5

Recurrence relations

5.1 Kick off with CAS
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5.1 Kick off with CAS
Generating terms of sequences with CAS

First-order recurrence relations relate a term in a sequence to the previous term in the same sequence. You can use recurrence relations to generate all of the terms in a sequence, given a starting (or initial) value.

1 Use CAS to generate a table of the first 10 terms of a sequence if the starting value is 7 and each term is formed by adding 3 to the previous term.

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<th>Term</th>
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2 Use CAS to generate the first 10 terms of a sequence if the starting value is 14 and each term is formed by subtracting 5 from the previous term.

<table>
<thead>
<tr>
<th>Term</th>
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<tbody>
<tr>
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3 Use CAS to generate the first 10 terms of a sequence where the starting value is 1.5 and each term is formed by multiplying the previous term by 2.

4 Use CAS to generate the first 10 terms of a sequence where the starting value is 4374 and each term is formed by dividing the previous term by 3.
Generating the terms of a first-order recurrence relation

A first-order recurrence relation relates a term in a sequence to the previous term in the same sequence, which means that we only need an initial value to be able to generate all remaining terms of a sequence.

In a recurrence relation the $n$th term is represented by $u_n$, with the next term after $u_n$ being represented by $u_{n+1}$, and the term directly before $u_n$ being represented by $u_{n-1}$.

The initial value of the sequence is represented by either $u_0$ or $u_1$.

We can define the sequence $1, 5, 9, 13, 17, \ldots$ with the following equation:

$$u_{n+1} = u_n + 4 \quad u_1 = 1.$$  

This expression is read as ‘the next term is the previous term plus 4, starting at 1’.

Or, transposing the above equation, we get:

$$u_{n+1} - u_n = 4 \quad u_1 = 1.$$  

A first-order recurrence relation defines a relationship between two successive terms of a sequence, for example, between:

- $u_n$, the previous term, and $u_{n+1}$, the next term.
- $u_{n-1}$, the previous term, and $u_n$, the next term.

The first term can be represented by either $u_0$ or $u_1$.

Throughout this topic we will use either notation format as short hand for next term, previous term and first term.

WORKED EXAMPLE 1

The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?

- a $u_n = u_{n-1} + 2 \quad u_1 = 3 \quad n = 1, 2, 3, \ldots$
- b $u_n = 4 + 2n \quad n = 1, 2, 3, \ldots$
- c $f_{n+1} = 3f_n \quad n = 1, 2, 3, \ldots$

THINK

- a The equation contains the consecutive terms $u_n$ and $u_{n-1}$ to describe a pattern with a known term.
- b The equation contains only the $u_n$ term. There is no $u_{n+1}$ or $u_{n-1}$ term.
- c The equation contains the consecutive terms $f_n$ and $f_{n+1}$ to describe a pattern but has no known first or starting term.

WRITE

- a This is a first-order recurrence relation. It has the pattern $u_n = u_{n-1} + 2$ and a starting or first term of $u_1 = 3$.
- b This is not a first-order recurrence relation because it does not describe the relationship between two consecutive terms.
- c This is an incomplete first-order recurrence relation. It has no first or starting term, so a sequence cannot be commenced.
Given a fully defined first-order recurrence relation (pattern and a known term) we can generate the other terms of the sequence.

**Starting term**

Earlier, it was stated that a starting term was required to fully define a sequence. As can be seen below, the same pattern with a different starting point gives a different set of numbers.

\[
\begin{align*}
    u_{n+1} &= u_n + 2 & u_1 &= 3 & \text{gives } 3, 5, 7, 9, 11, \ldots \\
    u_{n+1} &= u_n + 2 & u_1 &= 2 & \text{gives } 2, 4, 6, 8, 10, \ldots 
\end{align*}
\]

**WORKED EXAMPLE 2**

Write the first five terms of the sequence defined by the first-order recurrence relation:

\[ u_n = 3u_{n-1} + 5 \quad u_0 = 2. \]

**THINK**

1. Since we know the \( u_0 \) or starting term, we can generate the next term, \( u_1 \), using the pattern: The next term is \( 3 \times \) the previous term + 5.

2. Now we can continue generating the next term, \( u_2 \), and so on.

**WRITE**

\[
\begin{align*}
    u_n &= 3u_{n-1} + 5 \\
    u_1 &= 3u_0 + 5 \\
    &= 3 \times 2 + 5 \\
    &= 11 \\
    u_2 &= 3u_1 + 5 \\
    &= 3 \times 11 + 5 \\
    &= 38 \\
    u_3 &= 3u_2 + 5 \\
    &= 3 \times 38 + 5 \\
    &= 119 \\
    u_4 &= 3u_3 + 5 \\
    &= 3 \times 119 + 5 \\
    &= 362 \\
\end{align*}
\]

3. Write your answer. The sequence is 2, 11, 38, 119, 362.

**WORKED EXAMPLE 3**

A sequence is defined by the first-order recurrence relation:

\[ u_{n+1} = 2u_n - 3 \quad n = 1, 2, 3, \ldots \]

If the fourth term of the sequence is \(-29\), that is, \( u_4 = -29 \), then what is the second term?

