1 Number systems: real and complex

1.1 Kick off with CAS
1.2 Review of set notation
1.3 Properties of surds
1.4 The set of complex numbers
1.5 Multiplication and division of complex numbers
1.6 Representing complex numbers on an Argand diagram
1.7 Factorising quadratic expressions and solving quadratic equations over the complex number field
1.8 Review eBookplus
1.1 Kick off with CAS

To come

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
1.2 Review of set notation

Sets contain elements. In this chapter the elements are numbers. For example, the following are six elements: 1, 2, 3, 4, 5, 6.

$\xi$ is the universal set — the set of all elements under consideration. So, in this example, $\xi = \{1, 2, 3, 4, 5, 6\}$.

$\emptyset$ is the empty or null set. This set contains no elements. $\emptyset = \{}$.

An upper case letter, such as $A$, represents a subset of $\xi$. In our example, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4\}$.

$\in$ is read as ‘is an element of’. For example, $3 \in A$.

$\notin$ is read as ‘is not an element of’. For example, $2 \notin A$.

$\subset$ is read as ‘is a subset of’. For example, $\{1, 3\} \subset A$.

$\supset$ is read as ‘is a superset of’. For example, $A \supset \{1, 3\}$.

Related symbols, such as $\supseteq$, $\subseteq$ and $\not\subset$, are also used.

$A'$ is the complement of $A$. This set contains all the elements not in $A$ that are in $\xi$.

For example, given $\xi = \{1, 2, 3, 4, 5, 6\}$, if $A = \{1, 3, 5\}$, then $A' = \{2, 4, 6\}$.

$A \cup B$ is the union of $A$ and $B$. This set contains all the elements in sets $A$ and $B$.

For the example above, $A \cup B = \{1, 2, 3, 4, 5\}$.

$A \cap B$ is the intersection of $A$ and $B$. This set contains all the elements in both $A$ and $B$.

For the example above, $A \cap B = \{1, 3\}$.

$C \setminus D$ is read as ‘$C$ slash $D$’. This set contains all the elements in $C$ that are not in $D$. If $C = \{1, 2, 5, 6\}$ and $D = \{2, 5\}$, then $C \setminus D = \{1, 6\}$.

This notation is particularly useful in modifying a given set to exclude a small number of elements.

A Venn diagram may be used to illustrate set notation.

WORKED EXAMPLE 1

$\xi = \{2, 4, 6, 8, 10, 12\}$, $C = \{4, 8, 12\}$ and $D = \{2, 6, 10, 12\}$.

a Illustrate these sets on a Venn diagram.

Then state:

b $C'$

d $C \cap D$

f $C' \cap D'$

c $C \cup D$

e $(C \cup D)'$

g $C \setminus \{2\}$.

THINK

a Draw a Venn diagram and enter the elements in the appropriate region.

WRITE

a
The set of real numbers

Natural numbers

Numbers were invented to quantify objects in the environment. Hunter-gatherers used counting or natural numbers to communicate how many of a particular animal were seen on a hunting trip. The set of natural numbers is given as \( N = \{1, 2, 3, \ldots\} \). Natural numbers are positive whole numbers.

<table>
<thead>
<tr>
<th>Topic 1 NUMBER SYSTEMS: REAL AND COMPLEX</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b</strong> The set ( C' ) is the complement of ( C ) and contains all elements not in the set ( C ).</td>
</tr>
<tr>
<td><strong>c</strong> The set ( C \cup D ) is the union of ( C ) and ( D ) and contains all elements in sets ( C ) and ( D ).</td>
</tr>
<tr>
<td><strong>d</strong> The set ( C \cap D ) is the intersection of ( C ) and ( D ) and contains elements common to sets ( C ) and ( D ).</td>
</tr>
<tr>
<td><strong>e</strong> The set ((C \cup D)') is the complement of the union of sets ( C ) and ( D ). It contains elements not in the union of sets ( C ) and ( D ). In this case, there are no elements not in the union of sets ( C ) and ( D ).</td>
</tr>
<tr>
<td><strong>f</strong> The set ( C' \cap D' ) is the intersection of ( C' ) and ( D' ). It contains elements common to the sets ( C' ) and ( D' ). There are no common elements to ( C' ) and ( D' ).</td>
</tr>
<tr>
<td><strong>g</strong> The set ( C' \setminus {2} ) is the set of ( C' ) without the element 2. It contains all the elements of the set ( C' ) but not 2.</td>
</tr>
</tbody>
</table>

**b** \( C' = \{2, 6, 10\} \)

**c** \( C \cup D = \{2, 4, 6, 8, 10, 12\} \)

**d** \( C \cap D = \{12\} \)

**e** \((C \cup D)' = \emptyset\)

**f** \( C' \cap D' = \emptyset\)

**g** \( C' \setminus \{2\} = \{6, 10\} \)

**The set of real numbers**

**Natural numbers**

Numbers were invented to quantify objects in the environment. Hunter-gatherers used counting or natural numbers to communicate how many of a particular animal were seen on a hunting trip. The set of natural numbers is given as \( N = \{1, 2, 3, \ldots\} \). Natural numbers are positive whole numbers.

**WORKED EXAMPLE 2**

Prove that \( 6^n + 4 \) is divisible by 5 for all \( n \in \mathbb{N} \).

**THINK**

1. Let \( P(n) \) be the statement that \( 6^n + 4 \) divides evenly by 5.
2. Let \( n = 1 \).
3. Assume that it is true for \( n = k \).
4. Consider \( n = k + 1 \)

**WRITE**

\[
\frac{6^n + 4}{5} = 0
\]

\[
P(1) = 6^1 + 4 = 10, \text{ which is divisible by 5} \\
\therefore P(1) \text{ is true.}
\]

\[
P(k) = 6^k + 4 \text{ is divisible by 5} \\
\therefore P(k) \text{ is true.}
\]

\[
P(k + 1) = 6^{k+1} + 4 \\
= (6^k \times 6^1) + 4
\]
A prime number is a positive natural number that has exactly two factors: itself and one.
Composite numbers are positive natural numbers that have more than two factors, including at least one prime factor.

Are there an infinite number of prime numbers?
Let us assume that there is a finite number of prime numbers, say $p_1, p_2, p_3, \ldots, p_n$.
Let $M = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$.

$M$ could either be a prime number or a composite number. If $M$ is a prime number, then $p_n$ cannot be the last prime number. If $M$ is a composite number, then none of the primes $p_1, p_2, p_3, \ldots, p_n$ can divide into it, since there will always be a remainder of 1. Hence a prime factor exists that is not in $p_1, p_2, p_3, \ldots, p_n$. Therefore, we can conclude that there are infinitely many prime numbers.

Integers and rational numbers
The systematic consideration of the concept of number in algebra, and the numbers required to solve equations of the form $x + 2 = 0$ and $3x + 1 = 0$, resulted in the invention of integers and rational numbers.
The set of integers is given by $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$. They are positive and negative whole numbers, including zero.
$\mathbb{Z}^-$ is the set of negative integers: $\mathbb{Z}^- = \{\ldots, -3, -2, -1\}$.
$\mathbb{Z}^+$ is the set of positive integers: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.
Therefore, $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.

The set of rational numbers is given by $Q$. These are numbers of the form $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$. Whole numbers are also rational numbers.
Consistent with the definition of $Q$, $\mathbb{Z} \subset Q$.
$Q^-$ is the set of negative rational numbers.
$Q^+$ is the set of positive rational numbers.
Therefore, $Q = Q^- \cup \{0\} \cup Q^+$.
Rational numbers in their simplest form with denominators such as 2, 4, 5, 8, 10 produce terminating decimals. Some examples include:

$$\frac{3}{8} = 0.375, \frac{7}{16} = 0.4375, \frac{89}{125} = 0.712, \frac{123}{64} = 1.921875$$
Rational numbers in their simplest form with denominators such as 3, 6, 7, 9, 11, 13, 14, 15, 17 produce non-terminating recurring or repeating decimals. Some examples include:

\[
\begin{align*}
\frac{1}{3} &= 0.333\ldots = 0.\overline{3}, \\
\frac{1}{6} &= 0.1666\ldots = 0.\overline{16}, \\
\frac{5}{12} &= 0.41666\ldots = 0.4\overline{16}, \\
\frac{17}{99} &= 0.171717\ldots = 0.\overline{17}. \\
\end{align*}
\]

Irrational numbers

Irrational numbers are given by \( \sqrt{I} \). They are numbers that can be placed on a number line and may be expressed as non-terminating, non-recurring decimals. For example:

\[-\sqrt{2}, -\sqrt{3}, \frac{\sqrt{5}}{2}, -\sqrt{5} + 1, 4^3, 5^3, \pi.\]

Irrational numbers cannot be written in the form \( \frac{p}{q} \), where \( p \in Z \) and \( q \in Z \setminus \{0\} \).

Many irrational numbers in decimal form, such as \( \sqrt{2} \) and \( \pi \), have digits that have no pattern. For these numbers, it is impossible to predict the next digit from the preceding digits. However, other irrational numbers can be constructed with a pattern; for example:

\[0.101100111111011100.00\ldots \text{ and } 0.0101101111111111\ldots\]

There are two important subsets of the set of irrational numbers: the set of algebraic numbers and the set of transcendental numbers.

**Algebraic numbers** are those that are the solution of an algebraic polynomial equation of the form:

\[a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,\]

where \( a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in Z \). For example, algebraic numbers include \( 3^{\frac{1}{2}} \) from one of the solutions of \( x^2 - 3 = 0 \) and \( 2^{\frac{1}{2}} \) from \( x^2 - 8 = 0 \).

**Transcendental numbers** occur in the evaluation of some functions, such as trigonometric functions. For example, \( \sin(32.1^\circ) \) and \( \pi \) are transcendental numbers. The functions that produce these numbers are often called *transcendental functions*. 

---

**WORKED EXAMPLE 3**

**Using a calculator, express the following rational numbers in decimal form.**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong></td>
<td><strong>b</strong></td>
</tr>
<tr>
<td>( \frac{5}{16} )</td>
<td>( \frac{4}{7} )</td>
</tr>
</tbody>
</table>

**THINK**

a Since the denominator is 16, expect a terminating decimal.

b 1 Since the denominator is 7, expect a non-terminating, repeating decimal.

2 Indicate the repeating sequence using dot notation.

**WRITE**

a \( \frac{5}{16} = 0.3125 \)

b \( \frac{4}{7} = 0.5714285714\ldots \)
**Why is \( \sqrt{7} \) an irrational number?**

Assume that \( \sqrt{7} \) is a rational number and can be written in the form \( \frac{p}{q} \), a fraction in simplest form, where \( p \in \mathbb{Z} \) and \( q \in \mathbb{Z} \setminus \{0\} \).

\[
\sqrt{7} = \frac{p}{q}
\]

\[
7 = \frac{p^2}{q^2}
\]

\[
p^2 = 7q^2
\]

Therefore, \( p^2 \) is divisible by 7, which means that \( p \) is divisible by 7.

Let \( p = 7k, k \in \mathbb{Z} \)

\[
(7k)^2 = 7q^2
\]

\[
49k^2 = 7q^2
\]

\[
q^2 = 7k^2
\]

Therefore, \( q^2 \) is divisible by 7, which means that \( q \) is divisible by 7.

As \( p \) and \( q \) have a common factor of 7, this contradicts the fact that \( \frac{p}{q} \) is a fraction written in simplest form. The assumption that \( \sqrt{7} \) is rational is incorrect, hence \( \sqrt{7} \) is an irrational number.

**Real numbers**

Finally, the set of real numbers is given as \( \mathbb{R} \). \( \mathbb{R} \) includes all numbers that can be put on a number line, where \( R = Q \cup I \). The Venn diagram shows the relationships between \( R, Q, I, Z \) and \( N \).

---

**Absolute value of a real number**

The absolute value \( |a| \) of a real number \( a \) is the distance from \( a \) to zero.

\[
|-8| = 8
\]

\[
|11| = 11
\]

\[
|\sqrt{2}| = \sqrt{2}
\]
### WORKED EXAMPLE 4

For each of the numbers below, using \( R, Q, I, Z \) and \( N \), state all the sets for which they are a member.

<table>
<thead>
<tr>
<th>Number</th>
<th>Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>( Z ), ( Q ), ( R ), ( N )</td>
</tr>
<tr>
<td>( \frac{17}{3} )</td>
<td>( Q ), ( R )</td>
</tr>
<tr>
<td>3( \sqrt{2} )</td>
<td>( I ), ( R )</td>
</tr>
<tr>
<td>4.153</td>
<td>( Q ), ( R )</td>
</tr>
<tr>
<td>1.011 011 101 111...</td>
<td>( I ), ( R )</td>
</tr>
<tr>
<td>32( ^{\frac{1}{5}} )</td>
<td>( Z ), ( Q ), ( R )</td>
</tr>
</tbody>
</table>

**THINK**

<table>
<thead>
<tr>
<th>Number</th>
<th>Thought</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-5 is an integer.</td>
</tr>
<tr>
<td>( \frac{17}{3} )</td>
<td>( \frac{17}{3} ) is a rational number, as it can be written as a fraction.</td>
</tr>
<tr>
<td>3( \sqrt{2} )</td>
<td>3( \sqrt{2} ) is an irrational number.</td>
</tr>
<tr>
<td>27.179</td>
<td>27.179 is a rational number, as it is a recurring decimal.</td>
</tr>
<tr>
<td>4.153</td>
<td>4.153 is a rational number, as it is a terminating decimal.</td>
</tr>
<tr>
<td>17.1354...</td>
<td>17.1354... is an irrational number as there is no indication that there is a recurring pattern.</td>
</tr>
<tr>
<td>1.011 011 101 111...</td>
<td>1.011 011 101 111... is an irrational number.</td>
</tr>
<tr>
<td>32( ^{\frac{1}{5}} )</td>
<td>32( ^{\frac{1}{5}} ) can be simplified to 2 and is therefore a natural number.</td>
</tr>
<tr>
<td>17( ^{\frac{1}{3}} )</td>
<td>17( ^{\frac{1}{3}} ) is an irrational number.</td>
</tr>
</tbody>
</table>

**WRITE**

<table>
<thead>
<tr>
<th>Number</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-5 is a negative integer (( Z^- )). It is also a rational number (( Q )) and a real number (( R )).</td>
</tr>
<tr>
<td>( \frac{17}{3} )</td>
<td>( \frac{17}{3} ) is a rational number (( Q )) and a real number (( R )).</td>
</tr>
<tr>
<td>3( \sqrt{2} )</td>
<td>3( \sqrt{2} ) is an irrational number (( I )) and a real number (( R )).</td>
</tr>
<tr>
<td>27.179</td>
<td>27.179 is a rational number (( Q )) and a real number (( R )).</td>
</tr>
<tr>
<td>4.153</td>
<td>4.153 is a rational number (( Q )) and a real number (( R )).</td>
</tr>
<tr>
<td>17.1354...</td>
<td>17.1354... is an irrational number (( I )) and a real number (( R )).</td>
</tr>
<tr>
<td>1.011 011 101 111...</td>
<td>1.011 011 101 111... is an irrational number (( I )) and a real number (( R )).</td>
</tr>
<tr>
<td>32( ^{\frac{1}{5}} )</td>
<td>32( ^{\frac{1}{5}} ) is a natural number (( N )). It is also an integer (( Z )), a rational number (( Q )) and a real number (( R )).</td>
</tr>
<tr>
<td>17( ^{\frac{1}{3}} )</td>
<td>17( ^{\frac{1}{3}} ) is an irrational number (( I )) and a real number (( R )).</td>
</tr>
</tbody>
</table>

### WORKED EXAMPLE 5

Express each of the following in the form \( \frac{a}{b} \), where \( a \in Z \) and \( b \in Z \setminus \{0\} \).

