8 Vectors

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8.1 Kick off with CAS

To come

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
Introduction to vectors

A scalar quantity is one that is specified by size, or magnitude, only.

Distance is an example of a scalar quantity; it needs only a number to specify its size or magnitude. Time, length, volume, temperature and mass are scalars.

A vector quantity is specified by both magnitude and direction.

Displacement measures the final position compared to the starting position and requires both a magnitude (e.g. distance 800 m) and a direction (e.g. 230°T). Displacement is an example of a vector quantity. Force, velocity and acceleration are also vectors. They all require a size and a direction to be specified completely.

Representation of vectors

Vectors can be represented by directed line segments.

For example, if north is straight up the page and a scale of 1 cm = 20 m is used, then a displacement of 100 m south is represented by a 5 cm line straight down the page. We place an arrow on the line to indicate the direction of the vector, as shown at right.

The start and end points of a vector can be labelled with capital letters.

For example, the vector shown at right can have the starting point, or tail, labelled S and the end point, or head, labelled F.

This vector can then be referred to as SF.

The vector can also be represented by a lower-case letter over a tilde, for example, \( \vec{s} \).

Representing a vector as an ordered pair \((a, b)\)

A vector in the \(x-y\) plane can be described by an ordered pair \((a, b)\).

The values \(a\) and \(b\) are called components; \(a\) gives the change of position relative to the positive \(x\)-axis and \(b\) gives the change of position relative to the positive \(y\)-axis of the end of the vector compared to the start.

For example \((2, 4)\) represents a change of position of 2 units in the positive \(x\)-direction and 4 units in the positive \(y\)-direction.

Note that the vector represented by \((2, 4)\) doesn’t necessarily start at the origin. It can be in any position on the Cartesian plane.

Representing a vector as a column matrix

Any vector can be written as a column matrix, which is a matrix consisting of a single column with two elements. For example, the vector represented by the directed line segments described in the previous section can be written as the column matrix \[
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]. The top number gives the displacement relative to the positive direction on the \(x\)-axis and the bottom number gives the displacement relative to the positive direction on the \(y\)-axis.
**Position vectors**

A position vector describes a point in the Cartesian plane. Position vectors start at the origin O (0, 0). For example, for A (3, 1) the position vector $\overrightarrow{OA}$ is shown at right.

Note we can also use (3, 1) to describe any vector that travels three units across and one up, but it is only a position vector if it starts at (0, 0).

**WORKED EXAMPLE 1**

Write the following vectors in the form $(a, b)$ and $\begin{bmatrix} a \\ b \end{bmatrix}$.

- $\overrightarrow{OC}$
- $\overrightarrow{DA}$

**THINK**

- From O to C, we travel +4 units in the positive $x$-direction and +1 unit in the positive $y$-direction.

- From D to A, we travel −5 units in the positive $x$-direction and +3 units in the positive $y$-direction.

**WRITE**

- $\overrightarrow{OC} = (4, 1)$ and $\overrightarrow{OC} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

- $\overrightarrow{DA} = (-5, 3)$ and $\overrightarrow{DA} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

**WORKED EXAMPLE 2**

If we started at (5, −2), where would we end up after a displacement of (3, 2)?

**THINK**

1. Write $(5, -2) + (3, 2)$.

2. We start at $(5, -2)$ and move +3 units in the positive $x$-direction and +2 units in the positive $y$-direction.

3. Write the answer.

**WRITE**

- $(5, -2) + (3, 2) = (8, 0)$

- We would end up at the point $(8, 0)$.

**WORKED EXAMPLE 3**

Draw $\overrightarrow{d}$, the position vector of (−2, 3), on a set of axes.

**THINK**

1. A position vector must start at (0, 0) and end at the point specified. Make sure the arrow is pointing away from the origin.

2. Label the vector.
Equality of vectors
Two vectors are equal if they are:
1. equal in magnitude
2. parallel, and
3. point in the same direction.

WORKED EXAMPLE 4
Which of the vectors shown at right are equal?

THINK
1 Vectors \(a\) and \(e\) are of equal length, parallel and point in the same direction.
2 Vectors \(b\) and \(g\) are of equal length, parallel and point in the same direction.

WRITE
\[a = e\]
\[b = g\]

WORKED EXAMPLE 5
An aircraft flies 200 km north, then 400 km east.
Draw a vector diagram to represent the path taken by the aircraft and also the displacement of the aircraft from its starting point to its finishing point.

THINK
1 Take north as vertically up the page and east to the right.
2 Draw a short vertical directed line segment to represent a displacement of 200 km north.
3 Draw a horizontal directed line segment with its tail joined to the head of the first. This represents a displacement of 400 km east.
4 Draw a directed line segment from the tail of the ‘north’ vector (point \(S\)) to the head of the ‘east’ vector (point \(F\)). This represents the displacement of the aircraft from its starting point to its finishing point.

DRAW

EXERCISE 8.2
Introduction to vectors

1. Examine the diagram at right. Represent each of the following vectors as an ordered pair \((a, b)\).
   
   \[\overrightarrow{AB}\]
   \[\overrightarrow{AC}\]
   \[\overrightarrow{AF}\]
   \[\overrightarrow{BD}\]
2 Examine the diagram at right. Represent each of the following vectors as a column matrix \[
\begin{bmatrix}
a \\
b
\end{bmatrix}.
\]
\[
a \quad \overrightarrow{CD} \\
b \quad \overrightarrow{CA} \\
c \quad \overrightarrow{ED} \\
d \quad \overrightarrow{EF} \\
e \quad \overrightarrow{FE}
\]

3 If we started at \((1, 1)\) where would we end up after a displacement of \((-3, 6)\)?

4 If we started at the point \((2, -5)\), where would we end up after each of these displacements?

\[
a \quad (3, -2) \\
b \quad (-3, 5) \\
c \quad (0, 4) \\
d \quad (2, -5)
\]

5 Draw \(d\), the position vector \((-1, 4)\), on a set of axes.

6 Draw the position vector for each of the following points on the same set of axes.

\((4, 1), (-3, 2), (0, -3), (-2, -2)\)

7 Which of the vectors shown in the diagram at right are equal?

8 Which of the vectors shown in the diagram at right are equal?

9 An aeroplane flies 1000 km north from airport A to airport B. It then travels to airport C, which is 1200 km north-east of B. Draw a vector diagram to represent the path taken by the aeroplane and the displacement of the finishing point from the starting point.