**THINK**

1. Transpose the equation to make the previous term, \( u_n \), the subject.

2. Use \( u_4 \) to find \( u_3 \) by substituting into the transposed equation.

**WRITE**

\[
\begin{align*}
    u_n &= \frac{u_{n+1} + 3}{2} \\
    u_3 &= \frac{u_4 + 3}{2} \\
    &= \frac{-29 + 3}{2} \\
    &= -13
\end{align*}
\]

**Topic 5 RECURRENCE RELATIONS 223**
3 Use $u_3$ to find $u_2$. 

\[
\begin{align*}
    u_2 &= \frac{u_3 + 3}{2} \\
    &= \frac{-13 + 3}{2} \\
    &= -5
\end{align*}
\]

4 Write your answer. The second term, $u_2$, is $-5$.

---

**EXERCISE 5.2 Generating the terms of a first-order recurrence relation**

1. **WE1** The following equations each define a sequence. Which of them are first-order recurrence relations (defining a relationship between two consecutive terms)?
   - a) $u_n = u_{n-1} + 6 \quad u_1 = 7 \quad n = 1, 2, 3, \ldots$
   - b) $u_n = 5n + 1 \quad n = 1, 2, 3, \ldots$

2. The following equations each define a sequence. Which of them are first-order recurrence relations?
   - a) $u_n = 4u_{n-1} - 3 \quad n = 1, 2, 3, \ldots$
   - b) $f_{n+1} = 5f_n - 8 \quad f_1 = 0 \quad n = 1, 2, 3, \ldots$

3. **WE2** Write the first five terms of the sequence defined by the first-order recurrence relation:
   \[ u_n = 4u_{n-1} + 3 \quad u_0 = 5. \]

4. Write the first five terms of the sequence defined by the first-order recurrence relation:
   \[ f_{n+1} = 5f_n - 6 \quad f_0 = -2. \]

5. **WE3** A sequence is defined by the first-order recurrence relation:
   \[ u_{n+1} = 2u_n - 1 \quad n = 1, 2, 3, \ldots \]
   If the fourth term of the sequence is 5, that is, $u_4 = 5$, then what is the second term?

6. A sequence is defined by the first-order recurrence relation:
   \[ u_{n+1} = 3u_n + 7 \quad n = 1, 2, 3, \ldots \]
   If the seventh term of the sequence is 34, that is, $u_7 = 34$, then what is the fifth term?

7. Which of the following equations are complete first-order recurrence relation?
   - a) $u_n = 2 + n$
   - b) $u_n = u_{n-1} - 1 \quad u_0 = 2$
   - c) $u_n = 1 - 3u_{n-1} \quad u_0 = 2$
   - d) $u_n = 4u_{n-1} - 3$
   - e) $u_n = -u_{n-1}$
   - f) $u_n = n + 1 \quad u_1 = 2$
   - g) $u_n = 1 - u_{n-1} \quad u_0 = 21$
   - h) $u_n = a^{n-1} \quad u_0 = 2$
   - i) $f_{n+1} = 3f_n - 1$
   - j) $p_n = p_{n-1} + 7 \quad u_0 = 7$
8 Write the first five terms of each of the following sequences.
   a \( u_n = u_{n-1} + 2 \) \( u_0 = 6 \)  
   b \( u_n = u_{n-1} - 3 \) \( u_0 = 5 \)  
   c \( u_n = 1 + u_{n-1} \) \( u_0 = 23 \)  
   d \( u_{n+1} = u_n - 10 \) \( u_1 = 7 \)  
9 Write the first five terms of each of the following sequences.
   a \( u_n = 3u_{n-1} \) \( u_0 = 1 \)  
   b \( u_n = 5u_{n-1} \) \( u_0 = -2 \)  
   c \( u_n = -4u_{n-1} \) \( u_0 = 1 \)  
   d \( u_{n+1} = 2u_n \) \( u_1 = -1 \)  
10 Write the first five terms of each of the following sequences.
   a \( u_n = 2u_{n-1} + 1 \) \( u_0 = 1 \)  
   b \( u_n = 3u_{n-1} - 2 \) \( u_1 = 5 \)  
11 Write the first seven terms of each of the following sequences.
   a \( u_n = -u_{n-1} + 1 \) \( u_0 = 6 \)  
   b \( u_{n+1} = 5u_n \) \( u_1 = 1 \)  
12 Which of the sequences is generated by the following first-order recurrence relation?
   \( u_n = 3u_{n-1} + 4 \) \( u_0 = 2 \)  
   A 2, 3, 4, 5, 6, …  
   B 2, 6, 10, 14, 18, …  
   C 2, 10, 34, 106, 322, …  
   D 2, 11, 47, 191, 767, …  
   E 6, 10, 14, 18, 22, …  
13 Which of the sequences is generated by the following first-order recurrence relation?
   \( u_{n+1} = 2u_{n-1} \) \( u_1 = -3 \)  
   A \( -3, 5, 9, 17, 33, … \)  
   B \( -3, -5, -9, -17, -33, … \)  
   C \( -3, 5, -3, 5, -3, \ldots \)  
   D \( -3, -8, -14, -26, -54, \ldots \)  
   E \( -3, -7, -15, -31, -63, \ldots \)  
14 A sequence is defined by the first-order recurrence relation:
   \( u_{n+1} = 3u_n + 1 \) \( n = 1, 2, 3, \ldots \)  
   If the fourth term is 67 (that is, \( u_4 = 67 \)), what is the second term?  
15 A sequence is defined by the first-order recurrence relation:
   \( u_{n+1} = 4u_n - 5 \) \( n = 1, 2, 3, \ldots \)  
   If the third term is -41 (that is, \( u_3 = -41 \)), what is the first term?  
16 For the sequence defined in question 15, if the seventh term is -27, what is the fifth term?  
17 A sequence is defined by the first-order recurrence relation:
   \( u_{n+1} = 5u_n - 10 \) \( n = 1, 2, 3, \ldots \)  
   If the third term is -10, the first term is:
   A \( \frac{-14}{6} \)  
   B \( \frac{5}{6} \)  
   C \( 0 \)  
   D \( \frac{2}{3} \)  
   E 4  
18 Write the first-order recurrence relations for the following descriptions of a sequence and generate the first five terms of the sequence.
   a The next term is 3 times the previous term, starting at \( \frac{1}{4} \).
   b Next year’s attendance at a motor show is 2000 more than the previous year’s attendance, with a first year attendance of 200000.
   c The next term is the previous term less 7, starting at 100.
   d The next day’s total sum is double the previous day’s sum less 50, with a first day sum of $200.
5.3 First-order linear recurrence relations

First-order linear recurrence relations with a common difference

Consider the sequence 3, 7, 11, 15, 19, ...