<table>
<thead>
<tr>
<th>Number</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>0.23</td>
<td>0.23</td>
</tr>
</tbody>
</table>

**THINK**

<table>
<thead>
<tr>
<th>Number</th>
<th>Thought</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>Write 0.6 in expanded form.</td>
</tr>
<tr>
<td>0.23</td>
<td>Multiply [1] by 10.</td>
</tr>
</tbody>
</table>

**WRITE**

<table>
<thead>
<tr>
<th>Number</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.6 = 0.666666... [1]</td>
</tr>
<tr>
<td>0.23</td>
<td>0 × 0.23 = 0.23</td>
</tr>
</tbody>
</table>
The basic properties of number are assumed to be true if a counterexample cannot be found. For example, the statement ‘the product of two integers is an integer’ is accepted as true because a counterexample has not been found, but the statement ‘the quotient of two integers is an integer’ is false because a counterexample \( \frac{2}{3} \) is not an integer.

### WORKED EXAMPLE 6

Determine counterexamples for the following.

**a** The product of two irrational numbers is irrational.

**b** The sum of two irrational numbers is irrational.

**THINK**

**a** Take a simple irrational number such as \( \sqrt{2} \). Multiply by another irrational number, say \( \sqrt{2} \). State your answer.

**b** Take two irrational numbers such as 0.1011 001 11000... and 0.010011 000 111... Add these numbers.

**WRITE**

**a** Because \( \sqrt{2} \times \sqrt{2} = 2 \), which is a rational number, the statement ‘the product of two irrational numbers is irrational’ is shown to be false.

**b**

\[
0.101100111000... + 0.010011000111... = 0.111111111111...
\]

Because 0.111 111 111 111... is a rational number, the statement ‘the sum of two irrational numbers is irrational’ has been shown to be false.

### Standard form or scientific notation

Very large or very small numbers are conveniently expressed in standard form, \( a \times 10^b \), where \( a \in R, 1 \leq a < 10 \) and \( b \in Z \). For example, \( 1234111 = 1.234111 \times 10^6 \) and \( 0.000\,000\,000\,045 = 4.5 \times 10^{-11} \).

### Decimal places and significant figures

The numerical answer to a calculation may be required to be given correct to a set number of **decimal places**, and this is done through a process of rounding. To determine the number of decimal places contained in a number, count the number
of digits after the decimal point. For example, 0.35 has 2 decimal places. For numbers expressed to a given number of decimal places, remember to round up if the next digit is 5 or more. For example, rounded to 2 decimal places, 2.234 becomes 2.23 and 2.236 becomes 2.24.

To determine the number of significant figures contained in a number, count the number of digits from the first non-zero digit. For example, 0.035 contains 2 significant figures. Any zeros at the end of a number after a decimal point are considered to be significant. For example, 1.40 has 3 significant figures. The trailing zeros at the end of a number are not considered to be significant. For example, 24,000 has 2 significant figures.

For numbers expressed to a given number of significant figures, remember to round. For example, rounded to 2 significant figures, 2.234 becomes 2.2 and 2.236 also becomes 2.2.

Some examples are shown in the following table.

<table>
<thead>
<tr>
<th>Number</th>
<th>2 significant figures</th>
<th>3 significant figures</th>
<th>2 decimal places</th>
<th>3 decimal places</th>
</tr>
</thead>
<tbody>
<tr>
<td>471 860.2378</td>
<td>470 000</td>
<td>472 000</td>
<td>471 860.24</td>
<td>471 860.238</td>
</tr>
<tr>
<td>1.238 9</td>
<td>1.2</td>
<td>1.24</td>
<td>1.24</td>
<td>1.239</td>
</tr>
<tr>
<td>1.006 8</td>
<td>1.0</td>
<td>1.01</td>
<td>1.01</td>
<td>1.007</td>
</tr>
<tr>
<td>0.01678</td>
<td>0.017</td>
<td>0.0168</td>
<td>0.02</td>
<td>0.017</td>
</tr>
<tr>
<td>0.001556</td>
<td>0.0016</td>
<td>0.00156</td>
<td>0.00</td>
<td>0.002</td>
</tr>
<tr>
<td>0.1991</td>
<td>0.20</td>
<td>0.199</td>
<td>0.20</td>
<td>0.199</td>
</tr>
</tbody>
</table>

WORKED EXAMPLE 7

Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation correct to 1 significant figure.

\[ a \times 10^{24} \times 3 \times 10^{-10} \quad \text{and} \quad \frac{7 \times 10^{17}}{8 \times 10^{-19}} \]

THINK

a 1 Multiply the terms by using the properties of indices:
   \[ a^n \times a^m = a^{n+m} \].

2 Write the answer in standard form, correct to 1 significant figure.

b 1 Multiply the terms by using the properties of indices:
   \[ a^n \div a^m = a^{n-m} \].

2 Write the answer in standard form, correct to 1 significant figure.

WRITE

a \[ 8 \times 10^{24} \times 3 \times 10^{-10} = 24 \times 10^{14} \]

\[ 24 \times 10^{14} = 2.4 \times 10 \times 10^{14} \]

\[ = 2.4 \times 10^{15} \]

\[ = 2 \times 10^{15} \]

b \[ \frac{7 \times 10^{17}}{8 \times 10^{-19}} = 0.875 \times 10^{27} \]

\[ 0.875 \times 10^{27} = 0.9 \times 10^{27} \text{ or } \]

\[ = 9 \times 10^{26} \]
Subsets of the set of real numbers

Notation

There are different forms of notation for representing subsets.

1. Set notation

For example \( \{x: x \in Z, 1 \leq x \leq 5\} \), which is read as ‘the set of numbers \( x \) such that \( x \) is an element of the set of integers and \( x \) is greater than or equal to 1 and less than or equal to 5’.

If \( x \in R \), it is not necessary to include the nature of \( x \). For example, \( \{x: x \geq 2\} \) represents the set of real numbers greater than or equal to 2. The two sets above may be represented on a number line as follows.

If \( x \in Q \), the graph on the number line appears to look like the corresponding graph for \( x \in R \) because the number line appears to be continuous (although all irrational numbers are missing). For example, \( \{x: x \in Q, x \geq 2\} \) would appear to be identical to the graph of \( \{x: x \geq 2\} \) shown above.

If individual numbers are excluded from a given set, indicate this on a number line by an open circle. If individual numbers are included in a given set indicate this on the number line by a closed circle. For example, \( \{x: x \geq 2\}\{3\} \) is represented on a number line below.

A given set can be stated in more than one way using set notation.

For example, \( \{1, 2; 3, 4, 5\} \) can be written as \( \{x: x \in Z, 0 < x < 6\} \), \( \{x: 1 \leq x \leq 5\} \) or \( \{x: x \in Z^+, x \leq 5\} \).

2. Interval notation

Interval notation uses brackets, either square brackets \([-1, 6]\) or curved brackets \((1, 6)\) to describe a range of numbers between two numbers. Square brackets include the numbers at the end of the interval; curved brackets exclude them. For example: \( (3, 8) \) represents all real numbers between 3 and 8, excluding both 3 and 8.

\([-2, 6]\) represents all real numbers between \(-2\) and 6, including both \(-2\) and 6.

\([-4, 10)\) represents all real numbers between \(-4\) and 10, including \(-4\) and excluding 10.

Example sets are illustrated on number lines below.

\( [a, b] = \{x: a \leq x \leq b\} \)

\( (a, b] = \{x: a \leq x < b\} \)
 Topic 1 NUMBER SYSTEMS: REAL AND COMPLEX

(a, b) = \{x : a < x < b\}

(–\infty, a] = \{x : x \leq a\}

(a, \infty) = \{x : x > a\}

**WORKED EXAMPLE 8** List the following sets and then express each set using set notation. Illustrate each set on a number line.

**a** [Integers between –3 and 4]

**b** [Integers less than 2]

**THINK**

**a** 1 This set involves integers.

List the set of integers.

Express the set using set notation.

2 Draw a number line showing arrowheads on each end. Ensure that the numbers from –3 to 4 are shown using an appropriate scale.

Since the set of integers is to be represented, do not join the dots.

**WRITE/DRAW**

**a** \{–2, –1, 0, 1, 2, 3\} = \{x : x \in \mathbb{Z}, –3 < x < 4\}

**b** \{\ldots, –2, –1, 0, 1\} = \{x : x \in \mathbb{Z}, x < 2\}

**WORKED EXAMPLE 9** Use set notation to represent the following sets.

**a** [Rational numbers greater than 27]

**b** [Integers between and including both 100 and 300, except for 200]

**c** [Positive integers less than 9 and greater than 50]

**d** [Real numbers that are less than 7 and greater than 2]

**e** [Positive real numbers that are less than 2 or greater than 7]
14
MATHS QUEST 11 SPECIALIST MATHEMATICS VCE Units 1 and 2

Review of set notation

1 WE1 If \( \xi = \{1, 2, 3, 4, 5, 6\} \), \( A = \{1, 2\} \) and \( B = \{2, 3\} \), show these on a Venn diagram, and then state the following sets.
   a) \( A' \)
   b) \( A \cup c \)
   c) \( A \cap B \)
   d) \( A \setminus \{2\} \)

2 If \( \xi = \{4, 8, 12, 16, 20, 24, 28, 32, 36\} \), \( A = \{4, 8, 20\} \) and \( B = \{20, 24, 28, 32, 36\} \), show these on a Venn diagram, and then state the following sets.
   a) \( B' \)
   b) \( A \cup B' \)
   c) \( A' \cup B' \)
   d) \( (A \cap B)' \)

3 WE2 Prove that \( 4^n - 1 \) is divisible by 3 for all \( n \in \mathbb{N} \).

4 Prove that \( n^3 + 2n \) is divisible by 3 for all \( n \in \mathbb{N} \).
5. Use a calculator to express the following rational numbers in decimal form.
   \( \frac{213}{64} \)  \( \frac{15}{44} \)

6. Use a calculator to express the following rational numbers in decimal form.
   \( \frac{17}{12} \)  \( \frac{16}{13} \)

7. For each number, using \( \mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{I} \) and \( \mathbb{R} \), state all the sets of which it is a member.
   \(-2, \frac{16}{8}, \frac{21}{16}, -\frac{3}{2}, 6\sqrt{3}, 16^2 \)

8. Specify to which sets (\( \mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{I} \) and \( \mathbb{R} \)) each of the following numbers belong.
   5, \( \pi \), 21.72, 2.567, 4.135218974..., 4.232332333.

9. Express the following in their simplest rational form.
   \( a \) \( \frac{0.2}{4} \)  \( b \) \( \frac{1}{23} \)  \( c \) \( \frac{0.12}{3} \)  \( d \) \( \frac{2.11}{23} \)

10. Express each of the following in the form \( \frac{a}{b} \), where \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z}\setminus\{0\} \).
    \( a \) \( 0.41 \)  \( b \) \( 2.12 \)

11. Find a counterexample, if possible, for the following statements. If a counterexample is found, the statement is false (F). If a counterexample is not found, accept the statement to be true (T).
    a. The product of two integers is an integer.
    b. The division of an integer by an integer is a rational number.

12. a. The difference of two irrational numbers is irrational.
    b. The sum of an irrational number and a rational number is irrational.

13. Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation to 1 significant figure.
    \( a \) \( 1.5 \times 10^{16} \times 4 \times 10^{12} \)  \( b \) \( 1.2 \times 10^{24} \times 3 \times 10^{-10} \)  \( c \) \( 3.2 \times 10^{25} \times 2 \times 10^{15} \)

14. Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation to 1 significant figure.
    \( a \) \( 7 \times 10^{14} \times 9 \times 10^{-8} \)  \( b \) \( 8 \times 10^{-17} \div 4 \times 10^{-10} \)  \( c \) \( 2.5 \times 10^{12} \times 8 \times 10^{-7} \div 5 \times 10^8 \)

15. List the following sets and then express each set using set notation. Then illustrate each set on a number line.
    a. \{Integers between \(-6\) and \(1\)\}
    b. \{Integers from \(-3\) to \(4\)\}
    c. \{Integers greater than \(-6\) and less than or equal to \(4\)\}
    d. \{Positive integers less than \(5\)\}

16. List the following sets and then express each set using set notation. Then illustrate each set on a number line.
    a. \{Integers less than \(5\)\}
    b. \{Integers greater than \(2\)\}
    c. \{Negative integers greater than \(-5\)\}

17. Use set notation to represent the following sets.
    a. \{Rational numbers greater than \(5\)\}
    b. \{Rational numbers greater than \(5\) and less than or equal to \(20\)\}
    c. \{Positive rational numbers less than \(20\)\}
    d. \{Integers between \(5\) and \(20\), except for \(8\) and \(9\)\}
    e. \{Positive integers less than \(100\), except for integers between \(40\) and \(50\)\}
18 Use set notation to represent the following sets.
   a \{ Real numbers from 2 to 5, including 2 \}
   b \{ Real numbers that are less than 5 and greater than 3 \}
   c \{ Real numbers that are less than 3 and greater than 7 \}
   d \{ Positive real numbers that are less than 3 and greater than 7 \}

19 Use interval notation to represent the following sets, then illustrate the sets on a number line.
   a \( \{ x : -3 \leq x \leq 1 \} \)
   b \( \{ x : x < 2 \} \)
   c \( \{ x : -2 < x < 1 \} \)
   d \( \{ x : x \geq 2 \} \)

20 Use interval notation to represent the following sets, then illustrate the sets on a number line.
   a \( \{ x : 2 \leq x < 5 \} \cup \{ x : 4 \leq x < 6 \} \)
   b \( \{ x : x < 5 \} \cup \{ x : 4 \leq x < 6 \} \)
   c \( \{ x : 2 \leq x < 5 \} \cup \{ x : 4 < x \leq 6 \} \)
   d \( \{ x : x > 5 \} \cap \{ x : 4 < x \leq 6 \} \)

21 Copy the Venn diagram at right and then shade the region represented by each of the following sets.
   a \( A' \)
   b \( A \cup B \)
   c \( A \cap B \)
   d \( (A \cup B) \setminus (A \cap B) \)
   e \( A' \cap B \)
   f \( A' \cap B' \)
   g \( (A \cup B)' \)

22 Complete the following table.

<table>
<thead>
<tr>
<th>Number</th>
<th>3 significant figures</th>
<th>4 significant figures</th>
<th>2 decimal places</th>
<th>3 decimal places</th>
</tr>
</thead>
<tbody>
<tr>
<td>1267.1066</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.6699</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.00056</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.99987</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.076768</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00017495</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

23 Simplify the following.
   a \( -7|1 - 4| \)
   b \( \frac{|1-3| \times |5| \times |4|}{|1-6| + |6|} - |1-8| \)

24 The smallest subset of \( R \) in which \( 4 - 2\sqrt{27} \) belongs is:
   A \( \mathbb{Z}^+ \)
   B \( \mathbb{Z}^- \)
   C \( \mathbb{Q}^+ \)
   D \( \mathbb{Q}^- \)
   E \( \mathbb{I} \)

25 The smallest subset of \( R \) in which \( \frac{9}{4.4567} \) belongs is:
   A \( \mathbb{Z}^+ \)
   B \( \mathbb{Z}^- \)
   C \( \mathbb{Q}^+ \)
   D \( \mathbb{Q}^- \)
   E \( \mathbb{I} \)

26 If \( \xi = \{ 1, 2, 3, 4, 5, 6, 7, 8 \} \), \( A = \{ 1, 2, 3, 4 \} \) and \( B = \{ 5 \} \), then \( A \setminus B \) is:
   A \{ 1, 2, 3, 4, 5 \}
   B \{ 5, 6, 7, 8 \}
   C \varnothing
   D \{ 6, 7, 8 \}
   E \{ 1, 2, 3, 4, 5, 6, 7, 8 \}

27 3.0102 and 92457 to 4 significant figures are:
   A 3.01 and 92450
   B 3.010 and 92450
   C 3.01 and 92460
   D 3.010 and 92460
   E 3.0102 and 92457
28. 0.23, 0.23 and 0.232333... respectively belong which of the following sets?

A. \( \mathbb{Z}, \mathbb{Z}, I \)  
B. \( Q, Q, I \)  
C. \( Q, I, I \)  
D. \( Z, I, I \)  
E. \( Q, Q, Q \)

29. Which of the following sets is an incorrect representation of the set \{all integers from 1 to 5\}?

A. \{1, 2, 3, 4, 5\}  
B. \{x: x \in \mathbb{Z}, 1 \leq x \leq 5\}  
C. \{x: x \in \mathbb{Q}, -5 < x \leq 5\}  
D. \{\text{Real numbers from } -5 \text{ to } 5, \text{not including } -5\}  
E. \([-5, 5]\)

30. For the set illustrated on the given number line, which of the following cannot be true?

A. \((-5, 5]\)  
B. \{x: -5 < x \leq 5\}  
C. \{x: x \in \mathbb{Q}, -5 < x \leq 5\}  
D. \{\text{Real numbers from } -5 \text{ to } 5, \text{not including } -5\}  
E. \([-5, 5]\)

31. Calculate the following products and quotients using a calculator, expressing your answer in scientific notation to 3 significant figures.

a. \(1.4574 \times 10^{21} \times 3.6677 \times 10^9\)  
b. \(8.2583 \times 10^{25} \times 9.2527 \times 10^{-7}\)  
c. \(5.7789 \times 10^8 \div 4.6999 \times 10^9\)  
d. \(2.578 \times 10^{12} (8.775 \times 10^{-7} + 7.342 \times 10^{-6})\)  
e. \(5.878 \times 10^{13}\)

32. a. Using your calculator, investigate the percentage of prime numbers:
   i. between 1 and 9
   ii. between 10 and 99
   iii. between 100 and 999.

   b. What conclusion can you make about the percentage of prime numbers as they approach infinity?

1.3 Properties of surds

A surd is an irrational number of the form \(\sqrt[n]{a}\), where \(a > 0\) and \(n \in \mathbb{Z}^+\). In this section we will focus on the surds of the form \(\sqrt[n]{a}\), where \(a \in \mathbb{Q}\).

For example, \(\sqrt{21}\) is a surd, but \(\sqrt{36} = 6\) is a rational number and not a surd.

Simplifying surds

\(\sqrt{2}\) cannot be simplified because it does not have a perfect square factor, but \(\sqrt{8}\) can be simplified since \(\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}\). A surd is not simplified until all perfect square factors are removed, so the simplified version of \(\sqrt{32}\) is not \(2\sqrt{8}\) but \(4\sqrt{2}\).

<table>
<thead>
<tr>
<th>WORKED EXAMPLE 11</th>
<th>Simplify the following surds.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (\sqrt{384})</td>
<td>b. (3\sqrt{405})</td>
</tr>
<tr>
<td>THINK</td>
<td>WRITE</td>
</tr>
</tbody>
</table>
| a. 1 Express 384 as a product of two factors where one factor is the largest possible perfect square.  
2 Express \(\sqrt{64} \times 6\) as the product of two surds.  
3 Simplify the square root from the perfect square (that is, \(\sqrt{64} = 8\)). | a. \(\sqrt{384} = \sqrt{64} \times \sqrt{6}\)  
\(= \sqrt{64} \times \sqrt{6}\)  
\(= 8\sqrt{6}\) |
Addition and subtraction of surds

Only like surds may be added or subtracted. Like surds, in their simplest form, have the same number under the square root sign. For example,

\[
5\sqrt{3} + 7\sqrt{3} = (5 + 7)\sqrt{3} = 12\sqrt{3}
\]

and

\[
5\sqrt{3} - 7\sqrt{3} = (5 - 7)\sqrt{3} = -2\sqrt{3}.
\]

WORKED EXAMPLE 12

Simplify each of the following expressions involving surds. Assume that \(a\) and \(b\) are positive real numbers.

\[
a. 3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6}
\]

\[
b. 5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}
\]

\[
c. \frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}
\]

THINK

\[a. \text{ All three terms are like, since they contain the same surd (}\sqrt{6}\text{), so group the like terms together and simplify.}
\]

\[b. \text{ Simplify the surds where possible.}
\]

\[c. \text{ Simplify the surds where possible.}
\]

WRITE

\[a. 3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6} = (3 + 17 - 2)\sqrt{6} = 18\sqrt{6}
\]

\[b. 5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}
\]

\[= 5\sqrt{3} + 2\sqrt{4 \times 3} - 5\sqrt{2} + 3\sqrt{4 \times 2}
\]

\[= 5\sqrt{3} + 2 \times 2\sqrt{3} - 5\sqrt{2} + 3 \times 2\sqrt{2}
\]

\[= 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{2} + 6\sqrt{2}
\]

\[= 9\sqrt{3} + \sqrt{2}
\]

\[c. \frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}
\]

\[= \frac{1}{2} \times 10\sqrt{a^2 \times a \times b^2} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}
\]

\[= \frac{1}{2} \times 10 \times a \times b\sqrt{a} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}
\]

\[= 5ab\sqrt{a} + 6ab\sqrt{a} - 10a\sqrt{b}
\]

\[= 11ab\sqrt{a} - 10a\sqrt{b}
\]
Multiplication of surds

Using the property $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, where $a, b \in \mathbb{R}^+$, $\sqrt{2} \times \sqrt{6} = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$.

Using the distributive property $a(b + c) = ab + ac$,

$$\sqrt{2}(\sqrt{3} + \sqrt{6}) = \sqrt{2 \times 3} + \sqrt{2 \times 6} = \sqrt{6} + \sqrt{12} = \sqrt{6} + 2\sqrt{3}.$$ 

Using an extension of the distributive property,

$$(\sqrt{3} + 1) (\sqrt{3} - 2) = \sqrt{3 \times 3} - 2\sqrt{3} + \sqrt{3} - 2 = 3 - \sqrt{3} - 2 = 1 - \sqrt{3}.$$ 

When appropriate, the expansion of a perfect square may be used; that is,

$$(a + b)^2 = a^2 + 2ab + b^2 \text{ and } (a - b)^2 = a^2 - 2ab + b^2.$$ For example,

$$(\sqrt{3} - \sqrt{2})^2 = 3 - 2\sqrt{3 \times 2} + 2 = 5 - 2\sqrt{6}.$$ 

Definition of the conjugate

The conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$. The conjugate of $3 - 2\sqrt{5}$ is $3 + 2\sqrt{5}$. The product of a conjugate pair is rational if the numbers under the square root are rational. For example,

$$(\sqrt{3} + \sqrt{2}) (\sqrt{3} - \sqrt{2}) = \sqrt{3 \times 3} - \sqrt{3 \times 2} + \sqrt{2 \times 3} - \sqrt{2 \times 2}$$

$$= 3 - \sqrt{6} + \sqrt{6} - 2 = 1.$$ 

This is a special case of the difference of perfect squares expansion, $(a + b)(a - b) = a^2 - b^2$.

### WORKED EXAMPLE 13

Multiply the following surds, expressing answers in simplest form.

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong> 6$\sqrt{12} \times 2\sqrt{6}$</td>
<td><strong>a</strong> 6$\sqrt{12} \times 2\sqrt{6} = 6\sqrt{4 \times 3} \times 2\sqrt{6}$</td>
</tr>
<tr>
<td>1 Simplify $\sqrt{12}$.</td>
<td>$= 6 \times 2\sqrt{3} \times 2\sqrt{6}$</td>
</tr>
<tr>
<td>2 Multiply the coefficients and multiply the surds.</td>
<td>$= 12\sqrt{3 \times 2\sqrt{6}}$</td>
</tr>
<tr>
<td>3 Simplify the product surd.</td>
<td>$= 24\sqrt{18}$</td>
</tr>
<tr>
<td><strong>b</strong> $\frac{3}{5}\sqrt{70} \times \frac{1}{4}\sqrt{10}$</td>
<td><strong>b</strong> $\frac{3}{5}\sqrt{70} \times \frac{1}{4}\sqrt{10} = \frac{3}{5} \times \frac{1}{4} \times \sqrt{70 \times 10}$</td>
</tr>
<tr>
<td>1 Multiply the coefficients and multiply the surds.</td>
<td>$= \frac{3}{20} \sqrt{700}$</td>
</tr>
<tr>
<td>2 Simplify the product surd.</td>
<td>$= \frac{3}{20} \sqrt{100 \times 7}$</td>
</tr>
<tr>
<td>3 Simplify by dividing both 10 and 20 by 10 (cross-cancel).</td>
<td>$= \frac{3}{20} \times 10 \sqrt{7}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{3}{2} \sqrt{7}$ or $\frac{3\sqrt{7}}{2}$</td>
</tr>
</tbody>
</table>
Division of surds

\[ \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \text{ where } a, b \in R^+. \] For example, \( \sqrt{6}/\sqrt{2} = \sqrt{\frac{6}{2}} = \sqrt{3} \)

Using the property \( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b} \), where \( a \) and \( b \) are rational, we can express answers with rational denominators. For example,

\[ \frac{\sqrt{2}}{\sqrt{6}} = \sqrt{\frac{2}{6}} = \sqrt{\frac{2}{2} \times \frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}} \]

Using the property of conjugates, binomial surds in the denominator may be rationalised. For example,

\[ \frac{\sqrt{7} - 2\sqrt{2}}{\sqrt{7} + \sqrt{2}} = \frac{\sqrt{7} - 2\sqrt{2}}{\sqrt{7} + \sqrt{2}} \times \frac{\sqrt{7} - \sqrt{2}}{\sqrt{7} - \sqrt{2}} = \frac{7 - 2\sqrt{14} + 2 \times 2}{7 - 2} = \frac{11 - 3\sqrt{14}}{5} \]

By multiplying the original surd by \( \frac{\sqrt{7} - \sqrt{2}}{\sqrt{7} - \sqrt{2}} \), we are multiplying by 1, so the number is unchanged but is finally expressed in its rational denominator form.

