4 Kinematics

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4.1 Kick off with CAS

Kinematics involves the study of position, displacement, velocity and acceleration. From the study of calculus in Year 11, we know that if \( x \) is the position of an object moving in a straight line at time \( t \), then the velocity of the object at time \( t \) is given by \( v = \frac{dx}{dt} \) and the acceleration of the body at time \( t \) is given by \( a = \frac{dv}{dt} \).

1. If the position of a body moving in a straight line is given by \( x(t) = 2t^3 + 9t^2 - 12t + 10 \) where \( x \) is in centimetres and \( t \) is in seconds, use the ‘define’ function on the CAS calculator and calculate the:
   a. initial velocity and acceleration
   b. time when the body is at the origin
   c. velocity when the acceleration is zero
   d. acceleration when the velocity is zero.

2. A hot air balloon commences its descent at time \( t = 0 \) minutes. As it descends, the height of the balloon above the ground, in metres, is given by the equation, \( h(t) = 600 \times 2^{-\frac{t}{10}} \).
   a. Use CAS to sketch the height of the balloon above the ground at time \( t \) and the rate at which the balloon is descending at time \( t \).
   b. If the balloon is anchored by the crew on the ground when it is 2 metres above the ground, how long did it take until the balloon was secured?
   c. Find expressions for the velocity and acceleration of the balloon.
   d. Calculate the rate at which the balloon is descending after 20 minutes.
   e. Find the time when the balloon is descending at a rate of \(-9.0\) m/minute.

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
Constant acceleration

Kinematics is the study of the motion of objects. In this topic, the focus is on objects that move along a straight line, which is also known as rectilinear motion.

When we consider the motion of an object in a straight line with uniform acceleration, a number of rules can be used.

The diagram below represents the motion of an object with initial velocity $u$ and final velocity $v$ after $t$ seconds.

![Diagram](image_url)

The gradient of the line is calculated by $\frac{v - u}{t}$. On a velocity–time graph, the gradient is the acceleration of the object. Hence, $a = \frac{v - u}{t}$.

Transposing $a = \frac{v - u}{t}$, we get $v = u + at$ (equation 1).

Alternatively, by antidifferentiating $\frac{dv}{dt} = a$ with respect to $t$:

$$\int \frac{dv}{dt} dt = \int a dt$$

$$v = at + c$$

When $t = 0$, $c = u$ (initial velocity): $v = at + u$

The area under a velocity–time graph gives the displacement of the object. Therefore, using the rule for the area of a trapezium, we get $s = \frac{1}{2}(u + v)t$ (equation 2), where $\frac{1}{2}(u + v)$ is the average velocity of the object. Displacement is the average velocity of an object multiplied by time.

Substituting equation 1, $v = u + at$, into equation 2, $s = \frac{1}{2}(u + v)t$, gives:

$$s = \frac{1}{2}(u + u + at)t$$

$$s = \frac{1}{2}(2u + at)t$$

$$s = ut + \frac{1}{2}at^2$$ (equation 3)

Alternatively, by antidifferentiating $\frac{dx}{dt} = v = u + at$ with respect to $t$,

$x = ut + \frac{1}{2}at^2 + d$, where $d$ is the initial position.
When \( s = x - d \), which is the change in position of the particle from its starting point, \( s = ut + \frac{1}{2}at^2 \).

Using \( v = u + at \) (equation 1) and \( s = ut + \frac{1}{2}at^2 \) (equation 3):

\[
v^2 = (u + at)^2 \]
\[
v^2 = u^2 + 2uat + a^2t^2 \]
\[
v^2 = u^2 + 2a\left(ut + \frac{1}{2}at^2\right) \]
\[
v^2 = u^2 + 2as \quad \text{(equation 4)}
\]

If we substitute \( u = v - at \) into \( s = ut + \frac{1}{2}at^2 \), we get

\[
s = (v - at)t + \frac{1}{2}at^2,
\]
which simplifies to

\[
s = vt - \frac{1}{2}at^2 \quad \text{(equation 5)}.
\]

In summary, when considering rectilinear motion of an object in which the acceleration is constant, the following rules can be used:

- \( v = u + at \) \quad \text{\( v \) = velocity (m/s)}
- \( s = ut + \frac{1}{2}at^2 \) \quad \text{\( u \) = initial velocity (m/s)}
- \( v^2 = u^2 + 2as \) \quad \text{\( a \) = acceleration (m/s\(^2\))}
- \( s = \frac{1}{2}(u + v)t \) \quad \text{\( t \) = time (s)}
- \( s = vt - \frac{1}{2}at^2 \) \quad \text{\( s \) = displacement (m)}

In applying these rules, we must consider the following conditions.
- These quantities only apply when the acceleration is constant.
- Retardation or deceleration implies that acceleration is negative.
- The variable \( s \) is the displacement of an object. It is not necessarily the distance travelled by the object.