The common difference, \( d \), is the value between consecutive terms in the sequence:

\[
d = u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots
\]

\[
d = 7 - 3 = 11 - 7 = 15 - 11 = +4
\]

The common difference is +4.

This sequence may be defined by the first-order linear recurrence relation:

\[
u_{n+1} - u_n = 4 \quad u_1 = 3.
\]

Rewriting this, we obtain:

\[
u_{n+1} = u_n + 4 \quad u_1 = 3
\]

A sequence with a common difference of \( d \) may be defined by a first-order linear recurrence relation of the form:

\[
u_{n+1} = u_n + d \quad \text{(or } u_{n+1} - u_n = d\text{)}
\]

where \( d \) is the common difference and for

\( d > 0 \) it is an increasing sequence

\( d < 0 \) it is a decreasing sequence.

WORKED EXAMPLE 4

Express each of the following sequences as first-order recurrence relations.

a 7, 12, 17, 22, 27, ...

b 9, 3, -3, -9, -15, ...

THINK

a 1 Write the sequence.

2 Check for a common difference.

3 There is a common difference of 5 and the first term is 7.

b 1 Write the sequence.

2 Check for a common difference.

3 There is a common difference of -6 and the first term is 9.

WRITE

a 7, 12, 17, 22, 27, ...

\[
d = u_4 - u_3 = u_3 - u_2 = u_2 - u_1 \]

\[
= 22 - 17 = 17 - 12 = 12 - 7
\]

\[
= 5 \quad = 5 \quad = 5
\]

The first-order recurrence relation is given by:

\[
u_{n+1} = u_n + d \quad u_{n+1} = u_n + 5 \quad u_1 = 7
\]

b 9, 3, -3, -9, -15, ...

\[
d = u_4 - u_3 = u_3 - u_2 = u_2 - u_1 \]

\[
= -9 - (-3) = -3 - 3 = 3 - 9
\]

\[
= -6 = -6 = -6
\]

The first-order recurrence relation is given by:

\[
u_{n+1} = u_n - 6 \quad u_1 = 9
\]
Express the following sequence as a first-order recurrence relation.

\[ u_n = -3n - 2 \quad n = 1, 2, 3, 4, 5, \ldots \]

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
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</table>
| 1 Generate the sequence using the given rule. | \[ n = 1, 2, 3, 4, 5, \ldots \]
| | \[ u_n = -3n - 2 \] |
| | \[ n = 1 \quad u_1 = -3 \times 1 - 2 \]
| | \[ = -3 - 2 \]
| | \[ = -5 \] |
| | \[ n = 2 \quad u_2 = -3 \times 2 - 2 \]
| | \[ = -6 - 2 \]
| | \[ = -8 \] |
| | \[ n = 3 \quad u_3 = -3 \times 3 - 2 \]
| | \[ = -9 - 2 \]
| | \[ = -11 \] |
| | \[ n = 4 \quad u_4 = -3 \times 4 - 2 \]
| | \[ = -12 - 2 \]
| | \[ = -14 \] |
| | The sequence is \(-5, -8, -11, -14, \ldots\) |
| 2 There is a common difference of \(-3\) and the first term is \(-5\). | Write the first-order recurrence relation. |
| | The first order difference equation is: \[ u_{n+1} = u_n + 3 \quad u_1 = -5 \] |

**First-order linear recurrence relations with a common ratio**

Not all sequences have a common difference (i.e. a sequence increasing or decreasing by adding or subtracting the same number to find your next term). You might also find that a sequence increases or decreases by multiplying the terms by a common ratio. Consider the geometric \(1, 3, 9, 27, 81, \ldots\) The common ratio is found by:

\[
R = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots
\]

\[
R = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \ldots
\]

\[ = 3 \]

The common ratio is 3.

This sequence may be defined by the first-order linear recurrence relation:

\[ u_{n+1} = 3u_n \quad u_1 = 1 \]

A sequence with a common ratio of \(R\) may be defined by a first-order linear recurrence relation of the form:

\[ u_{n+1} = Ru_n \]

where \(R\) is the common ratio:

\( R > 1 \) is an increasing sequence
\( 0 < R < 1 \) is a decreasing sequence
\( R < 0 \) is a sequence alternating between positive and negative values.
WORKED EXAMPLE 6

Express each of the following sequences as first-order recurrence relations.

a 1, 5, 25, 125, 625, …

b 3, −6, 12, −24, 48 …

THINK

a 1 There is a common ratio of 5 and the first term is 1.

2 Write the first-order recurrence relation.

WRITE

a \( R = \frac{5}{1} = \frac{25}{5} = \frac{125}{25} = \ldots \)

\( u_1 = 1 \)

The first-order recurrence relation is given by:

\( u_{n+1} = 5u_n \quad u_1 = 1 \)

b 1 There is a common ratio of −2 and the first term is 3.

2 Write the first-order recurrence relation.

WRITE

b \( R = \frac{-6}{3} = \frac{12}{-6} = \frac{-24}{12} = \ldots \)

\( u_1 = 3 \)

The first-order recurrence relation is given by:

\( u_{n+1} = -2u_n \quad u_1 = 3 \)

WORKED EXAMPLE 7

Express each of the following sequences as first-order recurrence relations.

a \( u_n = 2(7)^{n-1} \quad n = 1, 2, 3, 4, \ldots \)

b \( u_n = -3(2)^{n-1} \quad n = 1, 2, 3, 4, \ldots \)

THINK

a 1 Generate the sequence using the given rule.