WORKED EXAMPLE

Expand and simplify the following where possible.

a \( \sqrt{7}(\sqrt{18} - 3) \)

b \(-2\sqrt{3}(\sqrt{10} - 5\sqrt{3}) \)

c \((\sqrt{5} + 3\sqrt{6})(2\sqrt{3} - \sqrt{2})\)

THINK

WRITE

a 1 Write the expression.
2 Simplify \( \sqrt{18} \).
3 Expand the bracket.
4 Simplify.

a \( \sqrt{7}(\sqrt{18} - 3) \)
\[ = \sqrt{7}(3\sqrt{2} - 3) \]
\[ = \sqrt{7} \times 3\sqrt{2} + \sqrt{7} \times -3 \]
\[ = 3\sqrt{14} - 3\sqrt{7} \]

b 1 Write the expression.
2 i Expand the brackets.
ii Be sure to multiply through with the negative.
3 Simplify.

b \(-2\sqrt{3}(\sqrt{10} - 5\sqrt{3}) \)
\[ = -2\sqrt{3} \times \sqrt{10} - 2\sqrt{3} \times -5\sqrt{3} \]
\[ = -2\sqrt{30} + 10\sqrt{9} \]
\[ = -2\sqrt{30} + 30 \]

8 1 Write the expression.
2 Expand the brackets.
3 Simplify.

c \((\sqrt{5} + 3\sqrt{6})(2\sqrt{3} - \sqrt{2}) \)
\[ = \sqrt{5} \times 2\sqrt{3} + \sqrt{5} \times -\sqrt{2} + 3\sqrt{6} \times 2\sqrt{3} + 3\sqrt{6} \times -\sqrt{2} \]
\[ = 2\sqrt{15} - \sqrt{10} + 6\sqrt{18} - 3\sqrt{12} \]
\[ = 2\sqrt{15} - \sqrt{10} + 6 \times 3\sqrt{2} - 3 \times 2\sqrt{3} \]
\[ = 2\sqrt{15} - \sqrt{10} + 18\sqrt{2} - 6\sqrt{3} \]
Express the following in their simplest form with a rational denominator.

\[
\begin{align*}
a & = \frac{9 \sqrt{88}}{6 \sqrt{99}} \\
b & = \frac{\sqrt{6}}{\sqrt{13}} \\
c & = \frac{1}{2\sqrt{6} - \sqrt{3}} + \frac{1}{3\sqrt{6} + 2\sqrt{3}}
\end{align*}
\]

**THINK**

a 1. Rewrite the surds, using \( \sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} \).

2. Simplify the fraction under the root.

3. Simplify the surds.

4. Multiply the whole numbers in the numerator and those in the denominator.

**WRITE**

a \[
\begin{align*}
&= \frac{9 \sqrt{88}}{6 \sqrt{99}} \\
&= \frac{9 \times 2\sqrt{2}}{6 \times 3} \\
&= \frac{\sqrt{78}}{13}
\end{align*}
\]

b \[
\begin{align*}
&= \frac{\sqrt{6}}{\sqrt{13}} \\
&= \frac{\sqrt{6} \cdot \sqrt{13}}{\sqrt{13} \cdot \sqrt{13}} \\
&= \frac{\sqrt{78}}{13}
\end{align*}
\]

c \[
\begin{align*}
&= \frac{1}{2\sqrt{6} - \sqrt{3}} \\
&= \frac{1 \times (2\sqrt{6} + \sqrt{3})}{(2\sqrt{6} - \sqrt{3}) \times (2\sqrt{6} + \sqrt{3})} \\
&= \frac{2\sqrt{6} + \sqrt{3}}{21} \\
&= \frac{1}{3\sqrt{6} + 2\sqrt{3}} \\
&= \frac{1 \times (3\sqrt{6} - 2\sqrt{3})}{(3\sqrt{6} + 2\sqrt{3}) \times (3\sqrt{6} - 2\sqrt{3})} \\
&= \frac{3\sqrt{6} - 2\sqrt{3}}{42}
\end{align*}
\]

**WORKED EXAMPLE**

1. Rewrite the surds, using \( \sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} \).
2. Simplify the fraction under the root.
3. Simplify the surds.
4. Multiply the whole numbers in the numerator and those in the denominator.

b 1. Write the fraction.

2. Multiply both the numerator and the denominator by the surd \( \sqrt{13} \).

4. Multiply the numerator and the denominator by the conjugate of the denominator.

3. Expand the denominator.

4. Simplify the denominator.

5. Write the second fraction.

6. Multiply the numerator and the denominator by the conjugate of the denominator.

7. Expand the denominator.

8. Simplify the denominator.
Properties of surds

1 WE11 Simplify the following surds.
   a $\sqrt{24}$  
   b $\sqrt{56}$  
   c $\sqrt{125}$  
   d $\sqrt{98}$  
   e $\sqrt{48}$

2 Simplify the following surds.
   a $\sqrt{300}$  
   b $7\sqrt{80}$  
   c $\sqrt{128}$  
   d $2\sqrt{18}$  
   e $-3\sqrt{50}$

3 WE12 Simplify the following expressions.
   a $7\sqrt{2} + 4\sqrt{3} - 5\sqrt{2} - 6\sqrt{3}$  
   b $2 + 5\sqrt{7} - 6 - 4\sqrt{7}$  
   c $3\sqrt{5} - 6\sqrt{3} + 5\sqrt{5} - 4\sqrt{2} - 8\sqrt{3}$  
   d $\sqrt{18} - \sqrt{12} + \sqrt{75} + \sqrt{27}$

4 Simplify the following expressions.
   a $\sqrt{50} - \sqrt{72} + \sqrt{80} + \sqrt{45}$  
   b $3\sqrt{12} - 5\sqrt{18} + 4\sqrt{27} + 5\sqrt{98}$  
   c $\frac{2\sqrt{3}}{4} - \frac{3\sqrt{2}}{8} + \frac{5\sqrt{3}}{8} - \frac{5\sqrt{2}}{4}$  
   d $\frac{2\sqrt{7}}{5} - \frac{3\sqrt{2}}{3} + \frac{5\sqrt{8}}{3} - \frac{5\sqrt{2}}{2}$

5 WE13 Express the following surds in their simplest form.
   a $\sqrt{6} \times \sqrt{15}$  
   b $2\sqrt{3} \times 5\sqrt{7}$

6 Simplify the following surds.
   a $4\sqrt{7} \times 3\sqrt{14}$  
   b $\frac{\sqrt{20}}{3} \times \sqrt{15}$

7 WE14 Expand, giving your answers in their simplest form.
   a $\sqrt{3}(\sqrt{5} - \sqrt{2})$  
   b $2\sqrt{3}(3\sqrt{3} + \sqrt{2})$  
   c $(\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{2})$  
   d $(\sqrt{18} - \sqrt{12})(\sqrt{3} - 2\sqrt{2})$

8 Expand and simplify.
   a $(\sqrt{5} + \sqrt{7})^2$  
   b $(2\sqrt{12} + 3\sqrt{18})^2$  
   c $(2\sqrt{5} - \sqrt{3})(2\sqrt{5} + \sqrt{3})$  
   d $(5\sqrt{5} - 10)(5\sqrt{5} + 10)$

9 WE15 Express the following surds in their simplest form with a rational denominator.
   a $\frac{\sqrt{18}}{\sqrt{3}}$  
   b $\frac{2\sqrt{24}}{3\sqrt{3}}$  
   c $\frac{\sqrt{5}}{\sqrt{3}}$  
   d $\frac{4\sqrt{3}}{7\sqrt{5}}$  
   e $\frac{2\sqrt{8}}{3\sqrt{12}}$  
   f $\frac{1}{\sqrt{5} - \sqrt{3}}$
10 Rationalise the denominators.

\[ \frac{\sqrt{3}}{\sqrt{3} - \sqrt{2}} \quad \frac{2\sqrt{2}}{2\sqrt{5} + 3\sqrt{2}} \quad \frac{5 + \sqrt{3}}{5 - \sqrt{3}} \]

\[ \frac{\sqrt{12} - \sqrt{8}}{\sqrt{12} + \sqrt{8}} \quad \frac{2\sqrt{5} - \sqrt{3}}{\sqrt{5} - 2\sqrt{3}} \quad \frac{2\sqrt{18} - \sqrt{24}}{3\sqrt{8} - \sqrt{54}} \]

11 Express the following surds in their simplest form with a rational denominator.

\[ \frac{1}{2\sqrt{2} - 3} + \frac{1}{2\sqrt{2} + 3} \quad \frac{1}{3\sqrt{2} - 2\sqrt{3}} - \frac{1}{2\sqrt{2} + 3\sqrt{3}} \]

\[ \frac{3\sqrt{5}}{3\sqrt{2} - 2\sqrt{3}} - \frac{2\sqrt{5} - 1}{2\sqrt{2} + 3\sqrt{3}} \quad \frac{2\sqrt{5} + 3\sqrt{3}}{3\sqrt{5} - 2\sqrt{3}} - \frac{3\sqrt{5} - 2\sqrt{3}}{2\sqrt{5} + 3\sqrt{3}} \]

\[ \frac{4\sqrt{2} + 3\sqrt{2}}{4\sqrt{2} - 2\sqrt{3}} \times \frac{5\sqrt{2} - 2\sqrt{3}}{6\sqrt{2} + 3\sqrt{3}} \quad \frac{2\sqrt{5} + 3\sqrt{3}}{3\sqrt{5} - 4\sqrt{3}} - \frac{3\sqrt{5} + 4\sqrt{3}}{2\sqrt{5} + 3\sqrt{3}} \]

12 Given that \( x = 2 - 3\sqrt{2} \), find each of the following, giving the answer in surd form with a rational denominator.

\[ \frac{x + 1}{x} \quad \frac{x - 1}{x} \quad \frac{x^2 - 2x}{x + 2} \quad \frac{x^2 + 2x}{x + 3} \quad \frac{x^2 - 4x - 14}{x + 3} \]

\[ 2x^2 - 2x - 9 \quad \text{Using your answers to e and f, state if } x = 2 - 3\sqrt{2} \text{ is a solution of } x^2 - 4x - 14 = 0 \text{ and } 2x^2 - 2x - 9 = 0. \]

13 Show that \( 5 - 2\sqrt{3} \) is a solution of one of the following equations:

\[ x^2 - 13x + 10 = 0 \text{ or } x^2 - 10x + 13 = 0. \]

14 Show that \( \sqrt{2} + 1 \) is a solution of both of the following equations:

\[ x^2 - 2\sqrt{2}x + 1 = 0 \text{ and } x^2 - (2\sqrt{2} + 3)x + 4 + 3\sqrt{2} = 0. \]

15 Expressed in its simplest form, \( \frac{2\sqrt{75} - 2\sqrt{27} - \frac{1}{2}\sqrt{48}}{3} \) equals:

\[ A \sqrt{3} \quad B \sqrt{3} \quad C \sqrt{3} \quad D \sqrt{3} \quad E -3\sqrt{3} \]

16 Expressed in its simplest form, \( \frac{\sqrt{14a^2b^2}}{\sqrt{7ab^2}} \) equals:

\[ A \sqrt{2a} \quad B 2a \quad C \sqrt{2ab} \quad D \sqrt{2a^2} \quad E \sqrt{2a^2} \]

17 Expressed in its simplest form, \( (3\sqrt{3} - 4\sqrt{8})(2\sqrt{3} - 3\sqrt{8}) \) equals:

\[ A 114 - 34\sqrt{6} \quad B 120 - 34\sqrt{6} \quad C -78 - 17\sqrt{24} \quad D 18 - 24\sqrt{2} \quad E -18 - 34\sqrt{6} \]

18 Expressed in its simplest form, \( \frac{15\sqrt{21}}{6\sqrt{14}} \) equals:

\[ A \frac{5\sqrt{3}}{2\sqrt{2}} \quad B \frac{5\sqrt{6}}{2\sqrt{2}} \quad C \frac{5\sqrt{3}}{2} \quad D \frac{5\sqrt{6}}{4} \quad E \frac{5\sqrt{6}}{2} \]
Expressed in its simplest form, $\frac{2\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ equals:

A $\frac{13 + 3\sqrt{15}}{2}$
B $\frac{12 - \sqrt{15}}{2}$
C $\frac{18 - 3\sqrt{15}}{2}$
D $\frac{12 + 3\sqrt{15}}{2}$
E $\frac{13 + \sqrt{15}}{2}$

Expressed in its simplest form, $\frac{3\sqrt{5} - 5}{\frac{3\sqrt{5} + 5}{3\sqrt{5} - 5}}$ equals:

A $3\sqrt{5}$
B $-30\sqrt{5}$
C $30\sqrt{5}$
D $52 - 30\sqrt{5}$
E $-3\sqrt{5}$

An ice-cream cone with measurements as shown is completely filled with ice-cream, and has a hemisphere of ice-cream on top.

a) Determine the height of the ice-cream cone in simplest surd form.

b) Determine the volume of the ice-cream in the cone.

c) Determine the volume of the ice-cream in the hemisphere.

d) Hence, find the total volume of ice-cream.

A gold bar with dimensions of $\frac{5\sqrt{20}}{2}$, $\frac{3\sqrt{12}}{2}$ and $2\sqrt{6}$ cm is to be melted down into a cylinder of height $4\sqrt{10}$ cm.

a) Find the volume of the gold, expressing the answer in the simplest surd form and specifying the appropriate unit.

b) Find the radius of cylinder, expressing the answer in the simplest surd form and specifying the appropriate unit.

c) If the height of the cylinder was $3\sqrt{40}$ cm, what would be the new radius? Express your answer in the simplest surd form.

The set of complex numbers

The need to invent further numbers became clear when equations such as $x^2 = -1$ and $x^2 = -9$ were considered. Clearly there are no real solutions, so imaginary numbers were invented, using the symbol $i$, where $i^2 = -1$. The equation $x^2 = -1$ has two solutions, $x = -i$ and $x = i$. As $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{-9} \times \sqrt{-1} = 3 \times \sqrt{1} = 3i, x^2 = -9$ has the solutions $x = \pm 3i$.

Quadratic equations such as $x^2 - 4x + 5 = 0$ were investigated further. Using the general formula for the solution of a quadratic equation, that is, if $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$ If the discriminant, $b^2 - 4ac$, is negative, the equation has no real solutions, but it does have two complex solutions.

A complex number is any number of the form $x + yi$, where $x, y \in R$.

$C$ is the set of complex numbers where $C = \{x + yi : x, y \in R\}$. 
Just as \( x \) is commonly used in algebra to represent a real number, \( z \) is commonly used to represent a complex number, where \( z = x + yi \).

If \( x = 0 \), \( z = yi \) is a pure imaginary number. If \( y = 0 \), \( z = x \) is a real number, so that \( R \subset C \). This is represented on the Venn diagram below.