In solving constant acceleration problems it is important to list the quantities given to determine what is required and which equation is appropriate. It is also important to check units. All quantities must be converted to compatible units such as m and m/s or km and km/h.

Remember that to convert km/h to m/s, we multiply by \( \frac{1000}{3600} \) or \( \frac{5}{18} \); to convert m/s to km/h, we multiply by \( \frac{18}{5} \) or 3.6.

**WORKED EXAMPLE 1**

A tram uniformly accelerates from a velocity of 10 m/s to a velocity of 15 m/s in a time of 20 s. Find the distance travelled by the tram.
THINK
1. List all the quantities given and check the units.
2. Determine the appropriate equation that will solve for distance, given all the quantities.
3. Substitute the quantities into the equation and solve.

WRITE
Units: m, s
\( u = 10, \ v = 15, \ t = 20 \)
\[ s = \frac{1}{2}(u + v)t \]
\[ s = \frac{1}{2}(10 + 15) \times 20 \]
\[ = 250 \]
The distance travelled by the tram is 250 m.

WORKED EXAMPLE 2
A triathlete on a bicycle reduces her speed from 10 m/s to 4 m/s over 28 m. Assuming the deceleration is constant, determine the deceleration and how long the triathlete will travel on her bicycle before she comes to rest.

THINK
1. List all the quantities given and check the units.
2. Determine the appropriate equation that will solve for deceleration given all the quantities.
3. Substitute the quantities into the equation and solve.

WRITE
Units: m, s
\( u = 10, \ v = 4, \ s = 28 \)
\[ v^2 = u^2 + 2as \]
\[ v^2 = u^2 + 2as \]
\[ 4^2 = 10^2 + 2 \times a \times 28 \]
\[ a = -1.5 \]
The deceleration of the triathlete is 1.5 m/s².
\( u = 10, \ v = 0, \ a = -1.5 \)
\[ v = u + at \]
\[ 0 = 10 - 1.5 \times t \]
\[ t = 6 \frac{2}{3} \]
The triathlete will come to rest after \( 6 \frac{2}{3} \) seconds.
Constant acceleration

1. A train uniformly accelerates from a velocity of 30 m/s to a velocity of 40 m/s in a time of 15 s. Find the distance travelled by the train.

2. A train uniformly accelerates from a velocity of 20 m/s to a velocity of 50 m/s in a time of 20 s. Find the acceleration of the train.

3. A triathlete on a bicycle reduces his speed from 10 m/s to 6 m/s over 100 m. Assuming the deceleration is constant, determine the deceleration.

4. A triathlete on a bicycle reduces her speed from 8 m/s to 5 m/s over 150 m. Assuming the deceleration is constant, determine the time the triathlete will travel on her bicycle before she comes to rest.

5. A train uniformly accelerates from a velocity of 10 m/s to a velocity of 30 m/s in a time of 25 s. Find the distance travelled by the train.

6. A motorcyclist reduces her speed from 20 m/s to 5 m/s over 200 m. Assuming the deceleration is constant, determine:
   a. the deceleration
   b. the time the motorcyclist will travel on her motorbike before she comes to rest
   c. how much further she will travel on her motorbike before she comes to rest.

7. A skateboarder starting from rest rolls down a skate ramp. After 10 seconds, his velocity is 15 m/s. Assuming constant acceleration, find:
   a. the distance travelled by the skateboarder
   b. the acceleration of the skateboarder.

8. A snowboarder starting from rest travels down a ski slope. After 15 seconds, her velocity is 20 m/s. Find the time taken for her to travel 200 m.

9. A truck initially travelling at a constant speed is subject to a constant deceleration of 2 m/s², bringing it to rest in 6 seconds. Find:
   a. the initial speed of the truck
   b. the distance covered before the truck comes to rest.

10. A train is travelling at 25 m/s when the brakes are applied, reducing the speed to 10 m/s in 2 minutes. Assuming constant acceleration, find how far the train will travel in total before stopping.

11. A jet plane lands at one end of a runway of length 1200 m. It takes 15 seconds to come to rest and its deceleration is 4.2 m/s². Is the runway long enough for the landing of the plane?
12 A motorcycle moving with uniform acceleration is observed to cover 60 m in 5 seconds and 22 m in the next second. Find the motorcycle’s:
   a acceleration  
   b initial velocity.

13 A cyclist travelling in a straight line passes a certain point at 5 m/s and is accelerating uniformly at 2 m/s² for a distance of 60 m. Find the time the cyclist takes to travel this distance.

14 A truck moving with constant acceleration is seen to move 21.5 m in 2 seconds and 15 m in the next second. What is the acceleration of the truck?

15 A tram moves for 3 seconds with constant acceleration. During this time it travels 26.5 m. It then travels with uniform velocity, and during the next 3 seconds it travels 21.8 m. Find the tram’s:
   a acceleration  
   b initial velocity.