WRITE

\( u_n = 2(7)^{n-1} \quad n = 1, 2, 3, 4, \ldots \)

\( u_1 = 2(7)^{1-1} \)

\( = 2 \times 7^0 \)

\( = 2 \times 1 \)

\( = 2 \)

\( n = 2 \)

\( u_1 = 2(7)^{2-1} \)

\( = 2 \times 7^1 \)

\( = 2 \times 7 \)

\( = 14 \)

\( n = 3 \)

\( u_1 = 2(7)^{3-1} \)

\( = 2 \times 7^2 \)

\( = 2 \times 49 \)

\( = 98 \)

\( n = 4 \)

\( u_1 = 2(7)^{4-1} \)

\( = 2 \times 7^3 \)

\( = 2 \times 343 \)

\( = 686 \)

2 There is a common ratio of 7 and the first term is 2.

\( R = 7, u_1 = 2 \)
3 Write the first-order recurrence relation.

The first-order recurrence relation is given by:

\[ u_{n+1} = 7u_n \quad u_1 = 2 \]

b 1 Generate the sequence using the given rule.

\[ b \quad n = 1, 2, 3, 4, \ldots \quad u_n = -3(2)^{n-1} \]

\[ u_1 = -3(2)^{1-1} = -3 \times 2^0 = -3 \]

\[ u_2 = -3(2)^{2-1} = -3 \times 2^1 = -6 \]

\[ u_3 = -3(2)^{3-1} = -3 \times 2^2 = -12 \]

\[ u_4 = -3(2)^{4-1} = -3 \times 2^3 = -24 \]

2 There is a common ratio of 2 and the first term is -3.

3 Write the first-order recurrence relation.

The first-order recurrence relation is given by:

\[ u_{n+1} = 2u_n \quad u_1 = -3 \]

---

**EXERCISE 5.3 First-order linear recurrence relations**

1 **WE4** Express each of the following sequences as first-order recurrence relations.

   a 1, 4, 7, 10, 13, …

   b 12, 8, 4, 0, -4, …

2 Express each of the following sequences as first-order recurrence relations.

   a -24, -19, -14, -9, -4, …

   b 55, 66, 77, 88, 99, …

3 **WE5** Express the following sequence as a first-order recurrence relation.

\[ u_n = -5n - 3 \quad n = 1, 2, 3, 4, 5, \ldots \]

4 Express the sequence as a first-order recurrence relation.

\[ u_n = \frac{1}{2}n + 4 \quad n = 1, 2, 3, 4, 5, \ldots \]

5 **WE6** Express each of the following sequences as first-order recurrence relations.

   a 2, 10, 40, 80, …

   b 4, -8, 16, -32, 64, …

6 Express each of the following sequences as first-order recurrence relations.

   a 3, 15, 45, 225, 1125, …

   b 200, -100, 50, -25, 12.5, …

7 **WE7** Express each of the following sequences as first-order recurrence relations.

   a \[ u_n = 2(5)^{n-1} \quad n = 1, 2, 3, 4, \ldots \]

   b \[ u_n = -3(4)^{n-1} \quad n = 1, 2, 3, 4, \ldots \]
8 Express each of the following sequences as first-order recurrence relations.
   a \( u_n = 3(2)^{n-1} \) \( n = 1, 2, 3, 4, \ldots \)  
   b \( u_n = -2(3)^{n-1} \) \( n = 1, 2, 3, 4, \ldots \)

9 Express each of the following sequences as first-order recurrence relations.
   a 1, 3, 5, 7, 9, …  
   b 3, 10, 17, 24, 31, …  
   c 12, 5, -2, -9, -16, …  
   d 1, 0.5, 0, -0.5, -1, …  
   e 2, 6, 10, 14, 18, …  
   f -2, 2, 6, 10, 14, …  
   g 6, 1, -4, -9, -14, …  
   h 4, 10.5, 17, 23.5, 30, …

10 The sequence -6, -3, 0, 3, 6, … can be defined by the first-order recurrence relation:
   A \( u_{n+1} = u_n - 3 \) \( u_0 = -6 \)  
   B \( u_{n+1} = u_n + 3 \) \( u_1 = -6 \)  
   C \( u_{n+1} = 3u_n \) \( u_1 = -3 \)  
   D \( u_{n+1} = 3u_n - 1 \) \( u_0 = -3 \)  
   E \( u_{n+1} = 3u_n \) \( u_1 = 3 \)

11 Express each of the following sequences as first-order recurrence relations.
   a \( u_n = -n - 3 \) \( n = 1, 2, 3, \ldots \)  
   b \( u_n = 2n + 1 \) \( n = 1, 2, 3, \ldots \)  
   c \( u_n = 3n - 4 \) \( n = 1, 2, 3, \ldots \)  
   d \( u_n = -2n + 6 \) \( n = 1, 2, 3, \ldots \)

12 The sequence defined by \( u_n = -2n + 3 \), \( n = 1, 2, 3, \ldots \), can be defined by the first-order recurrence relation:
   A \( u_{n+1} = u_n - 2 \) \( u_1 = 1 \)  
   B \( u_{n+1} = -2u_n \) \( u_1 = 3 \)  
   C \( u_{n+1} = -2u_n \) \( u_1 = 1 \)  
   D \( u_{n+1} = u_n + 3 \) \( u_1 = 1 \)  
   E \( u_{n+1} = -2u_n + 3 \) \( u_1 = 1 \)

13 Express each of the following sequences as a first-order recurrence relation.
   a 12, 10, 8, 6, 4, …  
   b 1, 2, 3, 4, 5, 6, …

14 Express each of the following sequences as first-order recurrence relations.
   a 5, 10, 20, 40, 80, …  
   b 1, 6, 36, 216, 1296, …  
   c -3, 3, -3, 3, -3, …  
   d -3, -12, -48, -192, -768, …  
   e 2, 6, 18, 54, 162, …  
   f 5, -5, 5, -5, 5, …  
   g -2, -8, -32, -128, -512, …  
   h 5, -15, 45, -135, 405, …