**Notation**

If \( z = a + bi \), the real component of \( z \) is \( \text{Re}(z) = a \), and the imaginary component of \( z \) is \( \text{Im}(z) = b \). For example, if \( z = -2 - 2\sqrt{3}i \), \( \text{Re}(z) = -2 \) and \( \text{Im}(z) = -2\sqrt{3} \) (not \( -2\sqrt{3}i \)). Similarly, \( \text{Re}(4 + 6i) = 4 \) and \( \text{Im}(4 + 6i) = 6 \).

**Equality of complex numbers**

If \( a + bi = c + di \), then \( a = c \) and \( b = d \).

For two complex numbers \( z_1 \) and \( z_2 \) to be equal, both their real and imaginary components must be equal.

**Multiplication of a complex number by a real constant**

If \( z = a + bi \), then \( kz = k(a + bi) = ka + kbi \).

For example, if \( z = -2 + 3i \), then \(-3z = -3(-2 + 3i) = 6 - 9i \).

**Adding and subtracting complex numbers**

If \( z_1 = a + bi \) and \( z_2 = c + di \), then \( z_1 + z_2 = (a + c) + (b + d)i \) and \( z_1 - z_2 = (a - c) + (b - d)i \).
**Worked Example 17**

If \(z_1 = 2 - 3i\) and \(z_2 = -3 + 4i\), find:

a. \(z_1 + z_2\)

b. \(z_1 - z_2\)

c. \(3z_1 - 4z_2\).

**Think**

a. Use the definition for addition of complex numbers:
\[z_1 + z_2 = (a + c) + (b + d)i\]

b. Use the definition for subtraction of complex numbers:
\[z_1 - z_2 = (a - c) + (b - d)i\]

c. First multiply each complex number by the constant and then use the definition for subtraction of complex numbers to answer the question.

**Write**

a. \(z_1 + z_2 = (2 - 3) + (-3 + 4)i = -1 + i\)

b. \(z_1 - z_2 = (2 + 3) + (-3 - 4)i = 5 - 7i\)

c. \(3z_1 - 4z_2 = 3(2 - 3i) - 4(-3 + 4i) = 6 - 9i - (-12 + 16i) = 18i - 25i\)

---

**Exercise 1.4**

**The set of complex numbers**

1. Solve to find \(x\) and \(y\) in the following.
   a. \((x + 1) + (y - 1)i = 2 + 3i\)
   b. \((x + 4) - (3 + yi) = 2 + 5i\)
   c. \((2x + i) + (3 - 2yi) = x + 3i\)
   d. \((x + 2i) + (2y + xi) = 7 + 4i\)

2. Solve the following.
   a. \((2x + 3yi) + 2(x + 2yi) = 3 + 2i\)
   b. \((x + i) + (2 + yi) = 2x + 3yi\)
   c. \((2x - 3i) + (-3 + 2yi) = y - xi\)

3. If \(z_1 = 2 - i\) and \(z_2 = 3 + 4i\) find:
   a. \(z_1 + z_2\)
   b. \(z_1 - z_2\)
   c. \(2z_1 + 3z_2\)

4. If \(z_1 = 3 - 4i\) and \(z_2 = 2 - 3i\), evaluate the following.
   a. \(2z_1 - 3z_2\)
   b. \(\sqrt{2}z_1 + 2\sqrt{2}z_2\)
   c. \(\sqrt{3}z_1 + \sqrt{3}z_2\)

5. Express the following in terms of \(i\).
   a. \(\sqrt{-16}\)
   b. \(\sqrt{-7}\)
   c. \(2 + \sqrt{-20}\)

   a. \(\sqrt{-10} + \sqrt{10}\)
   b. \(\frac{1 - \sqrt{-28}}{2}\)

7. Simplify the following numbers.
   a. \(-\sqrt{-25}\)
   b. \(\sqrt{-49} + \sqrt{4}\)
   c. \(11\sqrt{-81}\)

8. State the values of \(Re(z)\) and \(Im(z)\) for the following.
   a. \(3 + 4i\)
   b. \(-2 + \sqrt{2}i\)
   c. \((\sqrt{2} - 1) + (\sqrt{2} + 1)i\)

9. State the values of \(Re(z)\) and \(Im(z)\) for the following.
   a. \(\sqrt{8} - \sqrt{-40}\)
   b. \(-6\)
   c. \(13i\)
10 Find the following components.
   a \( \text{Re}(2 + 3i + 3(4 - 2i)) \)
   b \( \text{Re}(\sqrt{3} + 2\sqrt{2}i + \sqrt{2}(-3 - \sqrt{3}i)) \)
   c \( \text{Im}(2(2 - 3i) - 3(4 - 2i)) \)
   d \( \text{Im}(2\sqrt{3} - 2\sqrt{2}i + \sqrt{2}(-3 - \sqrt{6}i)) \)

11 If \( z_1 = 2 - i \) and \( z_2 = 3 - 2i \), then \( \text{Re}(2z_1 - 3z_2) \) equals:
   A 13  B -13  C 5  D -5  E 4

12 If \( z_1 = 2 - i \) and \( z_2 = 3 - 2i \), then \( \text{Im}(2z_1 - 3z_2) \) equals:
   A 4i  B 4  C -4  D -8  E -8i

13 If \( (2 + xi) + (4 - 3i) = x + 3yi \), then the respective values of \( x \) and \( y \) are:
   A 6, 1  B 3, 6  C 6, -3  D 6, 3  E 1, 6

14 Simplify \( (12 - 4i) + (3 + 6i) - (10 + 10i) \).

15 Find the error in the student’s work shown below.

16 Simplify \( (13 - 5i) + (7 - 5i) \), giving answer in simplest form of \( a + bi \).

17 Find the imaginary part of \( 21 - 2i + 2(10 - 5i) \).

1.5 Multiplication and division of complex numbers

Multiplication of complex numbers

If \( z_1 = a + bi \) and \( z_2 = c + di \) where \( a, b, c, d \in \mathbb{R} \), then

\[
z_1z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i
\]

Note that this is an application of the distributive property.

WORKED EXAMPLE

18 Simplify:
   a \( 2i(2 - 3i) \)
   b \( (2 - 3i)(-3 + 4i) \).

THINK
   a Expand the brackets.
   b Expand the brackets as for binomial expansion and simplify.

WRITE
   a \( 2i(2 - 3i) = 4i - 6i^2 = 6 + 4i \)
   b \( (2 - 3i)(-3 + 4i) = -6 + 8i + 9i - 12i^2 = 6 + 17i \)
The conjugate of a complex number

If \( z = a + bi \), then its conjugate, \( \bar{z} \), is \( \bar{z} = a - bi \).

The sum of a complex number and its conjugate \( z + \bar{z} = a + bi + a - bi = 2a \), which is a real number.

The product of a complex number and its conjugate

\[
zz = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2, \text{ which is a real number.}
\]

**WORKED EXAMPLE 19**

If \( z_1 = 2 + 3i \) and \( z_2 = -4 - 5i \), find:

\[
\begin{align*}
\text{a} & \quad \bar{z}_1 + \bar{z}_2 \\
\text{b} & \quad \bar{z}_1 + z_2 \\
\text{c} & \quad \bar{z}_1 \bar{z}_2 \\
\text{d} & \quad \bar{z}_1 \bar{z}_2.
\end{align*}
\]

**THINK**

1. Determine the conjugate of each complex number using the definition: if \( z = a + bi \), then its conjugate, \( \bar{z} \), is \( \bar{z} = a - bi \).

2. Evaluate \( \bar{z}_1 + \bar{z}_2 \).

**WRITE**

\[
\begin{align*}
\text{a} & \quad z_1 = 2 + 3i \\
\Rightarrow & \quad \bar{z}_1 = 2 - 3i \\
z_2 = -4 - 5i \\
\Rightarrow & \quad \bar{z}_2 = -4 + 5i \\
\bar{z}_1 + \bar{z}_2 & = (2 - 3i) + (-4 + 5i) \\
& = -2 + 2i \\
\text{b} & \quad z_1 + z_2 = (2 + 3i) + (-4 - 5i) \\
& = -2 - 2i \\
\bar{z}_1 + z_2 & = -2 + 2i \\
\text{c} & \quad \bar{z}_1 \bar{z}_2 = (2 - 3i)(-4 + 5i) \\
& = -8 + 10i + 12i - 15i^2 \\
& = 7 + 22i \\
\text{d} & \quad z_1z_2 = (2 + 3i)(-4 - 5i) \\
& = -8 - 10i - 12i - 15i^2 \\
& = 7 - 22i \\
\bar{z}_1 \bar{z}_2 & = 7 + 22i
\end{align*}
\]

**Division of complex numbers**

If \( z_1 = a + bi \) and \( z_2 = c + di \) where \( a, b, c, d \in R \), then using the conjugate

\[
\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}
\]
\[
\frac{ac - adi + bci - bdi^2}{c^2 - (di)^2} = \frac{(ac + bd) - (ad - bc)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} - \frac{ad - bc}{c^2 + d^2}i
\]

### WORKED EXAMPLE 20

Express each of the following in the form \( a + bi \).

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \frac{4 - i}{2} )</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>( (3 - 2i)^{-1} )</td>
<td>d</td>
</tr>
</tbody>
</table>

#### THINK

**a** Divide each term of the numerator by 2.

**b** Multiply the numerator and denominator by \( i \) and then divide each term of the numerator by 3. Write the answer in the required form \( a + bi \).

**c** 1 Express \( (3 - 2i)^{-1} \) as a reciprocal.

2 Multiply the numerator and the denominator by the conjugate of the denominator.

3 Write the answer in the required form \( a + bi \).

**d** 1 Multiply the numerator and the denominator by the conjugate of the denominator.

2 Write the answer in the required form \( a + bi \).

#### WRITE

**a** \[
\frac{4 - i}{2} = 2 - \frac{1}{2}i
\]

\[
\frac{3 - 4i}{3i} \times \frac{3 + 2i}{3 + 2i} = \frac{-3i - 4}{3} = \frac{-4}{3} - i
\]

**c** \[
(3 - 2i)^{-1} = \frac{1}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} = \frac{3 + 2i}{9 - (2i)^2} = \frac{3 + 2i}{9 - 4i^2} = \frac{3 + 2i}{13}
\]

\[
= \frac{3}{13} + \frac{2}{13}i
\]

**d** \[
\frac{2 - 3i}{2 + i} = \frac{2 - 3i}{2 + i} \times \frac{2 - i}{2 - i} = \frac{4 - 2i - 6i + 3i^2}{4 - i^2} = \frac{1 - 8i}{5} = \frac{1}{5} - \frac{8}{5}i
\]
EXERCISE 1.5

Multiplication and division of complex numbers

1. Evaluate the following, giving each answer in its simplest $a + bi$ form.
   a. $2i(2 + 3i)$
   b. $(2 - 3i)(1 + i)$
   c. $(-2 - i)(1 - 3i)$
   d. $(2 - 3i)^2$
   e. $(6 + 7i)(6 - 7i)$

2. Expand and simplify:
   a. $-2i(3 - 4i)$
   b. $(3 + 5i)(3 - 3i)$
   c. $(a + bi)(a - bi)$

3. Give the conjugate of the following complex numbers.
   a. $3 + 2i$
   b. $-4 + 3i$
   c. $\sqrt{2 - 2i}$
   d. $-8i$

4. If $z_1 = 4 - 3i$ and $z_2 = 3 - 4i$, evaluate the following, giving each answer in its simplest $a + bi$ form.
   a. $\bar{z}_1$
   b. $\bar{z}_1 z_2$
   c. $z_1 \bar{z}_2$
   d. $\bar{z}_1^2$
   e. $2i \bar{z}_2$
   f. $(z_1 + z_2)^2$

5. Express each of the following in the form $a + bi$.
   a. $\frac{3 - 4i}{-5i}$
   b. $\frac{3 + 4i}{3 - 4i}$
   c. $\frac{1 + 2i}{2 + i}$
   d. $\frac{(2 + i)^2}{1 + 2i}$
   e. $(3 + 2i)^{-1}$
   f. $(3 + 2i)^{-2}$

6. Simplify each of the following if $z_1 = 4 - 3i$ and $z_2 = 3 - 4i$.
   a. $\frac{z_1^{-1}}{z_2}$
   b. $\frac{z_1}{z_2}$
   c. $\left(\frac{z_1}{z_2}\right)^{-1}$
   d. $\frac{z_1 \bar{z}_2}{z_2}$
   e. $z_1 + \frac{1}{\bar{z}_1}$
   f. $\frac{z_1}{\bar{z}_1} - \frac{z_2}{\bar{z}_2}$

7. If $z_1 = a + bi$ and $z_2 = c + di$, prove that:
   a. $\bar{z}_1 \bar{z}_2 = \bar{z}_1 \bar{z}_2$
   b. $\bar{z}_1 + \bar{z}_2 = z_1 + z_2$
   c. $\frac{z_1}{z_2} = \frac{z_1}{z_2}$

8. Find $x$ and $y$ in each of the following.
   a. $(x + yi)(2 + i) = 3 + 6i$
   b. $\frac{x + yi}{1 + 2i} = 1 + i$

9. Solve for $z$.
   a. $(4 + 3i)z = 2 - i$
   b. $(2 - 3i)z = -3 - 2i$

10. For each of the following, state $\bar{z}$ and find $z^{-1}$, then state $z^{-1}$ in terms of $\bar{z}$.
    a. $z = 4 + 5i$
    b. $z = a + bi$

11. Expressed in $a + bi$ form, $(2\sqrt{3} - 3i) (3\sqrt{3} - 2i)$ equals:
    A. $24 - 13\sqrt{3}i$
    B. $12 - 13\sqrt{3}i$
    C. $(6\sqrt{3} + 6) - 13\sqrt{3}i$
    D. $12 - 5\sqrt{3}i$
    E. $12 - 5\sqrt{3}i$

12. Expressed in $a + bi$ form, $\frac{2\sqrt{3} - 3i}{3\sqrt{3} - 2i}$ equals:
    A. $\frac{24}{31} - \frac{5\sqrt{3}}{31}i$
    B. $\frac{12}{31} - \frac{5\sqrt{3}}{31}i$
    C. $\frac{24}{31} - \frac{13\sqrt{3}}{31}i$
    D. $\frac{24}{23} - \frac{5\sqrt{3}}{23}i$
    E. $\frac{212}{23} - \frac{5\sqrt{3}}{23}i$
Expressed in \(a + bi\) form, \((1 + i)^2 + (1 + i)^{-2}\) equals:

- **A** \(\frac{5}{2}i\)
- **B** \(\frac{2}{4} - \frac{7}{4}i\)
- **C** \(0\)
- **D** \(\frac{9}{4} - \frac{9}{4}i\)
- **E** \(\frac{3}{2}i\)

If \((2x + yi)(3 - 2i) = 4 + 5i\), then the respective values of \(x\) and \(y\) are:

- **A** \(\frac{2}{13}\) and \(\frac{23}{13}\)
- **B** \(\frac{1}{13}\) and \(\frac{23}{13}\)
- **C** \(\frac{7}{13}\) and \(\frac{23}{13}\)
- **D** \(\frac{1}{5}\) and \(\frac{23}{5}\)
- **E** \(\frac{2}{5}\) and \(\frac{23}{5}\)

Evaluate the following, giving each answer in the form \(a + bi\).