16 An eagle is moving with uniform acceleration. In the third second of its motion, it covers 30.2 m, and in the seventh second, it covers 17.8 m. Find the eagle’s:
   a acceleration  
   b initial velocity.

### 4.3 Motion under gravity

Vertical motion is when an object is projected vertically into the air, causing a one-dimensional movement along a straight line of force. The most common force around us is the Earth’s gravity, so vertical motion in relation to that gravity is towards or away from the centre of the Earth.

Acceleration due to gravity is the form of acceleration that is directed towards the centre of the Earth.

The **acceleration due to gravity** is approximately constant when an object’s motion is near the Earth’s surface. When considering vertical motion of an object in a straight line in which the acceleration is constant, the equations for constant acceleration can be applied.

In vertical motion problems, acceleration due to gravity is generally preferred as $g = 9.8 \text{ m/s}^2$ vertically downwards (towards the centre of Earth) rather than $g = -9.8 \text{ m/s}^2$ upwards.

In solving vertical motion problems, we can take either the upward or downward direction as positive. It also helps to sketch the situation.

- If the upward direction is taken as positive, velocity is upwards and positive, and acceleration is downwards and negative.
- If the downward direction is taken as positive, velocity is upwards and negative, and acceleration is downwards and positive.
**WORKED EXAMPLE 3**  
A tennis ball is thrown vertically upwards with a velocity of 15 m/s. What is the maximum height reached by the ball?

**THINK**
1. Draw a diagram and label the quantities.
   - Let the direction be positive upwards.
   - State the units.

2. List all the quantities given.

3. Determine the appropriate constant acceleration equation that will solve for the height of the tennis ball.

4. Substitute the quantities into the equation and solve.

**WRITE/DRAW**

<table>
<thead>
<tr>
<th>Units: m, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = 0$</td>
</tr>
<tr>
<td>$s = ?$</td>
</tr>
<tr>
<td>$a = -9.8$</td>
</tr>
<tr>
<td>$u = 15$</td>
</tr>
</tbody>
</table>

$v^2 = u^2 + 2as$

$0^2 = 15^2 + 2 \times -9.8 \times s$

$s = 11.48$

The tennis ball travels a distance of 11.48 m to its maximum height.

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**WORKED EXAMPLE 4**  
A rock is dropped down a well. It reaches the bottom of the well in 3.5 seconds. How deep is the well?

**THINK**
1. Draw a diagram and label the quantities.
   - Let the direction be positive downwards.
   - State the units.

2. List all the quantities given.

3. Determine the appropriate constant acceleration equation that will solve for the depth of the well.

**WRITE/DRAW**

<table>
<thead>
<tr>
<th>Units: m, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
</tr>
<tr>
<td>$s = ?$</td>
</tr>
<tr>
<td>$a = 9.8$</td>
</tr>
<tr>
<td>$t = 3.5$</td>
</tr>
</tbody>
</table>

$s = ut + \frac{1}{2}at^2$

$t = 3.5, u = 0, a = g = 9.8$

$s$
4 Substitute the quantities into the equation and solve.

\[ s = ut + \frac{1}{2}at^2 \]

\[ s = 0 + \frac{1}{2} \times 9.8 \times 3.5^2 \]

\[ s = 60.025 \]

The rock travels a distance of 60.025 m to the bottom of the well.

**EXERCISE 4.3**

**Motion under gravity**

1. **WE3** A basketball is thrown vertically upwards with a velocity of 5 m/s. What is the maximum height of the basketball in the air?

2. A skyrocket is projected vertically upwards to a maximum height of 154.34 m. Find the velocity of projection.

3. **WE4** A rock is dropped down a well. It reaches the bottom of the well in 3.2 seconds. How deep is the well?

4. A stone is dropped from the top of a cliff 122.5 m high. How long does it take for the stone to reach the bottom of the cliff?

5. A ball is thrown into the air with a velocity of 8 m/s. How long does the ball take to reach its maximum height?

6. A shot is fired vertically upward and attains a maximum height of 800 m. Find the initial velocity of the shot.

7. A boulder falls from the top of a cliff 45 metres high. Find the boulder’s speed just before it hits the ground.

8. A ball is thrown vertically upwards with a velocity of 15 m/s from the top of a building 20 metres high and then lands on the ground below. Find the time of flight for the ball.

9. A skyrocket is projected vertically upwards from the ground. It runs out of fuel at a velocity of 52 m/s and a height of 35 m. From this point on it is subject only to acceleration due to gravity. Find the maximum height of the skyrocket.

10. A stone is projected vertically upwards with a velocity of 12 m/s from the top of a cliff. If the stone reaches the bottom of the cliff in 8 seconds, find:
   a. the height of the cliff
   b. the velocity at which the stone must be projected to reach the bottom of the cliff in 4 s.

11. From a hot air balloon rising vertically upward with a speed of 8 m/s, a sandbag is dropped which hits the ground in 4 seconds. Determine the height of the balloon when the sandbag was dropped.