15 The sequence -2, 6, -18, 54, -162, … can be defined by the first-order recurrence relation:
   A \( u_{n+1} = -2u_n \) \( u_1 = -2 \)  
   B \( u_{n+1} = -2u_n \) \( u_1 = 3 \)  
   C \( u_{n+1} = -3u_n \) \( u_1 = 3 \)  
   D \( u_{n+1} = -3u_n \) \( u_1 = -2 \)  
   E \( u_{n+1} = 5u_n \) \( u_1 = 2 \)

16 Express each of the following sequences as first-order recurrence relations.
   a \( u_n = 2(3)^{n-1} \) \( n = 1, 2, 3, \ldots \)  
   b \( u_n = -3(4)^{n-1} \) \( n = 1, 2, 3, \ldots \)  
   c \( u_n = 0.5(-1)^{n-1} \) \( n = 1, 2, 3, \ldots \)  
   d \( u_n = 3(5)^{n-1} \) \( n = 1, 2, 3, \ldots \)  
   e \( u_n = -5(2)^{n-1} \) \( n = 1, 2, 3, \ldots \)  
   f \( u_n = 0.1(-3)^{n-1} \) \( n = 1, 2, 3, \ldots \)

17 The sequence \( u_n = -4(1)^{n-1} \) \( n = 1, 2, 3, \ldots \) can be defined by the first-order recurrence relation:
   A \( u_{n+1} = u_n \) \( u_1 = -4 \)  
   B \( u_{n+1} = u_n - 1 \) \( u_1 = -4 \)  
   C \( u_{n+1} = 4u_n \) \( u_1 = 1 \)  
   D \( u_{n+1} = -4u_n + 3 \) \( u_1 = -4 \)  
   E \( u_{n+1} = -4u_n \) \( u_1 = -1 \)

18 Express each of the following sequences as a first-order recurrence relation.
   a -4, 4, -4, 4, -4, …  
   b 2, -14, 98, -686, 4802, …  
   c 2, 6, 18, 54, 162, …  
   d -1, 1, -1, 1, -1, …  
   e 5, 10, 20, 40, 80, …
19 Express each of the following sequences as recurrence relations.
   \( a \): 4, 10.5, 17, 23.5, 30, … \( b \): 15, 10, 5, 0, -5, -10, …
   \( c \): 1, 5, 9, 13, 17, 21, …

20 Express each of the geometric sequences as recurrence relations.
   \( a \): \( u_n = 4(2)^n - 1 \), \( n = 1, 2, 3, \ldots \)
   \( b \): \( u_n = -3(4)^n - 1 \), \( n = 1, 2, 3, \ldots \)
   \( c \): \( u_n = 2(-6)^n - 1 \), \( n = 1, 2, 3, \ldots \)
   \( d \): \( u_n = -0.1(8)^n - 1 \), \( n = 1, 2, 3, \ldots \)
   \( e \): \( u_n = 3.5(-10)^n - 1 \), \( n = 1, 2, 3, \ldots \)

### 5.4 Graphs of first-order recurrence relations

Certain quantities in nature and business may change in a uniform way (forming a pattern).

This change may be an increase, as in the case of:

\[
    u_{n+1} = u_n + 2 \quad u_1 = 3,
\]

or it may be a decrease, as in the case of:

\[
    u_{n+1} = u_n - 2 \quad u_1 = 3.
\]

These patterns can be modelled by graphs that, in turn, can be used to recognise patterns in the real world.

A graph of the equation could be drawn to represent a situation, and by using the graph the situation can be analysed to find, for example, the next term in the pattern.

**First-order recurrence relations:** \( u_{n+1} = u_n + b \) (arithmetic patterns)

The sequences of a first-order recurrence relation \( u_{n+1} = u_n + b \) are distinguished by a straight line or a constant increase or decrease.

An increasing pattern or a positive common difference gives an upward straight line. A decreasing pattern or a negative common difference gives a downward straight line.

**WORKED EXAMPLE 8**

On a graph, show the first five terms of the sequence described by the first-order recurrence relation:

\[
    u_{n+1} = u_n - 3 \quad u_1 = -5.
\]

**THINK**

1. Generate the values of each of the five terms of the sequence.

**WRITE/DRAW**

\[
    u_{n+1} = u_n - 3 \\
    u_1 = -5 \\
    u_2 = u_1 - 3 \quad u_3 = u_2 - 3 \\
    = -5 - 3 \quad = -8 - 3 \\
    = -8 \quad = -11 \\
    u_4 = u_3 - 3 \quad u_5 = u_4 - 3 \\
    = -11 - 3 \quad = -14 - 3 \\
    = -14 \quad = -17
\]
First-order recurrence relations: $u_{n+1} = Ru_n$

The sequences of a first-order recurrence relation $u_{n+1} = Ru_n$ are distinguished by a curved line or a saw form.

An increasing pattern or a positive common ratio greater than 1 ($R > 1$) gives an upward curved line.

A decreasing pattern or a positive fractional common ratio ($0 < R < 1$) gives a downward curved line.

An increasing saw pattern occurs when the common ratio is a negative value less than $-1$ ($R < -1$).

A decreasing saw pattern occurs when the common ratio is a negative fraction ($-1 < R < 0$).