- **a** \(\frac{1}{2 - 3i} + \frac{1}{2 + 3i}\)
- **b** \(\frac{(3 - 2i)^2}{(2 - i)^2}\)
- **c** \(2 + \sqrt{3}i + \frac{1}{2 + \sqrt{3}i}\)

Evaluate the following, giving each answer in the form \(a + bi\).

- **a** \(\frac{\sqrt{3} - i}{\sqrt{3} + i} + \frac{\sqrt{3} + i}{\sqrt{3} - i}\)
- **b** \(\frac{2 - i}{3 + 2i} - \frac{3 + i}{2 + i}\)

If \(z = 1 + i\), simplify \(z^3 + z^2 + z + 1\), giving answer in the form \(a + bi\).

Write \(\frac{2 + 5i}{3 + i}\) in the form \(a + bi\).

### Representing complex numbers on an Argand diagram

Real numbers can be represented on a number line and complex numbers, with their real and imaginary components, require a plane. The Argand diagram or Argand plane has a horizontal axis, \(\text{Re}(z)\), and a vertical axis, \(\text{Im}(z)\). The complex number \(z = a + bi\) is represented by the point \((a, b)\). Because of the similarities with the Cartesian plane, \(a + bi\) is referred to as the Cartesian or rectangular form.

The complex numbers \(2 + 3i, 4, -3i\) and \(-2 - 4i\) are shown on the Argand plane at right.

#### WORKED EXAMPLE 21

**a** Express the following in their simplest form: \(i^0, i, i^2, i^3, i^4, i^5\).

**b** Use the pattern in these results to find the simplest form for \(i^8, i^{21}\) and \(i^{-63}\).

**c** Illustrate the points from part **a** on an Argand plane, and state their distance from the origin and the angle of rotation about the origin to rotate from one power of \(i\) to the next.

---

**THINK**

- Use \(i^2 = -1\) and index law rules to simplify each term.

**WRITE/DRAW**

- \(i^0 = 1\)
- \(i^1 = i\)
- \(i^2 = -1\)
- \(i^3 = i^2 \times i = -i\)
- \(i^4 = i^2 \times i^2 = 1\)
- \(i^5 = i^4 \times i = i\)
b The pattern repeats as shown in part a.

\[ i^8 = (i^4)^2 = 1 \]
\[ i^{21} = (i^5)^4 	imes i = i \]
\[ i^{-63} = (i^{-15})^4 	imes i^{-3} = i^{-3} \]
\[ = i \]

1 Rule up a pair of labelled, scaled axes for the Argand plane.
Place each of the points from part a onto the plane and label them.

2 Determine the distance of each point from the origin.
All points are 1 unit from the origin.

3 State the angle of rotation about the origin to rotate from one power of \( i \) to the next.
The angle of rotation about the origin to rotate from one power of \( i \) to the next is \( 90^\circ \) in an anticlockwise direction.

**EXERCISE 1.6**

### Representing complex numbers on an Argand diagram

1 Give the following in their simplest form.
   \[ a \quad i^7 \quad b \quad i^{37} \quad c \quad i^{-4} \quad d \quad i^{-15} \]

2 Give the following in their simplest form.
   \[ a \quad (2i)^6 \quad b \quad (-2i)^8 \quad c \quad -(2i)^9 \quad d \quad -(-2i)^{-9} \]

3 Plot the following points on an Argand plane.
   \[ a \quad 2 + 3i \quad b \quad 2 - 3i \quad c \quad -2 + 3i \quad d \quad -2 - 3i \quad e \quad -3i \quad f \quad 2 \]

4 Write down the complex number represented by the points A to F on the following Argand diagram.

5 If \( z_1 = 2 + 5i \) and \( z_2 = 5 - 7i \), find algebraically and represent on an Argand diagram.
   \[ a \quad z_1 + z_2 \quad b \quad z_1 - z_2 \]

6 Assume \( Z_1 = 3 + 5i \) and \( z_2 = -z_1 \), to answer the following.
   \[ a \quad \text{Sketch } z_1 \text{ and } z_2 \text{ on an Argand diagram.} \quad b \quad \text{State the angle of rotation about the origin to rotate from } z_1 \text{ to } z_2. \]

7 If \( z = 4 - 3i \), show on an Argand diagram:
   \[ a \quad z \quad b \quad -z \quad c \quad \bar{z}. \]
8 Represent the following on an Argand diagram if $z = 2 - i$.
   a $z_i$
   b $z^2$
   c $z^{-1}$

9 a If $z = 3 + 2i$, state $\bar{z}$ and calculate $z^{-1}$.
   b Plot $z$, $\bar{z}$, and $z^{-1}$ on an Argand plane.
   c What transformation plots $z$ onto $z^{-1}$?
   d What is the relationship between the origin and the points representing $\bar{z}$ and $z^{-1}$?

10 a If $z = 2 + 3i$, calculate $iz$, $i^2z$, $i^3z$ and $i^4z$.
   b Plot $z$, $iz$, $i^2z$, $i^3z$ and $i^4z$ on an Argand plane.
   c State a transformation that will plot the point $i^n z$ onto $i^{n+1}z$ for $n \in Z^+$.

11 a If $z = 1 + i$, calculate $z^{-2}$, $z^{-1}$, $z^0$, $z^2$, $z^3$ and $z^4$.
   b Plot $z^{-2}$, $z^{-1}$, $z^0$, $z^2$, $z^3$ and $z^4$ on an Argand plane.
   c State the rotation and the change in distance from the origin required to plot the point $z^n$ onto $z^{n+1}$ for $n \in Z$.
   d State the rotation and the distance from the origin required to plot the point $z^0$ onto $z^n$ for $n \in Z^+$.

12 State, using the results from question 11, the following powers of $z$ in their simplest Cartesian form.
   a $z^5$
   b $z^{-3}$
   c $z^{10}$
   d $z^{17}$
   e $z^{-13}$.

13 If $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$ which is the distance of $z$ from the origin. What does $|z| = 1$ represent?

14 Describe the graph of:
   a $|z - 1| = 1$
   b $|z + i| = 2$
   c $|z - 2 + 3i| = 3$.

1.7 Factorising quadratic expressions and solving quadratic equations over the complex number field

In the introduction to complex numbers, a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \in R \{0\}$, $b, c \in R$, was solved using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$ 

The expression under the square root sign is called the discriminant and is represented by $\Delta$, where $\Delta = b^2 - 4ac$. The discriminant can be used to determine the nature of the solutions. It can also be used to determine possible methods for factorising a quadratic expression.

The following table gives the method for factorising a quadratic expression and the nature of the solutions for given values of $\Delta$, where $a, b, c \in Q \{0\}$.

<table>
<thead>
<tr>
<th>Values of the discriminant</th>
<th>Factorising methods</th>
<th>Nature of solution(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta = 0$</td>
<td>A perfect square. State the answer.</td>
<td>Two equal rational solutions</td>
</tr>
<tr>
<td>$\Delta$ is a perfect square</td>
<td>Factorise over $Q$ or complete the square.</td>
<td>Two rational solutions</td>
</tr>
<tr>
<td>$\Delta &gt; 0$ and is not a perfect square</td>
<td>Complete the square.</td>
<td>Two irrational solutions</td>
</tr>
<tr>
<td>$\Delta &lt; 0$</td>
<td>Complete the square.</td>
<td>Two complex solutions</td>
</tr>
</tbody>
</table>
Factorising quadratic expressions over the complex number field

Factorising over \( R \) implies that all the coefficients must be real numbers and factorising over \( C \) implies that all the coefficients must be complex numbers. As factors over \( C \) are required in this section, the variable label will be \( z \). In Worked example 22 below, the factors for the expressions in parts \( a \) and \( b \) are factors over both \( R \) and \( C \), but the factors for the expression in part \( c \) are factors over \( C \) only. It is still correct to say that \( 2z^2 + 3 \) does not factorise over \( R \).

If \( c = 0 \) and \( a, b \in R\setminus\{0\} \), then factorise \( az^2 + bz \) by taking out the common factor \( z(az + b) \).

If \( b = 0 \) and \( a, c \in R\setminus\{0\} \), then factorise \( az^2 + c \) using the difference of squares to factorise.

Factorise each of the following quadratic expressions over \( C \).

<table>
<thead>
<tr>
<th>WORKED EXAMPLE 22</th>
<th>Factorise each of the following quadratic expressions over ( C ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a ) ( 2z^2 + 6z )</td>
<td>( b ) ( 2z^2 - 6 )</td>
</tr>
<tr>
<td><strong>THINK</strong></td>
<td><strong>WRITE</strong></td>
</tr>
<tr>
<td>( a ) Factorise ( 2z^2 + 6z ) by taking out the highest common factor.</td>
<td>( a ) ( 2z^2 + 6z = 2(z + 3) )</td>
</tr>
<tr>
<td>( b ) 1 Factorise ( 2z^2 - 6 ) by taking out the highest common factor.</td>
<td>( b ) ( 2z^2 - 6 = 2(z^2 - 3) )</td>
</tr>
<tr>
<td>2 Factorise further using the difference of two squares.</td>
<td>( = 2(z - \sqrt{3})(z + \sqrt{3}) )</td>
</tr>
<tr>
<td>( c ) 1 Factorise ( 2z^2 + 3 ) by taking out the common factor of 2.</td>
<td>( c ) ( 2z^2 + 3 = 2(z^2 + \frac{3}{2}) )</td>
</tr>
<tr>
<td>2 Factorise further using the difference of two squares. Let ( \frac{3}{2} = -\frac{3}{2}i^2 ).</td>
<td>( = 2(z^2 - \frac{3}{2}) )</td>
</tr>
<tr>
<td>3 Rationalise the denominators by multiplying the relevant terms by ( \frac{\sqrt{2}}{\sqrt{2}} ).</td>
<td>( = 2(z - \frac{\sqrt{3}}{\sqrt{2}})(z + \frac{\sqrt{3}}{\sqrt{2}}) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>WORKED EXAMPLE 23</th>
<th>Factorise each of the following quadratic expressions over ( C ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a ) ( z^2 - 6z + 9 )</td>
<td>( b ) ( z^2 - 4z - 60 )</td>
</tr>
<tr>
<td><strong>THINK</strong></td>
<td><strong>WRITE</strong></td>
</tr>
<tr>
<td>( a ) 1 Calculate the value of the discriminant to determine the nature of the factors.</td>
<td>( a ) ( \Delta = b^2 - 4ac )</td>
</tr>
<tr>
<td></td>
<td>( = (-6)^2 - 4(1)(9) )</td>
</tr>
<tr>
<td></td>
<td>( = 0 )</td>
</tr>
</tbody>
</table>
Solving quadratic equations over the complex number field

Two methods can be used to solve quadratic equations over the complex number field:

1. Factorise first and use the null factor property to state solutions.
2. Use the formula for the solution of a quadratic equation.

The null factor property states that if $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$. From Worked example 22 d,

**b 1** Calculate the value of the discriminant to determine the nature of the factors.

\[ \Delta = b^2 - 4ac \]
\[ \Delta = (-4)^2 - 4(1)(-60) \]
\[ \Delta = 256 \]

**2** Since $\Delta = 256$, which is a perfect square, the factors will be rational.

\[ z^2 - 4z - 60 = (z - 10)(z + 6) \]

**c 1** Calculate the value of the discriminant to determine the nature of the factors.

\[ \Delta = b^2 - 4ac \]
\[ \Delta = (-6)^2 - 4(2)(-6) \]
\[ \Delta = 84 \]

**2** Since $\Delta = 84$, which is not a perfect square but is positive, use the difference of two squares to find two factors over $R$.

\[ 2z^2 - 6z - 6 = 2(z^2 - 3z - 3) \]
\[ = 2 \left( z^2 - 3z + \left( \frac{3}{2} \right)^2 - \left( \frac{3}{2} \right)^2 \right) \]
\[ = 2 \left( z^2 - 3z + \frac{9}{4} - \frac{9}{4} \right) \]
\[ = 2 \left( z^2 - 3z + \frac{9}{4} \right) \]
\[ = z \left( z - \frac{3}{2} - \frac{\sqrt{31}}{2} \right) \left( z - \frac{3}{2} + \frac{\sqrt{31}}{2} \right) \]

**d 1** Calculate the value of the discriminant to determine the nature of the factors.

\[ \Delta = b^2 - 4ac \]
\[ \Delta = (-3)^2 - 4(-2)(-5) \]
\[ \Delta = -31 \]

**2** Since $\Delta = -31$, which is not a perfect square but is negative, use the difference of two squares to find two factors over $C$.

\[ -2z^2 - 3z - 5 = -2 \left( z^2 + \frac{3}{2}z + \frac{5}{2} \right) \]
\[ = -2 \left( z^2 + \frac{3}{2}z + \left( \frac{3}{4} \right)^2 + \frac{3}{4} \right) \]
\[ = -2 \left( z + \frac{3}{4} \right)^2 \]
\[ = -2 \left( z + \frac{3}{4} - \frac{\sqrt{31}}{4}i \right) \left( z + \frac{3}{4} + \frac{\sqrt{31}}{4}i \right) \]
$$-2z^2 - 3z - 5 = -2 \left( z + \frac{3}{4} - \frac{\sqrt{31}i}{4} \right) \left( z + \frac{3}{4} + \frac{\sqrt{31}i}{4} \right),$$ so the solutions of $$-2z^2 - 3z - 5 = 0$$ are from $$z + \frac{3}{4} - \frac{\sqrt{31}i}{4} = 0$$ and $$z + \frac{3}{4} + \frac{\sqrt{31}i}{4} = 0$$.