12. A missile is projected vertically upward with a speed of 73.5 m/s, and 3 seconds later a second missile is projected vertically upward from the same point with the same speed. Find when and where the two missiles collide.
13 A flare, A, is fired vertically upwards with a velocity of 35 m/s from a boat. Four seconds later, another flare, B, is fired vertically upwards from the same point with a velocity of 75 m/s. Find when and where the flares collide.

14 A skyrocket is launched upwards with a velocity of 60 m/s. Two seconds later, another skyrocket is launched upwards from the same point with the same initial velocity. Find when and where the two skyrockets meet.

15 A stone, A, is projected vertically upwards with a velocity of 25 m/s. After stone A has been in motion for 3 s, another stone, B, is dropped from the same point. Find when and where the two stones meet.

16 A worker climbs vertically up a tower to a certain height and accidently drops a small bolt. The man ascends a further 45 m and drops another small bolt. The second bolt takes 1 second longer than the first to reach the ground. Find:
   a the height above the ground at which the worker dropped the first bolt
   b the time it took the first bolt to reach the ground.

### Velocity–time graphs

Velocity–time graphs are a useful visual representation of the motion of an object in a straight line. We can use velocity–time graphs to solve kinematic problems. The following properties of velocity–time graphs make this possible.

- Because $a = \frac{dv}{dt}$, the gradient of the velocity–time graph at time $t$ gives the instantaneous acceleration at time $t$.
- Because $v = \frac{dx}{dt}$, the displacement is found by evaluating the definite integral $\int_{t_1}^{t_2} v \, dt = x_2 - x_1$. The distance is found by determining the magnitude of the signed area under the curve bounded by the graph and the $t$ axis, $\left| \int v(t) \, dt \right|$. Distance travelled cannot be a negative value.

Some useful formulas to assist in finding the displacement without using calculus are:
- area of a triangle: $A = \frac{1}{2}bh$
- area of a rectangle: $A = LW$
- area of a trapezium: $A = \frac{1}{2}(a + b)h$. 

![Velocity-Time Graph Diagram]
A train uniformly accelerates from a velocity of 25 m/s to a velocity of 50 m/s in a time of 20 seconds. Find the acceleration of the train and distance travelled by the train.

**THINK**

1. Draw a velocity–time graph.

2. To determine the acceleration of the train, we need to calculate the gradient of the graph.

\[
a = \frac{50 - 25}{20 - 0} = 1.25
\]

The acceleration of the train is 1.25 m/s\(^2\).

3. To determine the distance travelled we calculate the area under the graph. We can use the area of a trapezium.

\[
s = \frac{1}{2}(25 + 50) \times 20 = 750
\]

The distance travelled by the train is 750 m.

Puffing Billy runs on a straight track between Belgrave and Lakeside stations, which are 10 km apart. It accelerates at 0.10 m/s\(^2\) from rest at Belgrave station until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.15 m/s\(^2\) to rest at Lakeside station. Find the time taken for the Puffing Billy journey.

**THINK**

1. Draw a velocity–time graph.

2. To determine the time taken for Puffing Billy’s journey, we need to use the distance travelled and area under the graph.

Total distance:

\[
s = \frac{1}{2} \times 8 \times t_1 + (t_2 - t_1) \times 8 + \frac{1}{2} \times 8 \times (t_3 - t_2)
\]

\[= 10000\]
3 To determine \( t_1 \), the time taken for Puffing Billy to accelerate to 8 m/s, we can use the acceleration and slope equation.

\[
a = 0.1
\]
\[
\frac{8}{t_1}
\]
\[
t_1 = 80
\]

4 To determine the time taken to decelerate, \( t_3 - t_2 \), we can use the deceleration value and slope equation.

\[
a = 0.15
\]
\[
\frac{8}{t_2}
\]
\[
t_2 = 53\frac{1}{3}
\]

5 Substituting the time values for the train to accelerate and decelerate, we can find the time taken for the train to move at a constant speed.

Let \( T = t_2 - t_1 \).
\[
\frac{1}{2} \times 8 \times 80 + T \times 8 + \frac{1}{2} \times 8 \times 53\frac{1}{3} = 10000
\]
\[
T = 1183.33
\]

6 Calculating the total time.

Total time = 80 + 1183.33 + 53.33
\approx 1316.66

The time taken is 1316.66 seconds, or 21 minutes and 57 seconds.

**EXERCISE 4.4**

**Velocity–time graphs**

(Draw velocity–time graphs for all questions.)

1 **WE5** A train uniformly accelerates from a velocity of 15 m/s to a velocity of 40 m/s in a time of 15 seconds. Find the distance travelled by the train.

2 A train uniformly accelerates from a velocity of 25 m/s to a velocity of 55 m/s in a time of 10 seconds. Find the acceleration of the train and the distance travelled.