10 A boat travels 30 km north and then 40 km west. Draw a vector diagram showing the path of the boat and the displacement of the finishing point from the starting point.
11 Examine the diagram at right. Represent the change of position of each of the vectors shown in the form \((a, b)\).

12 Represent the change of position of each of the vectors shown in question 11 in the form \([a \ b]\).

13 Represent each of the following vectors on separate diagrams.
   a The position vector of \((2, 3)\)
   b The position vector of \((0, 5)\)
   c The position vector of \((-3, 2)\)

14 Represent each of the following vectors on separate diagrams.
   a A displacement of \((2, -8)\) starting from the point \((4, 4)\)
   b A displacement of \((-2, 5)\) starting from the point \((3, -6)\)
   c A displacement of \((0, 3)\) starting from the point \((2, 5)\)
   d The position vector of \((4, -2)\) followed by \((3, 5)\)

15 A vector that starts at the point \((-2, 1)\) and finishes at the point \((3, -3)\) is represented by a displacement of:
   A \((4, -5)\)
   B \((5, -4)\)
   C \((1, -2)\)
   D \((-5, 4)\)
   E \((3, 2)\)

In questions 20 to 22, draw vector diagrams to represent the paths described and the displacement of the finishing point from the starting point.

16 Draw two directed line segments represented by the vector \([2 \ 5]\).

17 The directed line segment shown in the diagram represents the vector \(d\).

Find \(a\) and \(b\) if \(d = \begin{bmatrix} a \\ b \end{bmatrix}\).

18 Sketch the following vectors on separate axes, if \(A = (2, -1), B = (0, 2), C = (4, 1)\) and \(O\) is the origin.
   a \(\overrightarrow{OA}\)
   b \(\overrightarrow{AB}\)
   c \(\overrightarrow{AC}\)
   d \(\overrightarrow{BC}\)

19 Express each of the vectors from question 18 in the form \([a \ b]\).

20 Marcus cycles 20 km in an easterly direction and then travels 30 km due south.

21 Bianca rows straight across a river in which a current is flowing at 3.5 km/h. Bianca can row at 11.5 km/h.

22 An aeroplane takes off and flies at an angle of elevation of \(25^\circ\) for 25 km. It then flies horizontally for 300 km.
8.3 Operations on vectors

Addition of vectors

If we travel from A to B and then from B to C, the combined effect is to start from A and finish at C. We write

\[ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \]

Notice that the tail of the second vector \( \overrightarrow{BC} \) is joined to the head of the first vector \( \overrightarrow{AB} \).

If the addition is reversed, so that the tail of the first vector is joined to the head of the second vector, the combined effect is also a vector equal to \( \overrightarrow{AC} \). So \( \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB} \).

This shows that changing the order in which vectors are added does not alter the combined effect of the vectors.

This method for adding two vectors is called the triangle rule for vectors.

The addition of vectors \( \overrightarrow{a} \) and \( \overrightarrow{b} \) can be shown by forming a vector from the tail of \( \overrightarrow{a} \) to the head of \( \overrightarrow{b} \).

Negative vectors

Just as moving \(-2\) units on the \(x\)-axis is opposite in direction to moving 2 units along the \(x\)-axis, the negative of a given vector is opposite in direction to the original vector.

The vector \(-\overrightarrow{b}\) has the same magnitude as \(\overrightarrow{b}\) but is in the opposite direction.

Subtraction of vectors

Subtraction of vectors can be performed by combining vector addition and negative vectors.

\[ \overrightarrow{a} - \overrightarrow{b} = \overrightarrow{a} + (-\overrightarrow{b}) \]

For example, if \(\overrightarrow{a}\) and \(\overrightarrow{b}\) are vectors as shown at right, then we can find \(\overrightarrow{a} - \overrightarrow{b}\) by:

1. expressing it as an addition:
   \[ \overrightarrow{a} - \overrightarrow{b} = \overrightarrow{a} + (-\overrightarrow{b}) \]

2. reversing the arrow on vector \(\overrightarrow{b}\) so that it becomes \(-\overrightarrow{b}\)

3. adding \(-\overrightarrow{b}\) to \(\overrightarrow{a}\) as shown to form \(\overrightarrow{a} - \overrightarrow{b}\).
Scalar multiplication
A displacement of $(2, 3)$ followed by another displacement of $(2, 3)$ equals a displacement of $(4, 6)$.

We could write this as $2(2, 3) = (4, 6)$.

The vector represented by $(2, 3)$ has been multiplied by the number 2 to give the vector represented by $(4, 6)$. 

WORKED EXAMPLE 6

Using $d, e$ and $f$ as shown in the diagram, draw vector diagrams to show:

- $a$ $d + e$
- $b$ $d + e + f$
- $c$ $e - f$

THINK

- $a$ Draw the vector $d$ and join the tail of $e$ to the head of $d$.

- $b$ $d + e + f$ is obtained by joining the head of $d + e$ (from part a) with the tail of $f$.

- $c$ Reverse the arrow on $f$ to obtain $-f$ and join the head of $e$ to the tail of $-f$.

WORKED EXAMPLE 7

If $a = (1, 4)$, $b = (-5, 2)$ and $c = (-2, 3)$, find each of the following:

- $a$ $a + b$
- $b$ $a - c$
- $c$ $a + b + c$

THINK

- $a$ Add the corresponding components of each vector to give the answer for $a + b$.

- $b$ Subtract the corresponding components of each vector to give the answer for $a - c$.

- $c$ $a + b + c$ may be calculated by adding the corresponding components of $a$ and $b$ and $c$.