WORKED EXAMPLE 9 On a graph, show the first six terms of the sequence described by the first-order recurrence relation:

$$u_{n+1} = 4u_n \quad u_1 = 0.5.$$
THINK

1 Generate the six terms of the sequence.

\[ u_{n+1} = 4u_n \quad u_1 = 0.5 \]

\[ u_2 = 4u_1 \quad u_3 = 4u_2 \]

\[ = 4 \times 0.5 \quad = 4 \times 2 \]

\[ = 2 \quad = 8 \]

\[ u_4 = 4u_3 \quad u_5 = 4u_4 \]

\[ = 4 \times 8 \quad = 4 \times 32 \]

\[ = 32 \quad = 128 \]

\[ u_6 = 4u_5 \]

\[ = 4 \times 128 \]

\[ = 512 \]

2 Graph these terms.

*Note: The sixth term is not included in this graph to more clearly illustrate the relationship between the terms.*

---

**Interpretation of the graph of first-order recurrence relations**

**Straight or linear**

A straight line or linear pattern is given by first-order recurrence relations of the form

\[ u_{n+1} = u_n + d \]

**Non-linear (exponential)**

A non-linear pattern is generated by first-order recurrence relations of the form

\[ u_{n+1} = Ru_n \]
Starting term

Earlier, the need for a starting term to be given to fully define a sequence was stated. As can be seen below, the same pattern but a different starting point gives a different set of numbers.

\[ u_{n+1} = u_n + 2 \quad u_1 = 3 \] gives 3, 5, 7, 9, 11, ...

\[ u_{n+1} = u_n + 2 \quad u_1 = 2 \] gives 2, 4, 6, 8, 10, ...

WORKED EXAMPLE 10

The first five terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.

THINK

1. Read from the graph the first five terms of the sequence.

2. Notice that the graph is linear and there is a common difference of \(-3\) between each term.

3. Write your answer including the value of one of the terms (usually the first), as well as the rule defining the first order difference equation.

WRITE

The sequence from the graph is:

\[ 18, 15, 12, 9, 6, \ldots \]

\[ u_{n+1} = u_n + d \]

Common difference, \(d = -3\)

\[ u_{n+1} = u_n - 3 \quad \text{or} \quad u_{n+1} - u_n = -3 \]

\[ u_{n+1} = u_n - 3 \quad u_1 = 18 \]

WORKED EXAMPLE 11

The first four terms of a sequence are plotted on the graph. Write the first-order recurrence relation that defines this sequence.
**THINK**
1. Read the terms of the sequence from the graph.
2. The graph is non-linear and there is a common ratio of 2, that is, for the next term, multiply the previous term by 2.
3. Define the first term.
4. Write your answer.

**WRITE**
The sequence is 2, 4, 8, 16, …

\[ u_{n+1} = R \times u_n \]

Common ratio, \( R = 2 \)
\[ u_{n+1} = 2u_n \]
\[ u_1 = 2 \]
\[ u_{n+1} = 2u_n \quad u_1 = 2 \]

---

**WORKED EXAMPLE 12**

The first five terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

A. \( u_{n+1} = u_n + 1 \) with \( u_1 = 1 \)
B. \( u_{n+1} = u_n + 2 \) with \( u_1 = 1 \)
C. \( u_{n+1} = 2u_n \) with \( u_1 = 1 \)
D. \( u_{n+1} = u_n + 1 \) with \( u_1 = 2 \)
E. \( u_{n+1} = u_n + 2 \) with \( u_1 = 2 \)

**THINK**
1. Eliminate the options systematically. Examine the first term given by the graph to decide if it is \( u_1 = 1 \) or \( u_1 = 2 \).

2. Observe any pattern between each successive point on the graph.

3. Option B gives both the correct pattern and first term.

**WRITE**
The coordinates of the first point on the graph are (1, 1). The first term is \( u_1 = 1 \). Eliminate options D and E.

There is a common difference of 2 or \( u_{n+1} = u_n + 2 \).

The answer is B.

---

**EXERCISE 5.4**

**Graphs of first-order recurrence relations**

1. **WE8** On a graph, show the first five terms of a sequence described by the first-order recurrence relation:

\[ u_{n+1} = u_n - 1 \quad u_1 = -2 \]

2. On a graph, show the first five terms of a sequence described by the first-order recurrence relation:

\[ u_{n+1} = u_n + 3 \quad u_1 = 1 \]

3. **WE9** On a graph, show the first six terms of a sequence described by the first-order recurrence relation:

\[ u_{n+1} = 3u_n \quad u_1 = 2 \]
4. On a graph, show the first six terms of a sequence described by the first-order recurrence relation:

\[ u_{n+1} = \frac{1}{2}u_n \quad u_1 = 36. \]

5. The first five terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

6. The first five terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

7. The first four terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

8. The first four terms of a sequence are plotted on the graph shown. Write the first-order recurrence relation that defines this sequence.

9. The first five terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

   \[
   \begin{array}{ll}
   A & u_{n+1} = u_n + 1 \quad u_1 = 2 \\
   B & u_{n+1} = u_n + 2 \quad u_1 = 2 \\
   C & u_{n+1} = u_n + 3 \quad u_1 = 1 \\
   D & u_{n+1} = u_n + 1 \quad u_1 = 1 \\
   E & u_{n+1} = u_n + 2 \quad u_1 = 1 \\
   \end{array}
   \]
10 The first four terms of a sequence are plotted on the graph shown. Which of the following first-order recurrence relations could describe the sequence?

- A $u_{n+1} = 2u_n$, $u_1 = 2$
- B $u_{n+1} = 2u_n$, $u_1 = 1$
- C $u_{n+1} = 2u_n$, $u_1 = 0$
- D $u_{n+1} = 3u_n$, $u_1 = 1$
- E $u_{n+1} = 3u_n$, $u_1 = 2$

11 For each of the following, plot the first five terms of the sequence defined by the first-order recurrence relation.

- a $u_{n+1} = u_n + 3$, $u_1 = 1$
- b $u_{n+1} = u_n + 7$, $u_1 = 5$
- c $u_{n+1} = u_n - 3$, $u_1 = 17$

12 For each of the following, plot the first five terms of the sequence defined by the first-order recurrence relation.

- a $u_{n+1} = -2u_n$, $u_1 = -0.5$
- b $u_{n+1} = 0.5u_{n-1}$, $u_1 = 16$
- c $u_{n+1} = 2.5u_n$, $u_1 = 2$