3 **WE6** Puffing Billy runs on a straight track between Emerald and Lakeside stations, which are 12 km apart. It accelerates at 0.05 m/s² from rest at Emerald station until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.10 m/s² to rest at Lakeside station. Find the time taken for Puffing Billy’s journey.

4 A tram runs on a straight track between two stops, a distance of 4 km. It accelerates at 0.1 m/s² from rest at stop A until it reaches its maximum speed of 15 m/s. It maintains this speed for a time, then decelerates at 0.05 m/s² to rest at stop B. Find the time taken for the tram journey.

5 A train uniformly accelerates from a velocity of 10 m/s to a velocity of 40 m/s in a time of 20 seconds. Find the distance travelled by the train.

6 A tram slows down to rest from a velocity of 15 m/s at a constant deceleration of 0.5 m/s². Find the distance travelled by the tram.
7 A car starting from rest speeds up uniformly to a velocity of 12 m/s in 15 seconds, continues at this velocity for 20 seconds and then slows down to a stop in 25 seconds. How far has the car travelled?

8 A rocket is travelling with a velocity of 75 m/s. The engines are switched on for 8 seconds and the rocket accelerates uniformly at 40 m/s². Calculate the distance travelled over the 8 seconds.

9 The current world record for the 100-metre men’s sprint is 9.58 seconds, run by Usain Bolt in 2009. Assuming that the last 40 m was run at a constant speed and that the acceleration during the first 60 m was constant, calculate Usain’s acceleration.

10 A tourist train runs on a straight track between Belvedere and Eureka stations, which are 8 km apart. It accelerates at 0.07 m/s² from rest at Belvedere until it reaches its maximum speed of 8 m/s. It maintains this speed for a time, then decelerates at 0.05 m/s² to rest at Eureka. Find the time taken for the journey.

11 Two racing cars are travelling along the same straight road. At time \( t = 0 \), car A passes car B. Car A is travelling at a constant velocity of 60 m/s. Car B accelerates from rest until it reaches 80 m/s after 20 seconds, and then it maintains that speed. What distance does car B travel before it overtakes car A?

12 A stationary unmarked police car is passed by a speeding car travelling at a constant velocity of 70 km/h. The police car accelerates from rest until it reaches 30 m/s after 15 seconds, which speed it then maintains. Find the time taken for the unmarked police car to catch up to the speeding car.

13 A bus takes 100 seconds to travel between two bus stops 1.5 km apart. It starts from rest and accelerates uniformly to a speed of 25 m/s, then maintains that speed until the brakes are applied to decelerate. If the time taken for acceleration is the same as deceleration, find the acceleration.

14 A particle travels in a straight line with a constant velocity of 40 m/s for 10 seconds. It is then subjected to a constant acceleration in the opposite direction for 20 seconds, which returns the particle to its original position. Sketch a velocity–time graph to represent the motion of the particle.

15 During a fireworks display, a skyrocket accelerates from rest to 30 m/s after 8 seconds. It is then subjected to a constant acceleration in the opposite direction for 4 seconds. It reaches its maximum height after 10 seconds.
   a Sketch a velocity–time graph to represent the motion of the skyrocket.
   b Find:
      i the maximum height the skyrocket will reach
      ii the time at which the skyrocket will hit the ground on its return.

16 At time \( t = 0 \), Jo is cycling on her bike at a speed of 6.5 m/s along a straight bicycle path and passes her friend Christina, who is stationary on her own bike.
Four seconds later, Christina accelerates in the direction of Jo for 8 seconds so that her speed, \( v \) m/s, is given by \( v = (t - 4)\tan\left(\frac{\pi}{48}\right) \). Christina then maintains her speed of 8 m/s.

a Show that Christina accelerates to a speed of 8 m/s.

b i On the velocity–time graph shown, draw the speed of Christina.

ii Find the time at which Christina passes Jo on her bike.

### Variable acceleration

There are many different types of motion in which acceleration is not constant. If we plot the velocity of an object against time, then the acceleration, \( \frac{dv}{dt} \), can be estimated by drawing the tangent to the graph at that time and finding the slope of the tangent. Alternatively, we can use calculus.

We know that instantaneous acceleration at time \( t \) can be found by \( a = \frac{dv}{dt} \). We also know that the distance covered between two time intervals is the area under a velocity–time graph, which can be calculated by the integral \( x = \int_{t_1}^{t_2} v \, dt \).

Furthermore, if we start with displacement, we know that velocity is the rate at which displacement varies and acceleration is the rate at which velocity varies, so we can write \( v = \frac{ds}{dt} \) and \( a = \frac{dv}{dt} \).

If we start with acceleration of a moving object as a function of time, velocity can be found by integrating \( a \) with respect to \( t \), and displacement can be found by integrating \( v \) with respect to \( t \). Therefore, we can write

\[
\begin{align*}
  v &= \int a \, dt = \int \frac{dv}{dt} \, dt \\
  s &= \int v \, dt = \int \frac{ds}{dt} \, dt.
\end{align*}
\]

In summary:
WORKED EXAMPLE 7

The motion of an object along a straight line is modelled by the equation \( v = -t(t - 5) \) where \( v \) m/s is the velocity and time is \( t \) seconds. What is the acceleration at \( t = 3 \)?