WRITE

- $a$ $a + b = (1, 4) + (-5, 2) = (-4, 6)$
- $b$ $a - c = (1, 4) - (-2, 3) = (3, 1)$
- $c$ $a + b + c = (1, 4) + (-5, 2) + (-2, 3) = (-6, 9)$
This process is called multiplication by a scalar or scalar multiplication. Scalar multiplication means that the vector is made larger or smaller by a scale factor. In the case above, the scalar is 2.

In general, we can say that if \( k \in \mathbb{R} \):

1. \( ka \) is a vector \( k \) times as big as \( a \) and in the same direction as \( a \) for \( k > 0 \).
2. \( ka \) is in the opposite direction to \( a \) for \( k < 0 \).

WORKED EXAMPLE 8

If \( a = (5, -4) \) and \( b = (-3, 2) \), calculate:

a 2\(a + b\)

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>a 1 Multiply each component of ( a ) by 2 to obtain 2(a).</td>
<td>a (2a = 2(5, -4)) (= (10, -8))</td>
</tr>
<tr>
<td>2 Add the components of 2(a) and (b) to obtain 2(a + b).</td>
<td>2(a + b = (10, -8) + (-3, 2)) (= (7, -6))</td>
</tr>
</tbody>
</table>

b 1 Subtract the components of \( a \) from \( b \) to obtain \( b - a \).

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>b 1 Subtract the components of ( a ) from ( b ) to obtain ( b - a ).</td>
<td>b (b - a = (-3, 2) - (5, -4)) (= (-8, 6))</td>
</tr>
</tbody>
</table>

b 2 Multiply the components of \( b - a \) by 3 to obtain 3\((b - a)\).

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>b 2 Multiply the components of ( b - a ) by 3 to obtain 3((b - a)).</td>
<td>3((b - a) = 3(-8, 6)) (= (-24, 18))</td>
</tr>
</tbody>
</table>

WORKED EXAMPLE 9

ABEF and BCDE are parallelograms with \( \overrightarrow{AB} \) represented by \( a \) and \( \overrightarrow{AF} \) represented by \( b \). The length of BC is twice the length of AB. Express the following vectors in terms of \( a \) and \( b \).

a \( \overrightarrow{BC} \)

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>a 1 ( \overrightarrow{BC} ) and ( \overrightarrow{AB} ) are in the same direction and ( \overrightarrow{BC} ) is twice as big as ( \overrightarrow{AB} ).</td>
<td>a ( \overrightarrow{BC} = 2\overrightarrow{AB} ) (= 2a)</td>
</tr>
<tr>
<td>2 Replace ( \overrightarrow{AB} ) by ( a ).</td>
<td></td>
</tr>
</tbody>
</table>

b \( \overrightarrow{AC} \)

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>b 1 ( \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} ) using vector addition.</td>
<td>b ( \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} ) (= a + 2a)</td>
</tr>
<tr>
<td>2 Replace ( \overrightarrow{AB} ) and ( \overrightarrow{BC} ) by ( a ) and 2(a) respectively.</td>
<td></td>
</tr>
<tr>
<td>3 Simplify.</td>
<td>(= 3a)</td>
</tr>
</tbody>
</table>

C \( \overrightarrow{CD} \)

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>c 1 ( \overrightarrow{CD} = \overrightarrow{AF} ) because opposite sides of a parallelogram are parallel and the same size.</td>
<td>c ( \overrightarrow{CD} = b)</td>
</tr>
<tr>
<td>2 ( \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} ), using the triangle rule to add vectors.</td>
<td>( \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} ) (= 2a + b)</td>
</tr>
<tr>
<td>3 Replace ( \overrightarrow{BC} ) and ( \overrightarrow{CD} ) by 2(a) and ( b ) respectively.</td>
<td></td>
</tr>
</tbody>
</table>
Simplify the expression \( \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{EC} \).

**THINK**

1. \( \overrightarrow{AB} + \overrightarrow{BC} \) represents a vector from A to B with the vector from B to C added on. This is the same as the vector from A to C.
2. \( -\overrightarrow{EC} \) is the same as vector \( \overrightarrow{CE} \) because the negative of a vector reverses the direction.
3. \( \overrightarrow{AC} + \overrightarrow{CE} \) represents a vector starting at A going to C and then from C to E. This is the same as \( \overrightarrow{AE} \).

**WRITE**

\[
\begin{align*}
\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{EC} & = \overrightarrow{AC} - \overrightarrow{EC} \\
& = \overrightarrow{AC} + \overrightarrow{CE} \\
& = \overrightarrow{AE}
\end{align*}
\]

**EXERCISE 8.3**

**Operations on vectors**

1. **WE6** Using vectors \( a \), \( b \) and \( c \) as shown, sketch:
   - \( 3a \)
   - \( 2b \)
   - \( c \)
   - \( a + b \)
   - \( a + c \)
   - \( b + c \)

2. Using \( a \), \( b \) and \( c \) from question 1, sketch:
   - \( a + 2b \)
   - \( 2a + 3c \)
   - \( a - c \)
   - \( a + b + c \)
   - \( a - b - c \)

3. **WE7** If \( m(-2, 3) \), \( n = (4, 0) \) and \( p = (-1, 5) \), find each of the following.
   - \( m + n \)
   - \( m + n + p \)

4. Using \( m \), \( n \) and \( p \) from question 3, find each of the following.
   - \( n - p \)
   - \( m - n - p \)

5. **WE8** Using \( m \), \( n \) and \( p \) from question 3, calculate the following.
   - \( 3m - p \)
   - \( 2m + n - p \)

6. Using \( m \), \( n \) and \( p \) from question 3, calculate the following.
   - \( 2(m + n) \)
   - \( 4p - 3n \)

7. **WE9** Refer to the cube as shown on the right. Let \( a = \overrightarrow{OA} \), \( c = \overrightarrow{OC} \) and \( d = \overrightarrow{OD} \). Write, in terms of \( a \), \( c \) and \( d \), the vectors representing:
   - \( \overrightarrow{DE} \)
   - \( \overrightarrow{AC} \)
   - \( \overrightarrow{OB} \)
   - \( \overrightarrow{AE} \)

8. Using \( a \), \( b \) and \( c \) from question 7, including the cube. Write the following vectors in terms of \( a \), \( c \) and \( d \):
   - \( \overrightarrow{EG} \)
   - \( \overrightarrow{DF} \)
   - \( \overrightarrow{OF} \)
   - \( \overrightarrow{AG} \)
   - \( \overrightarrow{DB} \)

9. **WE10** Show that \( \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC} \).