13 For each of the following, plot the first four terms of the sequence defined by the first-order recurrence relation.

- a $u_{n+1} = 3u_n - 4$, $u_1 = -3$
- b $u_{n+1} = 2u_{n-1} + 0.5$, $u_1 = 2$
- c $u_{n+1} = 2 + 5u_n$, $u_1 = 2$

14 For each of the following, plot the first four terms of the sequence defined by the first-order recurrence relation.

- a $u_{n+1} = 100 - 3u_n$, $u_1 = 20$
- b $u_{n+1} = u_n - 50$, $u_1 = 100$
- c $u_{n+1} = 0.1u_{n-1}$, $u_1 = 0$

15 On graphs, show the first 5 terms of the sequences defined by the following recurrence relations.

- a $u_{n+1} = u_n + 3$, $u_1 = 1$, $n = 1, 2, 3, \ldots$
- b $u_{n+1} = u_n - 3$, $u_1 = 17$, $n = 1, 2, 3, \ldots$
- c $u_{n+1} = u_n - 15$, $u_1 = 75$, $n = 1, 2, 3, \ldots$
- d $u_{n+1} = u_n + 10$, $u_1 = 80$, $n = 1, 2, 3, \ldots$

16 On graphs, show the first 4 terms of the sequences defined by the following recurrence relations.

- a $u_{n+1} = 6u_n$, $u_1 = 1$, $n = 1, 2, 3, \ldots$
- b $u_{n+1} = 3u_n$, $u_1 = -1$, $n = 1, 2, 3, \ldots$
- c $u_{n+1} = 1.5u_n$, $u_0 = 1$, $n = 0, 1, 2, 3, \ldots$
- d $u_{n+1} = 0.5u_n$, $u_1 = 10$, $n = 1, 2, 3, \ldots$

17 For each of the following graphs, write a first-order recurrence relation that defines the sequence plotted on the graph.
For each of the following graphs, write a first-order recurrence relation that defines the sequence plotted on the graph.

19 The first five terms of a sequence are plotted on the graph. Which of the following first-order recurrence relations could describe the sequence?

A  \( u_{n+1} = u_n - 8 \quad u_1 = 8 \)
B  \( u_{n+1} = u_n + 8 \quad u_1 = 8 \)
C  \( u_{n+1} = u_n - 8 \quad u_1 = 45 \)
D  \( u_{n+1} = u_n + 8 \quad u_1 = 45 \)
E  \( u_{n+1} = 8u_n \quad u_1 = 45 \)

20 Write the first-order recurrence relation that defines the sequence plotted on the graph shown.

21 Graphs of the first five terms of first-order recurrence relations are shown below together with the first-order recurrence relations. Match the graph with the first-order recurrence relation by writing the letter corresponding to the graph together with the number corresponding to the first-order recurrence relation.
The first few terms of four sequences are plotted on the graphs below. Match the following first-order recurrence relations with the graphs.

i \( u_{n+1} = u_n + \frac{1}{2} \), \( u_1 = 8 \)  

ii \( u_{n+1} = u_n - 2 \), \( u_1 = 1 \)  

iii \( u_{n+1} = 2u_n \), \( u_1 = 1 \)  

iv \( u_{n+1} = \frac{1}{2}u_n \), \( u_1 = 16 \)  

v \( u_{n+1} = 2u_n \), \( u_1 = 2 \)  

vi \( u_{n+1} = u_n - \frac{1}{2} \), \( u_1 = 4 \)
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5 Answers

EXERCISE 5.2
1 a This is a first-order recurrence relation.
   b This is not a first-order recurrence relation.
2 a This is not a first-order recurrence relation.
   b This is a first-order recurrence relation.
3 \( u_n = 4u_{n-1} + 3, \quad u_0 = 5 \)
   The sequence is 5, 23, 95, 383, 1535.
4 \( f_{n+1} = 5f_n - 6, \quad f_0 = -2 \)
   The sequence is \(-2, -16, -86, -436, -2186.\)
5 The second term is 2.
6 The fifth term is \( \frac{2}{3} \).
7 b, c, g, j
8 a 6, 8, 10, 12, 14
   b 5, 2, -1, -4, -7
   c 23, 24, 25, 26, 27
   d 7, -3, -13, -23, -33
9 a 1, 3, 9, 27, 81
   b -2, -10, -50, -250, -1250
   c 1, -4, 16, -64, 256
   d -1, -2, -4, -8, -16
10 a 1, 3, 7, 15, 31
     b 5, 13, 37, 109, 325
11 a The first seven terms are 6, -5, 6, -5, 6, -5, 6.
     b The first seven terms are 1, 5, 25, 125, 625, 3125.15625.
12 C
13 E
14 7
15 -1
16 \(-\frac{1}{8}\)
17 D
18 a \( u_{n+1} = 3u_n, \quad u_0 = \frac{1}{4} \) \( \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{20}{4}, \frac{1}{4}, \frac{20}{4} \)
   b \( u_{n+1} = u_n + 2000, \quad u_0 = 200000; \quad 200000, \quad 202000, \quad 204000, \quad 206000, \quad 208000 \)
   c \( u_{n+1} = u_n - 7, \quad u_0 = 100; \quad 100, \quad 93, \quad 86, \quad 79, \quad 72 \)
   d \( u_{n+1} = 2u_n - 50, \quad u_0 = \$200; \quad \$200, \quad \$350, \quad \$650, \quad \$1250, \quad \$2450 \)