THINK

1. Draw a velocity–time graph.

WRITE/DRAW

2. If we draw a tangent at \( t = 3 \), we can approximate the acceleration.

   We can see the acceleration is negative and approximately \(-1 \) m/s\(^2\) by using the coordinates (3, 6) and (7, 2) to calculate the slope.

3. To calculate the instantaneous acceleration at time \( t \), we need to use \( a = \frac{dv}{dt} \).

4. To calculate the acceleration at \( t = 3 \), substitute into \( a = -2t + 5 \).

\[
\begin{align*}
  v &= -t(t - 5) \\
  v &= -t^2 + 5t \\
  \frac{dv}{dt} &= -2t + 5 \\
  a &= -2t + 5 \\
  a &= -2 \times 3 + 5 \\
  a &= -6 + 5 \\
  a &= -1 \\

\text{The acceleration at } t = 3 \text{ for the motion of an object modelled by } v = -t(t - 5) \text{ is } -1 \text{ m/s}^2.
\]
A dog starts from rest to go on a walk. Its acceleration can be modelled by the equation \( a = 3t - t^2 \), where the units are metres and seconds. What is the distance the dog has travelled in the first 5 seconds of its walk?

**THINK**

1. Given \( a = 3t - t^2 \), velocity can be found by integrating \( a \) with respect to \( t \).

\[
\int a\,dt = \int (3t - t^2)\,dt
\]

\[
v = \frac{3}{2}t^2 - \frac{1}{3}t^3 + c
\]

When \( t = 0 \), \( v = 0 \), so \( c = 0 \).

\[
\therefore v = \frac{3}{2}t^2 - \frac{1}{3}t^3
\]

2. Draw a velocity–time graph of \( v = \frac{3}{2}t^2 - \frac{1}{3}t^3 \), representing the motion of the dog.

3. The distance travelled is the area between the curve and the \( t \)-axis on a velocity–time graph.
Acceleration can be expressed in four different ways.

As stated earlier, acceleration is the rate at which velocity varies as a function of time. When acceleration is given as a function of time, \( a = f(t) \), we use the expression \( a = \frac{dv}{dt} \).

Knowing that \( a = \frac{dv}{dt} \) and \( v = \frac{dx}{dt} \), we have \( a = \frac{d}{dt} \left( \frac{dx}{dt} \right) \), which gives another expression, \( a = \frac{d^2x}{dt^2} \).

When acceleration is written as a function of displacement, \( a = f(x) \), we know that \( a = \frac{dv}{dx} \). Using the chain rule, we get \( \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} \).

Since \( v = \frac{dx}{dt} \), we have \( a = \frac{dv}{dx} \times v \), which simplifies to the expression \( a = v \frac{dv}{dx} \).

We use this expression when a relationship between velocity and displacement is required.

Knowing that \( a = \frac{dv}{dx} \times v \) and \( v = \frac{d}{dv} \left( \frac{1}{2} v^2 \right) \), we can derive \( a = \frac{dv}{dx} \times \frac{d}{dv} \left( \frac{1}{2} v^2 \right) \), which simplifies to the expression \( a = \frac{d}{dx} \left( \frac{1}{2} v^2 \right) \). We use this expression when acceleration is given a function of displacement.

So acceleration can be expressed in any of these forms:

\[ a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v \frac{dv}{dx} = \frac{d}{dx} \left( \frac{1}{2} v^2 \right) \]

The form to use in a particular mathematical problem will depend on the form of the equation defining acceleration.

### Calculations

When \( v = 0 \),

\[ \frac{3}{2} t^2 - \frac{1}{3} t^3 = 0 \]

\[ t^2 \left( \frac{3}{2} - \frac{1}{3} t \right) = 0 \]

\[ t = 0 \text{ or } t = \frac{9}{2} \]

\[ s = \int_{0}^{4.5} \frac{3}{2} t^2 - \frac{1}{3} t^3 \, dt - \int_{4.5}^{5} \frac{3}{2} t^2 - \frac{1}{3} t^3 \, dt \]

\[ = \left[ \frac{t^3}{2} - \frac{t^4}{12} \right]_{0}^{4.5} - \left[ \frac{t^3}{2} - \frac{t^4}{12} \right]_{4.5}^{5} \]

\[ = 11.39 + 0.97 \]

\[ = 12.36 \]

The distance travelled by the dog in the first 5 seconds is 12.36 m.
• If \( a = f(t) \), then use \( \frac{dv}{dt} \) or \( \frac{d^2x}{dt^2} \).

• If \( a = f(x) \), then use \( \frac{d}{dx}\left(\frac{1}{2}v^2\right) \).