10. In simplest form, \( \overrightarrow{MN} - \overrightarrow{QP} + \overrightarrow{NP} + \overrightarrow{QR} \) equals:
    - \( \overrightarrow{0} \)
    - \( \overrightarrow{MR} \)
    - \( \overrightarrow{MQ} \)
    - \( \overrightarrow{QN} \)
    - \( \overrightarrow{NR} \)

11. Draw two vectors \( u \) and \( v \) such that \( u + v = (0, 0) \).

12. Draw two possible representations of \( u + v = (3, 5) \).
13 Draw two possible representations of \( \mathbf{u} + \mathbf{v} = (-3, 2) \).

14 Using the cube shown question 7. Write all the vectors that are equal to the following vectors.

\[
\begin{align*}
\text{a} & \quad \overrightarrow{OA} \\
\text{b} & \quad \overrightarrow{OC} \\
\text{c} & \quad \overrightarrow{OD} \\
\text{d} & \quad \overrightarrow{GF} \\
\text{e} & \quad \overrightarrow{AB} \\
\text{f} & \quad \overrightarrow{AD}
\end{align*}
\]

15 ABCDEF is a regular hexagon with vectors \( \overrightarrow{OA} \) and \( \overrightarrow{OB} \) represented by \( \mathbf{a} \) and \( \mathbf{b} \) respectively. Write, in terms of \( \mathbf{a} \) and \( \mathbf{b} \), the following vectors:

\[
\begin{align*}
\text{a} & \quad \overrightarrow{DO} \\
\text{b} & \quad \overrightarrow{DA} \\
\text{c} & \quad \overrightarrow{CD} \\
\text{d} & \quad \overrightarrow{AB} \\
\text{e} & \quad \overrightarrow{BC} \\
\text{f} & \quad \overrightarrow{AC} \\
\text{g} & \quad \overrightarrow{DE} \\
\text{h} & \quad \overrightarrow{ED} \\
\text{i} & \quad \overrightarrow{EA} \\
\text{j} & \quad \overrightarrow{DF}
\end{align*}
\]

16 Show that \( \overrightarrow{EF} + \overrightarrow{GH} - \overrightarrow{GF} - \overrightarrow{EH} = \mathbf{0} \).

17 Express in simplest form \( \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DE} - \overrightarrow{DC} \).

18 In terms of vectors \( \mathbf{a} \) and \( \mathbf{b} \) in the figure at right, the vector joining O to D is given by:

\[
\begin{align*}
\text{A} & \quad 3\mathbf{a} + 3\mathbf{b} \\
\text{B} & \quad 2\mathbf{a} + 4\mathbf{b} \\
\text{C} & \quad 3\mathbf{b} - 2\mathbf{a} \\
\text{D} & \quad 2\mathbf{a} - 3\mathbf{b} \\
\text{E} & \quad \text{none of these}
\end{align*}
\]

19 In terms of vectors \( \mathbf{a} \) and \( \mathbf{b} \), the vector joining E to O at right is:

\[
\begin{align*}
\text{A} & \quad 3\mathbf{a} + 4\mathbf{b} \\
\text{B} & \quad 4\mathbf{b} - 3\mathbf{a} \\
\text{C} & \quad 3\mathbf{a} - 4\mathbf{b} \\
\text{D} & \quad -3\mathbf{a} - 4\mathbf{b} \\
\text{E} & \quad \text{none of these}
\end{align*}
\]

20 The parallelogram ABCD can be defined by the two vectors \( \mathbf{b} \) and \( \mathbf{c} \). In terms of these vectors, find:

\[
\begin{align*}
\text{a} & \quad \text{the vector from A to D} \\
\text{b} & \quad \text{the vector from C to D} \\
\text{c} & \quad \text{the vector from D to B}.
\end{align*}
\]

21 A rectangular prism CDEFGHIJ can be defined by the vectors \( \mathbf{r}, \mathbf{s} \) and \( \mathbf{t} \), as shown on the right. Express in terms of \( \mathbf{r}, \mathbf{s} \) and \( \mathbf{t} \):

\[
\begin{align*}
\text{a} & \quad \text{the vector joining C to H} \\
\text{b} & \quad \text{the vector joining C to J} \\
\text{c} & \quad \text{the vector joining G to D} \\
\text{d} & \quad \text{the vector joining F to I} \\
\text{e} & \quad \text{the vector joining H to E} \\
\text{f} & \quad \text{the vector joining D to J} \\
\text{g} & \quad \text{the vector joining C to I} \\
\text{h} & \quad \text{the vector joining J to C}.
\end{align*}
\]

22 A cube PQRSTUVW can be defined by the three vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) as shown at right. Express in terms of \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \):

\[
\begin{align*}
\text{a} & \quad \text{the vector joining P to V} \\
\text{b} & \quad \text{the vector joining P to W} \\
\text{c} & \quad \text{the vector joining U to Q} \\
\text{d} & \quad \text{the vector joining S to W} \\
\text{e} & \quad \text{the vector joining Q to T}.
\end{align*}
\]
8.4 Magnitude, direction and components of vectors

Magnitude

The magnitude of a vector can be calculated from the length of the line segment representing the vector.

The magnitude of a vector \( \vec{a} \) is denoted by \(| \vec{a} | \) or \( a \).

Direction

The direction of a vector can be found by applying appropriate trigonometric ratios to find a relevant angle.

This angle is usually the angle that the vector makes with a given direction such as north, the positive x-axis, or the horizontal or vertical.