EXERCISE 5.3
1 a \( u_{n+1} = u_n + 3, \quad u_1 = 1 \)
   b \( u_{n+1} = u_n - 4, \quad u_1 = 12 \)
2 a \( u_{n+1} = u_n + 5, \quad u_1 = -24 \)
   b \( u_{n+1} = u_n + 11, \quad u_1 = 55 \)
3 The sequence is \(-8, -13, -18, -23, -28, \ldots \)
   The first-order recurrence relation is \( u_{n+1} = u_n - 5, \quad u_1 = -8 \).
4 The sequence is \( 4\frac{1}{2}, 5\frac{1}{2}, 6\frac{1}{2}, \ldots \)
   The first-order recurrence relation is \( u_{n+1} = u_n + \frac{1}{2}, \quad u_1 = 4 \).
5 a \( u_{n+1} = 5u_n, \quad u_1 = 1 \)
   b \( u_{n+1} = -2u_n, \quad u_1 = 4 \)
6 a \( u_{n+1} = 5u_n, \quad u_1 = 3 \)
   b \( u_{n+1} = -\frac{1}{2}u_n, \quad u_1 = 200 \)
7 a \( u_{n+1} = 5u_n, \quad u_1 = -2 \)
   b \( u_{n+1} = 4u_n, \quad u_1 = -3 \)
8 a \( u_{n+1} = 2u_n, \quad u_1 = 3 \)
   b \( u_{n+1} = 3u_n, \quad u_1 = -2 \)
9 a \( u_{n+1} = u_n + 2, \quad u_1 = 1 \)
   b \( u_{n+1} = u_n + 7, \quad u_1 = 3 \)
   c \( u_{n+1} = u_n - 7, \quad u_1 = 12 \)
   d \( u_{n+1} = u_n - 0.5, \quad u_1 = 1 \)
   e \( u_{n+1} = u_n + 4, \quad u_1 = 2 \)
   f \( u_{n+1} = u_n + 4, \quad u_1 = -2 \)
   g \( u_{n+1} = u_n - 5, \quad u_1 = 6 \)
   h \( u_{n+1} = u_n + 6.5, \quad u_1 = 4 \)
10 B
11 a \( u_{n+1} = u_n - 1, \quad u_1 = -4 \)
   b \( u_{n+1} = u_n + 2, \quad u_1 = 3 \)
   c \( u_{n+1} = u_n + 3, \quad u_1 = -1 \)
   d \( u_{n+1} = u_n - 2, \quad u_1 = 4 \)
12 A
13 a \( u_{n+1} = u_n - 2, \quad u_1 = 12, \quad n = 1, 2, 3, \ldots \)
   b \( u_{n+1} = u_n + 1, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots \)
14 a \( u_{n+1} = 2u_n, \quad u_1 = 5 \)
   b \( u_{n+1} = 6u_n, \quad u_1 = 1 \)
   c \( u_{n+1} = -u_n, \quad u_1 = -3 \)
   d \( u_{n+1} = 4u_n, \quad u_1 = -3 \)
   e \( u_{n+1} = 3u_n, \quad u_1 = 2 \)
   f \( u_{n+1} = -u_n, \quad u_1 = 5 \)
   g \( u_{n+1} = 4u_n, \quad u_1 = -2 \)
   h \( u_{n+1} = -3u_n, \quad u_1 = 5 \)
15 D
16 a \( u_{n+1} = 3u_n, \quad u_1 = 2 \)
   b \( u_{n+1} = 4u_n, \quad u_1 = -3 \)
   c \( u_{n+1} = -u_n, \quad u_1 = 0.5 \)
   d \( u_{n+1} = 5u_n, \quad u_1 = 3 \)
   e \( u_{n+1} = 2u_n, \quad u_1 = -5 \)
   f \( u_{n+1} = -3u_n, \quad u_1 = 0.1 \)
EXERCISE 5.4

1. \[
7u_{n+1} = 2u_n, \quad u_1 = 1.5
\]

2. \[
u_{n+1} = 2u_n, \quad u_1 = 3
\]

3. \[
u_{n+1} = u_n + 6.5, \quad u_1 = 4, \quad n = 1, 2, 3, \ldots
\]

4. \[
u_{n+1} = u_n - 5, \quad u_1 = 15, \quad n = 1, 2, 3, \ldots
\]

5. \[
u_{n+1} = u_n + 4, \quad u_1 = 1, \quad n = 1, 2, 3, \ldots
\]

6. \[
u_{n+1} = 2u_n, \quad u_1 = 5, \quad n = 1, 2, 3, \ldots
\]

7. \[
u_{n+1} = 2u_n, \quad u_1 = 3
\]

8. \[
u_{n+1} = u_n - 1, \quad n = 1, 2, 3, \ldots
\]

9. \[
u_{n+1} = 2u_n, \quad u_1 = 3
\]

10. \[
u_{n+1} = 4u_n, \quad u_1 = -3, \quad n = 1, 2, 3, \ldots
\]

11. \[
u_{n+1} = -6u_n, \quad u_1 = 2, \quad n = 1, 2, 3, \ldots
\]

12. \[
u_{n+1} = 8u_n, \quad u_1 = -0.1, \quad n = 1, 2, 3, \ldots
\]

13. \[
u_{n+1} = -10u_n, \quad u_1 = 3.5, \quad n = 1, 2, 3, \ldots
\]
Topic 5 RECURRENCE RELATIONS

14 a  

Value of term

Term number

15 a  

Value of term

Term number

16 a  

Value of term

Term number

17 a  

\[ u_{n+1} = u_n + 3 \]

\[ u_1 = 2 \]

b  

\[ u_{n+1} = u_n - 20 \]

\[ u_1 = 90 \]

c  

\[ u_{n+1} = u_n - 4 \]

\[ u_1 = 8 \]

18 a  

\[ u_{n+1} = 2u_n \]

\[ u_1 = 5 \]

b  

\[ u_{n+1} = -u_n \]

\[ u_1 = 6 \]

c  

\[ u_{n+1} = -0.5u_n \]

\[ u_1 = 8 \]

19 C

\[ u_{n+1} = 3u_n \]

\[ u_1 = -1 \]

21 a  

ii

b  

v

c  

iii

d  

iv

e  

vi

f  

i

22 i  

matches to d.

ii  

matches to b.

iii  

matches to a.

iv  

matches to c.