• If \( a = f(v) \), then use \( \frac{dv}{dt} \) if the initial conditions are given in terms of \( t \) and \( v \), or \( a = v \frac{dv}{dx} \) if the initial conditions relate to \( v \) and \( x \).

WORKED EXAMPLE 9

A particle moves in a straight line. When the particle’s displacement from a fixed origin is \( x \) m, its velocity is \( v \) m/s and its acceleration is \( a \) m/s\(^2\). Given that \( a = 4x \) and that \( v = -3 \) when \( x = 0 \), find \( v \) in terms of \( x \).

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Decide which acceleration form to use to find the relationship between ( v ) and ( x ). ( a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) )</td>
<td>( a = 4x ) ( \frac{dv}{dx} = 4x )</td>
</tr>
<tr>
<td>2 Solve for ( v ) in terms of ( x ) by using integration. ( \int\left(\frac{dv}{dx}\right)dx = \int (4x),dx ) ( \int (v),dv = \int (4x),dx )</td>
<td>( \frac{1}{2}v^2 = 2x^2 + c ) ( v^2 = 4x^2 + d ) ( v = \pm\sqrt{4x^2 + d} )</td>
</tr>
<tr>
<td>3 Calculate the constant by using the initial conditions. ( v = -3 ) and ( x = 0 ), ( v = \pm\sqrt{4x^2 + d} ) ( -3 = -\sqrt{d} ) ( d = 9 )</td>
<td>The relationship between ( v ) and ( x ) is given by ( v = -\sqrt{4x^2 + 9} ).</td>
</tr>
<tr>
<td>4 Determine the relationship between ( v ) and ( x ).</td>
<td></td>
</tr>
</tbody>
</table>

EXERCISE 4.5 Variable acceleration

Throughout this exercise, units are metres and seconds unless otherwise stated.

1 WE7 The motion of an object along a straight line is modelled by the equation \( v = -t(t - 6) \), where \( v \) m/s is the velocity and time is \( t \) seconds. What is the acceleration at \( t = 2 \)?

2 The motion of an object along a straight line is modelled by the equation \( v = -\frac{1}{2}t^2 - 4t \), where \( v \) m/s is the velocity and time is \( t \) seconds. What is the acceleration at \( t = 7 \)?

3 WE5 An object starting from rest accelerates according to the equation \( a = 4 - t \), where the units are metres and seconds. What is the distance the object has travelled in the first 3 seconds?
4 A sprinter starting from rest accelerates according to the equation \( a = 3t - t^2 \), where the units are metres and seconds. What is the distance the sprinter has travelled in the first 2 seconds?

5 A particle moves in a straight line. When the particle’s displacement from a fixed origin is \( x \) m, its velocity is \( v \) m/s and its acceleration is \( a \) m/s\(^2\). Given that \( a = 16x \) and that \( v = 4 \) when \( x = 0 \), find the relationship between \( v \) and \( x \).

6 A particle moves in a straight line. When the particle’s displacement from a fixed origin is \( x \) m, its velocity is \( v \) m/s and its acceleration is \( a \) m/s\(^2\). Given that \( a = 2v^3 \) and that \( v = 2 \) when \( x = 0 \), find the relationship between \( x \) and \( v \).

7 The motion of an object along a straight line is modelled by the equation \( v = -t(t - 6) \), where \( v \) is the velocity and \( t \) is the time. What is the acceleration at \( t = 3.5 \)?

8 The motion of an object along a straight line is modelled by the equation \( v = 10\left(\frac{1}{2}t^2 + e^{-0.1t}\right) \), where \( v \) is the velocity and \( t \) is the time. What is the acceleration at \( t = 2 \)?

9 A jet plane starting from rest accelerates according to the equation \( a = 30 - 2t \). What is the distance the jet has travelled in the first 15 seconds?

10 The diagram below shows the motion of an object along a straight line.

![Graph showing motion of an object]

a When is the object at rest?

b When is the acceleration equal to zero?

c Approximate the acceleration at \( t = 1.5 \), correct to 1 decimal place.

11 A golf ball is putted on the green with an initial velocity of 6 m/s and decelerates uniformly at a rate of 2 m/s\(^2\). If the hole is 12 m away, will the golf ball reach it?
12 An object travels in a line so that the velocity, $v$ m/s, is given by $v^2 = 10 - 2x^2$. Find the acceleration at $x = 2$.

13 The velocity, $v$ m/s, and the acceleration, $a$ m/s$^2$, of a particle at time $t$ seconds after the particle is dropped from rest are given by $a = \frac{1}{50}(490 - v)$, $0 \leq v < 490$. Express $v$ in terms of $t$.