WORKED EXAMPLE 11

Find the magnitude and direction, relative to the positive x-axis, of the vector \((3, -2)\).

THINK

1. Draw a diagram of the vector and denote it as \( \vec{a} \) with the angle between \( \vec{a} \) and the positive x-axis as \( \theta \).
2. The magnitude of \( \vec{a} \) is the length of the line segment representing the vector.
3. Use Pythagoras’ theorem to calculate this length.
4. Calculate the angle \( \theta \) using trigonometry.
5. State the solution with the angle down from the positive x-axis given as a negative.

DRAW/WRITE

\[|\vec{a}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}\]

\[\tan(\theta) = \frac{2}{3}\]

\[\theta = 33.7^\circ\]

The vector \((3, -2)\) has a magnitude of \(\sqrt{13}\) units and makes an angle of \(-33.7^\circ\) with the positive x-axis.

The angle that a vector makes with the positive x-axis can be found using trigonometry. If the vector points in the negative x-direction, then you will need to add your found angle \( \theta \) to \( 90^\circ \) or subtract it from \( 180^\circ \) to find the required angle. See the diagram at right.

Upward vectors are expressed as positive angles anticlockwise from the positive x-axis. Downward vectors are expressed as negative angles clockwise from the positive x-axis.

In general, if \( \vec{r} = (a, b) \), then the direction of \( \vec{r} \) compared to the positive x-axis is found by appropriately adjusting \( \theta \) where \( \tan(\theta) = \frac{b}{a} \).
Vector components

We have seen that two vectors may be added to give one resultant vector. The reverse process may be used to express one vector as the sum of two other vectors. This process is called ‘breaking the vector into two components’.

A vector can be broken into two perpendicular components such as $x$ and $y$ or north and east.

It may be convenient to find the effect of a vector in a particular direction. We do this by breaking the vector into two components.

A force $F$ acting as shown will move an object to the right and upwards. The force $F$ can be separated into two component parts; one in the horizontal direction, $H$, and the other in the vertical direction, $V$.

$$F = H + V$$

The effect of the force in the horizontal direction is given entirely by $H$ and the effect in the vertical direction is given by $V$.

By breaking $F$ into component parts in two perpendicular directions, we can analyse the effect of the vector in one or both of these directions.

**WORKED EXAMPLE 12** Write the horizontal and vertical components of a vector of magnitude 5 and angle of $120^\circ$ with the positive $x$-axis.

**THINK**

1. Represent the vector on the Cartesian plane.

2. Construct a right-angled triangle with the vector as the hypotenuse and the other sides $H$ for horizontal and $V$ for vertical.

3. Calculate the angle between the vector and the $x$-axis and indicate it on the graph.

4. Calculate $V$ using the sine ratio.

**DRAW/WRITE**

Angle $= 180^\circ - 120^\circ$

$= 60^\circ$

$$\sin (60^\circ) = \frac{V}{5}$$

$$V = 5 \sin (60^\circ) = \frac{5\sqrt{3}}{2} \text{ (or 4.33)}$$
5 Calculate $H$ using the cosine ratio.

\[
\cos (60^\circ) = \frac{H}{5} \\
H = 5 \cos (60^\circ) \\
= \frac{5}{2} \text{ (or 2.5)}
\]

6 State the solution, adding negative signs where necessary.

The vector has a horizontal component of $-\frac{5}{2}$ and a vertical component of $\frac{5\sqrt{3}}{2}$.

---

**THINK**

a 1 Draw a vector diagram representing the motion of the car. Call the vector $\vec{a}$ and its eastern and northern components $\vec{e}$ and $\vec{n}$ respectively.

2 Calculate $n$ (the magnitude of $\vec{n}$) using the cosine ratio.

\[
\cos (30^\circ) = \frac{n}{12} \\
\Rightarrow n = 12 \cos (30^\circ) \\
\Rightarrow n = 10.4
\]

The car has travelled to approximately 10.4 km north of its starting point.

b 1 Calculate $e$ (the magnitude of $\vec{e}$) using the sine ratio.

\[
\sin (30^\circ) = \frac{e}{12} \\
\Rightarrow e = 12 \sin (30^\circ) \\
\Rightarrow e = 6
\]

The car has travelled to 6 km east of its starting point.

---

**WORKED EXAMPLE 13**

A car travels 12 km in a direction N30°E. From its starting point, how far has it travelled:

a north

b east?

---

**EXERCISE 8.4**

**Magnitude, direction and components of vectors**

1 Calculate the magnitude and direction, relative to the positive $x$-axis, of the following displacements.

a (6, 2) b (4, -1) c (2, 4) d (1, 1)
2 Calculate the magnitude and direction, relative to the positive x-axis, of the following displacements.
   \( \text{a} \ (-2, 1) \quad \text{b} \ (-1, 4) \quad \text{c} \ (1, 0) \quad \text{d} \ (-2, -2) \)

3 \textbf{WE12} Write the horizontal and vertical components of these vectors. Write your answers in exact form where possible.
   \( \text{a} \) Magnitude 2, angle of 60° with the x-axis
   \( \text{b} \) Magnitude 3, angle of 150° with the x-axis
   \( \text{c} \) Magnitude 10, angle of −60° with the x-axis
   \( \text{d} \) Magnitude 2, angle of −120° with the x-axis
   \( \text{e} \) Magnitude 20, angle of 45° with the x-axis
   \( \text{f} \) Magnitude 4, parallel to the y-axis
   \( \text{g} \) Magnitude 12, parallel to the x-axis

4 Write the horizontal and vertical components of these vectors. Write your answers in exact form where possible.
   \( \text{a} \) A speed of 30 m/s vertically downwards
   \( \text{b} \) A move of 10 m to the left at an angle of 30° downwards from the x-axis
   \( \text{c} \) A move of 20 m to the right at an angle of 30° upwards from the x-axis
   \( \text{d} \) A speed of 50 m/s horizontally to the right
   \( \text{e} \) A force of 40 N at an angle of 20° to the horizontal
   \( \text{f} \) A force of 98 N vertically downwards
   \( \text{g} \) A force of 1250 N at an angle of 15° to the horizontal.