The solutions are $$z = -\frac{3}{4} \pm \frac{\sqrt{31}}{4}i$$.

If $$az^2 + bz + c = 0$$, where $$a \in \mathbb{C} \setminus \{0\}, b, c \in \mathbb{C}$$, the formula for the solution of the quadratic equation is $$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$.

### WORKED EXAMPLE 24
Solve the following using the formula for the solution of a quadratic equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>a $$2z^2 + 4z + 5 = 0$$</td>
<td>1 Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve over $\mathbb{C}$, where $a = 2, b = 4, c = 5$.</td>
<td>a $z = \frac{-4 \pm \sqrt{16 - 40}}{4}$</td>
</tr>
<tr>
<td></td>
<td>2 Express the answer in the form $a + bi$.</td>
<td>a $z = \frac{-4 \pm \sqrt{-24}}{4}$</td>
</tr>
<tr>
<td>b $$2iz^2 + 4z - 5i = 0$$</td>
<td>1 Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve over $\mathbb{C}$, where $a = 2i, b = 4, c = -5i$.</td>
<td>b $z = \frac{-4 \pm \sqrt{16 - 4 \times -10i^2}}{4i}$</td>
</tr>
<tr>
<td></td>
<td>2 Express the answer in the form $a + bi$.</td>
<td>b $z = \frac{-4 \pm \sqrt{-24}}{4i}$</td>
</tr>
</tbody>
</table>

$$= \frac{-4 \pm 2\sqrt{6}i}{4i} \times \frac{i}{i}$$

$$= \frac{(-4 \pm 2\sqrt{6}i)i}{-4}$$

$$= i \pm \frac{\sqrt{6}}{2}$$

$$= \frac{\sqrt{6}}{2} + i \text{ or } -\frac{\sqrt{6}}{2} + i$$
EXERCISE 1.7

Factorising quadratic expressions and solving quadratic equations over the complex number field

PRACTISE

1. Factorise the following quadratic expressions over \( C \).
   \[ \begin{align*}
   a & \quad 2z^2 - 6 \\
   b & \quad 2z^2 - 3 \\
   c & \quad 3z^2 + 6 \\
   d & \quad 2z^2 + \frac{1}{2}
   \end{align*} \]

2. Factorise the following quadratic expressions over \( C \).
   \[ \begin{align*}
   a & \quad z^2 - 4z \\
   b & \quad 6z^2 - 2z \\
   c & \quad 2\sqrt{2}z^2 - \sqrt{2}z \\
   d & \quad -4z^2 - 3z
   \end{align*} \]

3. Factorise the following quadratic expressions over \( C \) using the completion of the square method.
   \[ \begin{align*}
   a & \quad z^2 + 4z + 14 \\
   b & \quad z^2 + 10z + 16 \\
   c & \quad 2z^2 + 5z - 3 \\
   d & \quad z^2 + z - 3 \\
   e & \quad z^2 + 8z + 16
   \end{align*} \]

4. Factorise the following quadratic expressions over \( C \) using the completion of the square method.
   \[ \begin{align*}
   a & \quad z^2 + 2z + 3 \\
   b & \quad 2z^2 - 5z + 2 \\
   c & \quad 2z^2 + 8z + 8 \\
   d & \quad -2z^2 + 5z + 4 \\
   e & \quad -4z^2 + 4z - 1
   \end{align*} \]

5. Factorise the following quadratic expressions over \( C \), and then solve the given quadratic equations.
   \[ \begin{align*}
   a & \quad 3z^2 - 2 = 0 \\
   b & \quad 2z^2 + 5 = 0 \\
   c & \quad 2z^2 - 7z = 0 \\
   d & \quad z^2 - 6z + 5 = 0 \\
   e & \quad z^2 - 5z + 6 = 0
   \end{align*} \]

6. Solve the following quadratic equations over \( C \) using the formula for the solution of a quadratic equation.
   \[ \begin{align*}
   a & \quad z^2 - 10z + 25 = 0 \\
   b & \quad z^2 - 10z + 5 = 0 \\
   c & \quad z^2 + 4z + 7 = 0 \\
   d & \quad 2z^2 - 7z + 6 = 0 \\
   e & \quad 3z^2 - 7z + 7 = 0 \\
   f & \quad -2z^2 + 4z - 6 = 0
   \end{align*} \]

7. Factorise the following quadratic expressions over \( C \) without using the completion of the square method.
   \[ \begin{align*}
   a & \quad z^2 + 8z + 16 \\
   b & \quad 2z^2 - 8z + 8 \\
   c & \quad 2z^2 + 3z - 2 \\
   d & \quad 2z^2 - 2z - 3 \\
   e & \quad 2z^2 - 2z - 24 \\
   f & \quad -12z^2 + 10z + 12
   \end{align*} \]

8. Factorise the following quadratic expressions over \( C \), and then solve the given quadratic equations.
   \[ \begin{align*}
   a & \quad 2z^2 - 5z + 3 = 0 \\
   b & \quad z^2 - 4z + 2 = 0 \\
   c & \quad 2z^2 + 5z + 4 = 0 \\
   d & \quad z^2 - 6z + 5 = 0 \\
   e & \quad -3z^2 - 2z - 1 = 0
   \end{align*} \]

9. Expand the following.
   \[ \begin{align*}
   a & \quad (z - (2 + 3i)) (z - (2 - 3i)) \\
   b & \quad (z - (2 + 3i))^2 \\
   c & \quad (z - 2 + 3i) (z - 3 - 2i)
   \end{align*} \]

10. Solve the following quadratic equations over \( C \), using the formula for the solution of a quadratic equation.
    \[ \begin{align*}
    a & \quad iz^2 - 6z + 5i = 0 \\
    b & \quad (2 + i)z^2 - iz - (2 - i) = 0 \\
    11. & \quad Solve -3iz^2 - (1 + i)z + 5i = 0
    \end{align*} \]
12 Using the smallest set from \( Q, I \) and \( C \), the solutions of \( 2z^2 - 5z + 6 = 0 \) and \( 5z^2 - 11z + 5 = 0 \), respectively, belong to the sets:

A \( C, C \)  
B \( C, Q \)  
C \( C, I \)  
D \( I, I \)  
E \( I, Q \)

13 The factors of \( z^2 + 6z + 11 \) and \( 2z^2 - 4z + 3 \), respectively, are:

A \( (z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), 2\left(z - 1 - \frac{\sqrt{2}}{2}i\right)(z - 1 + \frac{\sqrt{2}}{2}i) \)  
B \( (z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), \left(z - 1 - \frac{\sqrt{2}}{2}i\right)(z - 1 + \frac{\sqrt{2}}{2}i) \)  
C \( (z - 3 - \sqrt{2}i)(z - 3 + \sqrt{2}i), 2\left(z - 1 - \frac{\sqrt{2}}{2}i\right)(z - 1 + \frac{\sqrt{2}}{2}i) \)  
D \( (z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), \left(z + 1 - \frac{\sqrt{2}}{2}i\right)(z + 1 + \frac{\sqrt{2}}{2}i) \)  
E \( (z - 3 - \sqrt{2}i)(z - 3 + \sqrt{2}i), 2\left(z + 1 - \frac{\sqrt{2}}{2}i\right)(z + 1 + \frac{\sqrt{2}}{2}i) \)

14 \( 2 - i \) is a solution of \( x^2 - 4x + k = 0 \). What is the value of \( k \)?

A \( 3 \)  
B \( 5 \)  
C \( \sqrt{5} \)  
D \(-3\)

15 The solutions to the quadratic equation: \( x^2 - 2x + 3 = 0 \) are:

A \( x = 3 \) or \( x = -1 \)  
B \( x = 2 + \sqrt{2}i \) or \( x = 2 - \sqrt{2}i \)  
C \( x = 1 + \sqrt{2}i \) or \( x = 1 - \sqrt{2}i \)  
D \( x = 1 + 2i \) or \( x = 1 - 2i \)

16 Solve \( x^2 + 4x + 7 = 0 \).

17 If \( \frac{10x^2 - 2x + 4}{x^2 + x} = \frac{A(Bx^2 +Cx + D)}{(x-i)(x+i)} \), Find \( A \), \( B \), \( C \) and \( D \).

18 Solve \( x^4 + 13x^2 + 36 = 0 \).
studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

studyON

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

**REVIEW QUESTIONS**

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

- **Number systems: real and complex**
1 Answers

EXERCISE 1.2

1

\[ \begin{array}{c}
A & B \\
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array} \]

a \{3, 4, 5, 6\}
b \{1, 2, 3\}
c \{2\}
d \{1\}

2

\[ \begin{array}{c}
A & B \\
4 & 24 & 20 \\
8 & 28 & 32 \\
12 & 16 & 36 \\
\end{array} \]

a \{4, 8, 12, 16\}
b \{4, 8, 12, 16, 20\}
c \{4, 8, 12, 16, 24, 28, 32, 36\}
d \{4, 8, 12, 16, 24, 28, 32, 36\}

3 Answers will vary.

4 Answers will vary.

5 a \(3.328125\)
b \(0.3409\)

6 a \(1.416\)
b \(1.230769\)

7 \(-2 \Rightarrow \text{Integer: } Z, Q, R\)

a \(\frac{10}{5} \Rightarrow \text{Natural number: } N, Z, Q, R\)
b \(\frac{21}{10} \Rightarrow \text{Rational: } Q, R\)
c \(-3 \frac{1}{2} \Rightarrow \text{Rational: } Q, R\)
d \(6 \sqrt{3} \Rightarrow \text{Irrational: } I, R\)

16 \(\frac{1}{4} \Rightarrow \text{Natural number: } N, Z, Q, R\)

8 \(5 \frac{1}{3} \Rightarrow \text{Irrational: } I, R\)

a \(\pi \Rightarrow \text{Irrational: } I, R\)
b \(21.72 \Rightarrow \text{Rational: } Q, R\)
c \(2.567 \Rightarrow \text{Rational: } Q, R\)

4.135 218 976 \Rightarrow \text{Irrational: } I, R

4.232 332 333 \Rightarrow \text{Irrational: } I, R

9 a \(\frac{8}{33}\)
b \(\frac{374}{333}\)

10 a \(\frac{61}{95}\)
b \(3517\)

c \(\frac{51}{90}\)
d \(\frac{2357}{1110}\)

11 a True
b True

12 a False e.g. \(\sqrt{2} - \sqrt{2} = 0\)
b True

13 a \(6 \times 10^{28}\)
b \(4 \times 10^{14}\)
c \(6 \times 10^{40}\)

14 a \(6 \times 10^7\)
b \(2 \times 10^{27}\)
c \(4 \times 10^{-3}\)

15 a \(\{x: x \in \mathbb{Z}, -6 < x < 1\}\)
b \(\{x: x \in \mathbb{Z}, -3 \leq x \leq 4\}\)
c \(\{x: x \in \mathbb{Z}, -6 \leq x \leq 4\}\)
d \(\{x: x \in \mathbb{Z}, 0 < x < 5\}\)

16 a \(\{x: x \in \mathbb{Z}, x < 5\}\)
b \(\{x: x \in \mathbb{Z}, x > 2\}\)
c \(\{x: x \in \mathbb{Z}, -5 < x < 0\}\)

17 a \(\{x: x \in \mathbb{Q}, x > 5\}\)
b \(\{x: x \in \mathbb{Q}, 5 < x \leq 20\}\)
c \(\{x: x \in \mathbb{Q}^+, x < 20\}\)
d \(\{x: x \in \mathbb{Z}, 5 < x < 20\}\) \(\{8, 9\}\)
e \(\{x: x \in \mathbb{Z}^+, x < 100\}\) \(\{40 < x < 50\}\)

18 a \(\{x: 2 \leq x < 5\}\)
b \(\{x: 3 < x < 5\}\)
c \(\{x: x < 3\} \cup \{x: x > 7\}\)
d \(\{x: x \in \mathbb{R}, x < 3\} \cup \{x: x \in \mathbb{R}, x > 7\}\)
19 a \( x : -3 \leq x \leq 1 \)
Interval notation \((-3, 1)\)

b \( x : x < 2 \)
Interval notation \((-\infty, 2)\)

c \( x : -2 < x < 1 \)
Interval notation \((-2, 1)\)

d \( x : x \geq 2 \)
Interval notation \([2, \infty)\)

20 a \( x : 2 \leq x < 5 \) ∪ \( x : 4 \leq x < 6 \)
Interval notation
\([2, 5) \cup [4, 6) = [2, 6)\)

b \( x : x < 5 \) ∪ \( x : 4 \leq x < 6 \)
Interval notation
\((-\infty, 5) \cup [4, 6) = (-\infty, 6)\)

c \( x : 2 \leq x < 5 \) ∪ \( x : 4 < x \leq 6 \)
Interval notation
\([2, 5) \cup (4, 6] = [2, 6] \cap (4, 5)\)

d \( x : x > 5 \) ∩ \( x : 4 < x \leq 6 \)
Interval notation
\((5, \infty) \cap (4, 6] = (5, 6]\)

21 a \( A' \)

22 See table at foot of page.*

<table>
<thead>
<tr>
<th>Number</th>
<th>3 sig. fig.</th>
<th>4 sig. fig.</th>
<th>2 D.P.</th>
<th>3 D.P.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1267.1066</td>
<td>1270</td>
<td>1267</td>
<td>1267.11</td>
<td>1267.107</td>
</tr>
<tr>
<td>7.6699</td>
<td>7.67</td>
<td>7.670</td>
<td>7.67</td>
<td>7.670</td>
</tr>
<tr>
<td>8.000 56</td>
<td>8.00</td>
<td>8.001</td>
<td>8.00</td>
<td>8.001</td>
</tr>
<tr>
<td>0.999 87</td>
<td>1.00</td>
<td>1.000</td>
<td>1.00</td>
<td>1.000</td>
</tr>
<tr>
<td>0.076 768</td>
<td>0.0768</td>
<td>0.07677</td>
<td>0.08</td>
<td>0.077</td>
</tr>
<tr>
<td>0.000 174 95</td>
<td>0.000 175</td>
<td>0.000 175 0</td>
<td>0.00</td>
<td>0.000</td>
</tr>
</tbody>
</table>
23 a $-11$
b $-3$
24 B
25 E
26 D
27 D
28 B
29 E
30 C
31 a $5.35 \times 10^{30}$
b $7.64 \times 10^{19}$
c $1.23 \times 10^7$
d $3.60 \times 10^{-7}$
32 a i 40%
ii 23%
iii 16%
b Percentage of prime number decrease as they approach infinity.

