2} \)
4. Solve for \( x \): \( \frac{1}{6}x = \frac{1}{5}x + 11 \)
5. Perform the operation, simplify, and express in standard form: \( (8/\sqrt{11} - 9i)(8/\sqrt{11} + 9i) \)
6. Solve for \( x \), and give any excluded values: \( \frac{x}{x^2 - 2} = \frac{1}{3}x - 2 \)
7. The original price of a hiking stick is \$59.50. The sale price is \$35.70. Find the percent of the markdown.
8. Solve by factoring: \( x(6x + 1) = 12 \)
9. Solve by completing the square: \( \frac{x^2}{2} - x = \frac{1}{5} \)
10. Solve and check: \( \sqrt{x + 2} = -3 \)
11. Solve using substitution: \( x^3 - x^2 - 12 = 0 \)

Solve and express the solution in interval notation.

12. \(-7 < 3 - 2x \leq 5\)
13. \( \frac{x}{x - 5} < 0 \)
14. \( |2.7 - 3.2x| \leq 1.3 \)
15. Calculate the distance and midpoint between the segment joining the points \((-2.7, -1.4)\) and \((5.2, 6.3)\).
16. Find the slope of the line passing through the points \((0.3, -1.4)\) and \((2.7, 4.3)\).
17. Write an equation of a line that passes through the points \((1.2, -3)\) and \((-0.2, -3)\).
18. Transform the equation into standard form by completing the square, and state the center and radius of the circle: \( x^2 + y^2 + 12x - 18y - 4 = 0 \)
19. Find the equation of a circle with center \((-2, -1)\) and passing through the point \((-4, 3)\).
20. If a cellular phone tower has a reception radius of 100 miles and you live 85 miles north and 23 miles east of the tower, can you use your cell phone at home? Explain.
21. Use interval notation to express the domain of the function \( g(x) = \frac{1}{x - 1} \).
22. Find the average rate of change for \( f(x) = 5x^2 \), from \( x = 2 \) to \( x = 4 \).
23. Evaluate \( g(f(-1)) \) for \( f(x) = |6 - x| \) and \( g(x) = x^2 - 3 \).
24. Find the inverse of the function \( f(x) = x^2 + 3 \) for \( x \geq 0 \).
25. Write an equation that describes the variation: \( r \) is inversely proportional to \( t \). \( r = 45 \) when \( t = 3 \).
26. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) \( x \) intervals where the function is increasing, decreasing, and constant.

\[
f(x) = \begin{cases} 
1 - |x| & -1 \leq x < 1 \\
1 - |x - 2| & 1 < x \leq 3 
\end{cases}
\]
27. Use a graphing utility to graph the function \( f(x) = x^2 - 3x \) and \( g(x) = x^2 + x - 2 \) in the same screen. Find the function \( h \) such that \( g \circ h = f \).