5 \textbf{WE13} A yacht sails 32 km in a direction S25°E. From its starting point how far has it travelled:
   \( \text{a} \) south \qquad \text{b} \ east?

6 A car travels 20 km in a direction N45°W. From its starting point, how far has it travelled:
   \( \text{a} \) north \qquad \text{b} \ west?

7 Refer to the diagram of the cube shown. If the sides of the cube are 1 unit in length, write the magnitudes of these vectors in exact form.
   \( \text{a} \) \( \overrightarrow{OA} \)
   \( \text{b} \) \( \overrightarrow{AB} \)
   \( \text{c} \) \( \overrightarrow{OB} \)
   \( \text{d} \) \( \overrightarrow{OD} \)
   \( \text{e} \) \( \overrightarrow{AD} \)
   \( \text{f} \) \( \overrightarrow{DF} \)
   \( \text{g} \) \( \overrightarrow{OE} \)
   \( \text{h} \) \( \overrightarrow{EF} \)
   \( \text{i} \) \( \overrightarrow{OF} \)
   \( \text{j} \) \( \overrightarrow{AG} \)

8 A vector has a horizontal component of \(-x \ (x > 0)\) and a vertical component of \(y \ (y > 0)\). Write the magnitude and direction from the positive x-axis of the vector.

9 Find the magnitude and direction of each of the following vectors. Express the direction relative to the positive x-axis.
10 Write the horizontal and vertical components of a vector of magnitude 30 on an angle of 310° with the positive x-axis. Give answers correct to 1 decimal place.

11 For each of the following, find:
   i the magnitude of the vector
   ii the direction of each vector. (Express the direction with respect to the positive x-axis)
   a \((6, 6)\)
   b \((-4, 7)\)
   c \((-3.4, -3.5)\)
   d \((320, -10)\)

12 Using the vector shown at right, find:
   a the magnitude of \(\vec{u}\)
   b the direction of \(\vec{u}\) (express the angle with respect to the positive x-axis)
   c the true bearing of \(\vec{u}\).

13 A vector with a bearing of 60 degrees from N and a magnitude of 10 has:
   A \(x\)-component = \(\frac{\sqrt{3}}{2}\), \(y\)-component = \(\frac{1}{2}\)
   B \(x\)-component = \(\frac{1}{2}\), \(y\)-component = \(\frac{\sqrt{3}}{2}\)
   C \(x\)-component = \(5\sqrt{3}\), \(y\)-component = 5
   D \(x\)-component = 5, \(y\)-component = \(5\sqrt{3}\)
   E none of the above

14 Consider the vector \(\vec{w}\), shown on the right, with a magnitude of 100 and on a bearing of 210°T. Find the \(x\) and \(y\) components of \(\vec{w}\). Express answers as exact values.

15 Justine cycles 8 km in a northerly direction. She then travels 6 km in an easterly direction. Calculate the magnitude and direction of her displacement.

16 Express the horizontal and vertical components of a vector represented by a ship that sails on a bearing of 331° for 125 km.

17 For the following pairs of vectors, calculate the magnitude and direction of \(a + b\).
   a \(a = 10\) km north and \(b = 6\) km north-east
   b \(a = 25\) units east and \(b = 20\) units S30°W
   c \(a = 10\) units and \(b = 8\) units in the opposite direction
18 For the following pairs of vectors, calculate the magnitude and direction of \( \mathbf{a} + \mathbf{b} \) and \( \mathbf{a} - \mathbf{b} \).

a \( \mathbf{a} = 12 \text{ km west} \) and \( \mathbf{b} = 12 \text{ km south} \)

b \( \mathbf{a} = 20 \text{ km} \) and \( \mathbf{b} = 15 \text{ km} \) in the same direction

c \( \mathbf{a} = 50 \text{ units} \) in a direction \( 300^\circ \text{T} \) and \( \mathbf{b} = 40 \text{ units} \) in a direction \( 30^\circ \text{T} \)

8.5 \( \mathbf{i}, \mathbf{j} \) notation

Unit vectors

1. A unit vector is any vector with a magnitude or length of 1 unit.
2. The vector \( \mathbf{i} \) is defined as the unit vector in the positive x-direction.
3. The vector \( \mathbf{j} \) is defined as the unit vector in the positive y-direction.

For example, a displacement of \( \mathbf{d} = (2, 5) \) represents a move of 2 units in the positive x-direction and 5 units in the positive y-direction.

An alternative way of representing this is:

\[ \mathbf{d} = 2 \mathbf{i} + 5 \mathbf{j} \]

Any vector in two dimensions can be represented as a combination of \( \mathbf{i} \) and \( \mathbf{j} \) vectors, the coefficient of \( \mathbf{i} \) representing the magnitude of the horizontal component and the coefficient of \( \mathbf{j} \) representing the magnitude of the vertical component.

In general we may represent any two-dimensional vector \( \mathbf{r} \) as:

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} \]

where \( x, y \in \mathbb{R} \)

This vector, \( \mathbf{r} \), may also be written as a column matrix:

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

where \( \mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

For example,

\[ \mathbf{d} = 2 \mathbf{i} + 5 \mathbf{j} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

---

**WORKED EXAMPLE 14**

a Draw a vector to represent \( \mathbf{a} = 3\mathbf{i} - \mathbf{j} \).

b Find the magnitude and direction of the vector \( \mathbf{a} \).

**THINK**

a 1 Draw axes with \( \mathbf{i} \) and \( \mathbf{j} \) as unit vectors in the x- and y-directions respectively.

2 Represent \( 3 \mathbf{i} - \mathbf{j} \) as a vector from 0 that is 3 units in the positive x-direction and 1 unit in the negative y-direction, and mark the angle between \( \mathbf{a} \) and the x-axis as \( \theta \).