**EXERCISE 1.3**

1 a $2\sqrt{5}$
b $2\sqrt{7}$
c $5\sqrt{3}$
d $7\sqrt{2}$
e $4\sqrt{3}$
2 a $10\sqrt{3}$
b $28\sqrt{3}$
c $2\sqrt{2}$
d $6\sqrt{2}$
e $-3\sqrt{2}$
f $\frac{-3\sqrt{2}}{2}$
3 a $2\sqrt{2} - 2\sqrt{3}$
b $\sqrt{7} - 4$
c $-2(3\sqrt{3} + 2\sqrt{2})$
d $3(\sqrt{2} + 2\sqrt{3})$
4 a $7\sqrt{3} - \sqrt{2}$
b $2(9\sqrt{3} + 10\sqrt{2})$
c $\frac{9\sqrt{3}}{8} - \frac{13\sqrt{2}}{8}$
d $\frac{118\sqrt{3}}{15} - \frac{49\sqrt{2}}{10}$
5 a $3\sqrt{10}$
b $10\sqrt{21}$
6 a $84\sqrt{2}$
b $\frac{5\sqrt{3}}{6}$
7 a $\sqrt{15} - \sqrt{6}$
b $18 + 2\sqrt{6}$
c $5 - \sqrt{10} - \sqrt{15} + \sqrt{6}$
d $7\sqrt{6} - 18$
8 a $2(6 + \sqrt{35})$
b $2(105 + 36\sqrt{6})$
c 17
d 25
9 a $\sqrt{6}$
b $\frac{4\sqrt{2}}{3}$
c $\frac{\sqrt{15}}{3}$
d $\frac{4\sqrt{15}}{35}$
e $\frac{2\sqrt{6}}{9}$
f $\frac{\sqrt{5} + \sqrt{3}}{2}$
10 a $3 + \sqrt{6}$
b $2\sqrt{10} - 6$
c $14 + 5\sqrt{3}$
d $\frac{11}{11}$
e $-4 - 3\sqrt{15}$
f $\frac{6 + 2\sqrt{3}}{3}$
11 a $-4\sqrt{2}$
b $\frac{69\sqrt{2} + 20\sqrt{3}}{114}$
c $195\sqrt{10} + 78\sqrt{15} - 12\sqrt{2} + 18\sqrt{3}$
\[\frac{114}{231}\]
d $1920 - 338\sqrt{15}$
e $\frac{35 - 7\sqrt{6}}{15}$
f $\frac{-47 - 12\sqrt{15}}{3}$
12 a $26 - 45\sqrt{2}$
\[\frac{14}{14}\]
b $30 - 39\sqrt{3}$
\[\frac{14}{14}\]
c $-3(6 + 5\sqrt{2})$
d $\frac{22 - 12\sqrt{2}}{7}$
e 0
31 \div 18 \sqrt{2}
\quad g \quad x = 2 - 3 \sqrt{2} \text{ is a solution to } x^2 - 4x - 14. \text{ However, since there was no remainder, it is not a solution to } 2x^2 - 2x - 9.

13 Answers will vary.

14 Answers will vary.

15 B

16 A

17 A

18 D

19 A

20 E

21 a \quad 2\sqrt{37} \text{ cm}
\quad b \quad 18\pi\sqrt{37} \text{ cm}^3
\quad c \quad 54\pi\sqrt{3} \text{ cm}^3
\quad d \quad 18\pi(\sqrt{37} + 3\sqrt{3}) \text{ cm}^3

22 a \quad 360\sqrt{10} \text{ cm}^3
\quad b \quad \frac{3\sqrt{10}\pi}{\pi} \text{ cm}
\quad c \quad \frac{2\sqrt{15}\pi}{\pi} \text{ cm}

**EXERCISE 1.4**

1 a \quad x = 1 \text{  } y = 4
\quad b \quad x = 1 \text{  } y = -5
\quad c \quad x = -3 \text{  } y = -1
\quad d \quad x = 1 \text{  } y = 3

2 a \quad x = \frac{3}{4} \text{  } y = \frac{4}{7}
\quad b \quad x = 2 \text{  } y = \frac{3}{2}
\quad c \quad x = \frac{6}{5} \text{  } y = \frac{12}{5}

3 a \quad 5 + 3i
\quad b \quad -1 - 5i
\quad c \quad 13 + 10i

4 a \quad i
\quad b \quad 7\sqrt{2} - 10\sqrt{2}i
\quad c \quad (3\sqrt{2} + 2\sqrt{3}) - (4\sqrt{2} + 3\sqrt{3})i

5 a \quad 4i
\quad b \quad \sqrt{7}i
\quad c \quad 2 + 2\sqrt{3}i

6 a \quad \sqrt{10} + \sqrt{10}i
\quad b \quad 1 - 2\sqrt{7}i

7 a \quad -5i
\quad b \quad 7i + 2
\quad c \quad 99i

8 a \quad \text{Re} (3 + 4i) = 3
\quad \text{Im} (3 + 4i) = 4

**EXERCISE 1.5**

1 a \quad -6 + 4i
\quad b \quad 5 - i
\quad c \quad -5 + 5i
\quad d \quad -5 - 12i
\quad e \quad 85

2 a \quad -8 - 6i
\quad b \quad 24 + 6i
\quad c \quad a^2 + b^2

3 a \quad 3 - 2i
\quad b \quad -4 - 3i
\quad c \quad \sqrt{2} + 2i
\quad d \quad 8i

4 a \quad 4 + 3i
\quad b \quad 24 + 7i
\quad c \quad 24 + 7i
\quad d \quad 7 + 24i
\quad e \quad -8 + 6i
\quad f \quad 28i

5 a \quad \frac{1}{5} + \frac{3}{5}i
\quad b \quad \frac{-7}{25} + \frac{24}{25}i
\quad c \quad \frac{4}{3} + \frac{3}{5}i
\quad d \quad \frac{11}{5} - \frac{2}{5}i
13 e $\frac{3}{13} + \frac{2}{13}i$

f $\frac{5}{169} - \frac{12}{169}i$

6 a $\frac{4}{25} + \frac{3}{25}i$

b $-i$

c $\frac{24}{25} + \frac{7}{25}i$

d $\frac{527}{625} - \frac{326}{625}i$

e $\frac{104}{25} + \frac{78}{25}i$

f $\frac{14}{25}$

7 a Answers will vary.

b Answers will vary.

c Answers will vary.

8 a $x = \frac{12}{5}$

$y = \frac{9}{5}$

$y - 2x = 1$

$y = 3$

$x = -1$

9 a $z = \frac{1}{5} - \frac{2}{5}i$

b $z = -1$

10 a $z^{-1} = \frac{\bar{z}}{41}$

b $z^{-1} = \frac{\bar{z}}{a^2 + b^2}$ or $z^{-1} = \frac{\bar{z}}{zz}$

11 B

12 A

13 E

14 B

15 a $\frac{4}{13}$

b $\frac{63}{25} - \frac{16}{25}i$

c $\frac{16}{7} - \frac{8\sqrt{3}}{7}i$

16 a 1

b $\frac{9}{13} - \frac{20}{13}i$

17 $0 + 5i$

18 $\frac{11}{10} + \frac{13}{10}i$

EXERCISE 1.6

1 a $-i$

b $i$

2 a $-64$

b 256

c $-512i$

d $\frac{-i}{512}$

3 a

b

c

d

e

4 A $1 + i$

B $-3 - 2i$

C $1 - 3i$

D $3i$

E 3

F $-2 + i$
5 a $7 - 2i$
   b $-3 + 12i$

6 a
   b A rotation of $180^\circ$ about the origin.

7 a $z = 4 - 3i$
   b $-z = -4 + 3i$
   c $\bar{z} = 4 + 3i$

8
   b $\overline{z}$
   c $\bar{z}$
   d $\overline{\bar{z}}$

9 a $\frac{3}{13} - \frac{2i}{13}$

10 a $z = 2 + 3i$
   $iz = i(2 + 3i) = -3 + 2i$
   $i^2z = i(-3 + 2i) = -2 - 3i$
   $i^3z = i(-2 - 3i) = 3 - 2i$
   $i^4z = i(3 - 2i) = 2 + 3i$

11 a $z^{-2} = \frac{1}{(1 + i)^2} = \frac{1}{1 + 2i - 1} = \frac{1}{2i} = -\frac{1}{2} = -\frac{1}{2}i$
   $z^{-1} = \frac{1}{1 + i} \times \frac{1 - i}{1 - i} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i$
   $z^0 = (1 + i)^0 = 1$
   $z^1 = 1 + i$
   $z^2 = (1 + i)^2 = 1 + 2i - 1 = 2i$
   $z^3 = (1 + i)^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$
   $z^4 = (1 + i)^4 = (-2 + 2i)(1 + i)$
   $= -2 - 2i + 2i - 2 = -4$

b
   c Plotted $z^n$ compared to $z^{n+1}$ where $n \in \mathbb{Z}$, undergoes a rotation $45^\circ$ anticlockwise and its distance from the origin is increased by a factor of $\sqrt{2}$.
   d Rotated $n \times 45^\circ$ or $\frac{n\pi}{4}$ anticlockwise
   Distance from the origin multiplied by a factor of $(\sqrt{2})^n$
12 a $-4 - 4i$
  b $-1 - rac{1}{4}i$
  c $32i$
  d $256 + 256i$
13 A circle with centre $0 + 0i$ and a radius of 1
14 a A circle with centre $1 + 0i$ and a radius of 1
  b A circle with centre $2 - 3i$ and a radius of 3

EXERCISE 1.7
1 a $2(z - \sqrt{3})(z + \sqrt{3})$
  b $(\sqrt{2}z - \sqrt{3})(\sqrt{2}z + \sqrt{3})$
  c $3(z - \sqrt{2}i)(z + \sqrt{2}i)$
  d $\frac{1}{2}(2i - 1)(2i + 1)$
2 a $z(z - 4)$
  b $2z(3z - 1)$
  c $\sqrt{2}z(2z - 1)$
  d $-z(4z + 3)$
3 a $(z + 2 - \sqrt{10}i)(z + 2 + \sqrt{10}i)$
  b $(z + 2)(z + 8)$
  c $2(z - \frac{1}{2})(z + 3)$
  d $\left(z + \frac{1}{2} - \frac{\sqrt{13}}{2}\right)\left(z + \frac{1}{2} + \frac{\sqrt{13}}{2}\right)$
  e $(z + 4)^2$
4 a $(z + 1 - \sqrt{2}i)(z + 1 + \sqrt{2}i)$
  b $2(z - 2)(z - 2)$
  c $2(z + 2)^2$
  d $-2\left(z - \frac{5}{4} - \frac{\sqrt{57}}{4}\right)\left(z - \frac{5}{4} + \frac{\sqrt{57}}{4}\right)$
  e $-4(z - \frac{1}{3})^2$
5 a $(\sqrt{3}z - \sqrt{2}) - (\sqrt{3}z + \sqrt{2}) = 0$
  b $(\sqrt{2}z - \sqrt{3}i)(\sqrt{2}z + \sqrt{3}i) = 0$
  c $z(2z - 7) = 0$
  d $(z - 5)(z - 1) = 0$
  e $(z - 3)(z - 2) = 0$

6 a $z = 5$
  b $z = 5 \pm 2\sqrt{3}$
  c $z = -2 \pm i\sqrt{3}$
  d $z = 2, z = \frac{1}{2}$
  e $z = \frac{7 \pm i\sqrt{35}}{6}$
  f $z = 1 \pm i\sqrt{2}$
7 a $(z + 4)^2$
  b $2(z - 2)(z - 2)$
  c $(2z - 1)(z + 2)$
  d $(z + 3)(z - 1)$
  e $2(z + 3)(z - 4)$
  f $-2(z - 3)(3z + 2)$
8 a $(2z - 3)(z - 1) = 0$
  b $(z - 2 - \sqrt{2})(z - 2 + \sqrt{2}) = 0$
  c $2\left(z + \frac{5}{4} + \frac{\sqrt{17}}{4}\right)\left(z + \frac{5}{4} + \frac{\sqrt{17}}{4}\right) = 0$
  d $(z - 5)(z - 1) = 0$
  e $-3\left(z + \frac{1}{3} - \frac{\sqrt{2}i}{3}\right)\left(z + \frac{1}{3} + \frac{\sqrt{2}i}{3}\right) = 0$
  f $z = \frac{1}{3} \pm \frac{\sqrt{2}i}{3}$
9 a $z^2 - 4z + 13$
  b $z^2 + 6i - 9$
  c $z^2 - 5z + iz - 5i + 12$
10 a $z = -3i \pm i\sqrt{14}$
  b $z = \frac{1 \pm 2\sqrt{19} + i(2 + \sqrt{19})}{10}$
11 a $(z - 1) \pm i\sqrt{2}(i - 30)$
  b $\frac{10}{6}$
12 A
13 A
14 B
15 C
16 $x = -2 \pm \sqrt{37}$
17 $A = 2, B = 5$
  c $C = -1, D = 2$
18 $x = \pm 3i$ or $x = \pm 2i$