14 If $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 3x^2$ and $v = 2$ when $x = 0$, find $v$ in terms of $x$.

15 A particle moves in a straight line. At time $t$ its displacement from a fixed origin is $x$.
   a If $v = 2x + 5$, find $a$ in terms of $x$.
   b If $v = \frac{(x - 1)^2}{2}$, and $x = 0$ when $t = 0$, find $x$ when $t = 2$.
   c If $a = \frac{1}{(x + 4)^2}$, and $v = 0$ when $x = 0$, find $x$ when $v = \frac{1}{2}$.
   d If $a = 3 - v$, and $v = 0$ when $x = 0$, find $x$ when $v = 2$.

16 A truck travelling at 20 m/s passes a stationary speed camera and then decelerates so that its velocity, $v$ m/s, at time $t$ seconds after passing the speed camera is given by $v = 20 - 2 \tan^{-1}(t)$.
   a After how many seconds will the truck's speed be 17 m/s?
   b Explain why $v$ will never equal 16.
   c i Write a definite integral that gives the distance, $x$ metres, travelled by the truck after $T$ seconds.
      ii Find the distance travelled by the truck at $t = 8$.

17 An object falls from a hovering surf-lifesaving helicopter over Port Phillip Bay at 500 m above sea level. Find the velocity of the object when it hits the water when the acceleration of the object is $0.2v^2 - g$.

18 a A particle moves from rest at the origin, O, with an acceleration of $v^3 + x^2v$ m/s$^2$, where $v$ is the particle’s velocity measured in m/s. Find the velocity of the particle when it is 0.75 m to the right of O.
   
   b A particle travels in a straight line with velocity $v$ m/s at time $t$ s. The acceleration of the particle, $a$ m/s$^2$, is given by $a = -2 + \sqrt{v^2 + 5}$. Find the time it takes for the speed of the particle to increase from 2 m/s to 8 m/s.
studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS
Download the Review questions document from the links found in the Resources section of your eBookPLUS.
4 Answers

EXERCISE 4.2
1 525 m
2 1.5 m/s²
3 −0.32 m/s²
4 61.5 s
5 500 m
6 a −0.94 m/s²  b 21.33 s  c 13.33 m
7 a 75 m  b 1.5 m/s²
8 17.32 s
9 a 12 m/s  b 36 m
10 2500 m
11 Yes; 472.5 m to rest
12 a 3 1/3 m/s²  b 3 2/3 m/s
13 5.64 s
14 2.83 m/s²
15 a 1.5 m/s²  b 10.4 m/s
16 a −3.1 m/s²  b 37.95 m/s

EXERCISE 4.3
1 1.28 m
2 55 m/s
3 50.176 m
4 5 s
5 0.816 s
6 125.22 m/s
7 29.7 m/s
8 4.06 s
9 172.96 m
10 a 217.6 m  b 34.8 m/s downward
11 46.4 m
12 6 s after missile 2 is projected; 264.6 m above the point of projection
13 55.37 m; 4.78 s after flare A is projected
14 7.122 s; 178.73 m above point
15 7.022 s after B is dropped; 241.66 m below the point of projection
16 a 32.14 m  b 2.56 s

EXERCISE 4.4
1 412.5 m
2 3 m/s²; 400 m
3 15 min
4 491 2/3 s or 8.19 min
5 500 m
6 225 m
7 480 m
8 1880 m
9 2.3245 m/s²
10 18.95 min or 1137.14 s
11 2.4 km
12 21.3158 s
13 0.625 m/s²
14 0
15 a 0
16 a 8 m/s

v (m/s) t (s)
−10 −20 −30 −40 −50 −60 −70 −80 −90 −100 −110
10 20 30 40 50 60 70 80 90 100
−120 −140
10 20 30 40 50 60
16 a 8 m/s

v (m/s) t (s)
0 2 4 6 8 10 12 14 16 18
0 2 4 6 8 10 12 14 16 18
Cristina Jo
ii 48.74 s
EXERCISE 4.5

1 2 m/s²
2 –11 m/s²
3 13.5 m
4 \( \frac{2}{3} \) m
5 \( v = 4\sqrt{x^2 + 1} \)
6 \( x = \frac{1}{2v} + \frac{1}{4} \)
7 –1 m/s²
8 19.18 m/s²
9 2250 m
10 a 0.5 s, 2 s, 3 s
   b 1 s and 2.5 s
   c Approximately –3 m/s²
11 No, only travels 9 m
12 –4 m/s²

13 \( v = 490 \left( 1 - e^{-\frac{t}{50}} \right) \)
14 \( v^2 = -2x^3 + 2x^2 + 4 \)
15 a \( a = 4x + 10 \)
   b \( \frac{1}{2} \) m
   c 4 m
   d 1.296 m or 3 log₃ 3 – 2 m
16 a 14.1 s
   b As \( t \to \infty \), \( v \to 20 - \pi \)
   c i \( \int_{0}^{T} 20 - 2 \tan^{-1}(t) \, dt \)
   ii 141.03 m
17 7 m/s
18 a –3.142 m/s
   b 2.19 s