**DRAW/WRITE**

a [Diagram showing \( \mathbf{i} \) and \( \mathbf{j} \) with vector \( \mathbf{a} = 3\mathbf{i} - \mathbf{j} \)]
As we have seen, angles are usually given with respect to the positive $x$-direction. We may generalise this procedure:

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$r = x\mathbf{i} + y\mathbf{j}$</td>
</tr>
<tr>
<td>2.</td>
<td>magnitude $r$, $</td>
</tr>
<tr>
<td>3.</td>
<td>the direction from the positive $x$-axis is given by appropriately adjusting $\theta$ where $\tan(\theta) = \frac{y}{x}$.</td>
</tr>
</tbody>
</table>

Addition, subtraction and multiplication by a scalar for a vector in $i$, $j$ form follow the rules of normal arithmetic, with each component treated separately.

\[
\begin{align*}
\mathbf{a} + \mathbf{b} & = (x_1\mathbf{i} + y_1\mathbf{j}) + (x_2\mathbf{i} + y_2\mathbf{j}) \\
& = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} \\
\mathbf{a} - \mathbf{b} & = (x_1\mathbf{i} + y_1\mathbf{j}) - (x_2\mathbf{i} + y_2\mathbf{j}) \\
& = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} \\
k\mathbf{a} & = k(x_1\mathbf{i} + y_1\mathbf{j})
\end{align*}
\]

WORKED EXAMPLE 15

If $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$, express in $i$, $j$ form:

<table>
<thead>
<tr>
<th>Part</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\mathbf{a} + \mathbf{b}$</td>
</tr>
<tr>
<td>b</td>
<td>$2\mathbf{a} - \mathbf{b}$</td>
</tr>
</tbody>
</table>

**THINK**

a. Add the $\mathbf{i}$ components and $\mathbf{j}$ components separately.

b. $2\mathbf{a}$ is calculated by multiplying the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{a}$ by 2.

2. $2\mathbf{a} - \mathbf{b}$ is calculated by subtracting the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{b}$ respectively from $2\mathbf{a}$.

**WRITE**

a. $\mathbf{a} + \mathbf{b} = (3\mathbf{i} + \mathbf{j}) + (-2\mathbf{i} + 5\mathbf{j}) = 3\mathbf{i} - 2\mathbf{i} + \mathbf{j} + 5\mathbf{j} = \mathbf{i} + 6\mathbf{j}$

b. $2\mathbf{a} = 2(3\mathbf{i} + \mathbf{j}) = 6\mathbf{i} + 2\mathbf{j}$

$2\mathbf{a} - \mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - (-2\mathbf{i} + 5\mathbf{j}) = 6\mathbf{i} + 2\mathbf{j} + 2\mathbf{i} - 5\mathbf{j} = 8\mathbf{i} - 3\mathbf{j}$
Multiplying two vectors

The dot product is one method of multiplying one vector by another vector. It is also called a scalar product as the result of this multiplication is a scalar (magnitude only). The product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$.

Consider the two vectors $\mathbf{u}$ and $\mathbf{v}$, as shown.

**By definition, the dot product $\mathbf{u} \cdot \mathbf{v}$ is given by:**

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos (\theta)$$

where $\theta$ is the angle between (the positive directions of) $\mathbf{u}$ and $\mathbf{v}$.

**Note:** The vectors are not aligned as for addition or subtraction; instead their two tails are joined.
Let $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 6\mathbf{i}$. Find $\mathbf{u} \cdot \mathbf{v}$.

**THINK**

1. Find the magnitudes of and $\mathbf{u}$ and $\mathbf{v}$.
   - $|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$
   - $|\mathbf{v}| = \sqrt{6^2} = 6$

2. Draw a right-angled triangle showing the angle that $\mathbf{u}$ makes with the positive $x$-axis since $\mathbf{v}$ is along the $x$-axis.

3. Find $\cos \theta$, knowing that $\mathbf{u} = 5$ and the $x$-component of $\mathbf{u}$ is 3.
   - $\cos (\theta) = \frac{3}{5}$

4. Find $\mathbf{u} \cdot \mathbf{v}$ using equation 1.
   - $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \times |\mathbf{v}| \times \cos (\theta)$
   - $= 5 \times 6 \times \frac{3}{5}$
   - $= 18$

**WORKED EXAMPLE 18**

Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$.

**THINK**

1. Write down $\mathbf{u} \cdot \mathbf{v}$.
   - $\mathbf{u} \cdot \mathbf{v} = (2\mathbf{i} + 5\mathbf{j}) \cdot (3\mathbf{i} + \mathbf{j})$

2. Multiply the corresponding components.
   - $\mathbf{u} \cdot \mathbf{v} = 2 \times 3 + 5 \times 1$
   - $= 11$

**Write**

- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \times |\mathbf{v}| \times \cos (90^\circ)$
- $= |\mathbf{u}| \times |\mathbf{v}| \times 0$ since $\cos (90^\circ) = 0$
- $= 0$

**Perpendicular vectors**

If two vectors are perpendicular then the angle between them is $90^\circ$.

- $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.
### Parallel vectors

If two vectors are parallel (\( \parallel \)) then the angle between them is either 0° (if acting in the same direction) or 180° (if acting in opposite directions).

For \( \theta = 0^\circ \) (same direction)
\[
\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta) = |\mathbf{u}| |\mathbf{v}| \cos(0^\circ) = |\mathbf{u}| |\mathbf{v}| \quad \text{as } \cos(0^\circ) = 1
\]

For \( \theta = 180^\circ \) (opposite directions)
\[
\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta) = |\mathbf{u}| |\mathbf{v}| \cos(180^\circ) = -|\mathbf{u}| |\mathbf{v}| \quad \text{as } \cos(180^\circ) = -1
\]

---

**WORKED EXAMPLE 19**

Find the constant \( a \) if the vectors \( \mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \) and \( \mathbf{v} = -3\mathbf{i} - a\mathbf{j} \) are perpendicular.

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Find the dot product.</td>
<td>( \mathbf{u} \cdot \mathbf{v} = (4\mathbf{i} + 3\mathbf{j}) \cdot (-3\mathbf{i} - a\mathbf{j}) )</td>
</tr>
<tr>
<td>2 Simplify.</td>
<td>( = -12 - 3a )</td>
</tr>
<tr>
<td>3 Set ( \mathbf{u} \cdot \mathbf{v} ) equal to zero since ( \mathbf{u} ) and ( \mathbf{v} ) are perpendicular.</td>
<td>( \mathbf{u} \cdot \mathbf{v} = -12 - 3a = 0 )</td>
</tr>
<tr>
<td>4 Solve the equation for ( a ).</td>
<td>( a = -4 )</td>
</tr>
</tbody>
</table>

---

**WORKED EXAMPLE 20**

Let \( \mathbf{u} = 5\mathbf{i} + 2\mathbf{j} \). Find a vector parallel to \( \mathbf{u} \) such that the dot product is 87.

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 The dot product is positive so the vectors are in the same direction, with ( \theta = 0^\circ ).</td>
<td>( \mathbf{u} \cdot \mathbf{v} =</td>
</tr>
<tr>
<td>2 Find the magnitude of ( \mathbf{u} ).</td>
<td>( \therefore</td>
</tr>
<tr>
<td>3 Substitute (</td>
<td>\mathbf{u}</td>
</tr>
<tr>
<td></td>
<td>( = \sqrt{25 + 4} )</td>
</tr>
<tr>
<td></td>
<td>( = \sqrt{29} )</td>
</tr>
<tr>
<td></td>
<td>( \therefore</td>
</tr>
<tr>
<td></td>
<td>( = 3\sqrt{29} )</td>
</tr>
</tbody>
</table>
Finding the angle between two vectors

The dot product formula can be used to find the angle between two vectors.

\[ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta) \]

By re-arranging this formula, we get:

\[ \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \]

\[ \therefore \theta = \cos^{-1}\left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) \]

WORKED EXAMPLE 21 Let \( \mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \) and \( \mathbf{v} = 2\mathbf{i} - 3\mathbf{j} \). Find the angle between them to the nearest degree.

**THINK**

1. Find the dot product \( \mathbf{u} \cdot \mathbf{v} \).

   \[ \mathbf{u} \cdot \mathbf{v} = (4\mathbf{i} + 3\mathbf{j}) \cdot (2\mathbf{i} - 3\mathbf{j}) \]

   \[ = 4 \times 2 + 3 \times -3 \]

   \[ = -1 \]

2. Find the magnitude of \( \mathbf{u} \).

   \[ |\mathbf{u}| = \sqrt{4^2 + 3^2} \]

   \[ = \sqrt{25} \]

   \[ = 5 \]

3. Find the magnitude of \( \mathbf{v} \).

   \[ |\mathbf{v}| = \sqrt{2^2 + (-3)^2} \]

   \[ = \sqrt{13} \]

4. Substitute results into the formula for the angle between two vectors.

   \[ \theta = \cos^{-1}\left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) \]

   \[ = \cos^{-1}\left( \frac{-1}{5\sqrt{13}} \right) \]

   \[ = \cos^{-1}(0.05547) \]

   \[ = 93.1798^\circ \]

   \[ \approx 93^\circ \]

**WRITE**

5. Answer to the nearest degree.

**EXERCISE 8.5 \( \mathbf{i}, \mathbf{j} \) notation**

**PRACTISE**

1. **WE14** Draw a vector to represent each of the following.

   a. \( 4\mathbf{i} + 3\mathbf{j} \)
   b. \( 4\mathbf{i} - 3\mathbf{j} \)
   c. \( 2\mathbf{i} + 2\mathbf{j} \)
   d. \( \mathbf{i} - \mathbf{j} \)
   e. \( 4\mathbf{i} + \mathbf{j} \)
   f. \( 5\mathbf{i} \)
   g. \( -6\mathbf{j} \)
   h. \( -2\mathbf{i} \)
   i. \( -8\mathbf{i} - 6\mathbf{j} \)
   j. \( -5\mathbf{i} + 12\mathbf{j} \)
2 Calculate the magnitude and direction of each of the vectors in question 1.

3 If \( a = 3i + 2j \), \( b = i - j \) and \( c = -2j \), find the following in \( i, j \) form.
   \( a \begin{align*} 3a & \quad b \quad a + b \quad c \quad a - c \quad d \quad 2b \\ a + b + c & \quad f \quad 2b - c \quad g \quad 3a + 2b + c \quad h \quad 4c \\ i \quad 4c - a & \quad j \quad 3c - a - b \end{align*} \)

4 If \( u = 2i - 3j \) and \( v = 3i + j \), find the following in exact form.
   \( a \begin{align*} |u| & \quad b \quad |u + v| \quad c \quad |3v| \quad d \quad |u - v| \\ |u| & \quad |u + v| \quad |3v| \quad |u - v| \end{align*} \)

5 \( \overrightarrow{OA} = 2i - j \) and \( \overrightarrow{OB} = 4i + 3j \).
   \( a \) Represent \( \overrightarrow{OA} \) and \( \overrightarrow{OB} \) on a diagram.
   \( b \) Find, in terms of \( i \) and \( j \), the vector \( \overrightarrow{AB} \).
   \( c \) If \( M \) is the midpoint of \( \overrightarrow{AB} \), find the vector \( \overrightarrow{OM} \) in terms of \( i \) and \( j \).

6 OACB is a rectangle in which the vector \( \overrightarrow{OA} = 4i \) and \( \overrightarrow{OB} = 6j \). Express the following in terms of \( i \) and \( j \).
   \( a \) \( \overrightarrow{OC} \)
   \( b \) \( \overrightarrow{OM} \) where \( M \) is the midpoint of \( \overrightarrow{OA} \)
   \( c \) \( \overrightarrow{AC} \)
   \( d \) \( \overrightarrow{ON} \) where \( N \) is the midpoint of \( \overrightarrow{OB} \)
   \( e \) \( \overrightarrow{AB} \)
   \( f \) \( \overrightarrow{MN} \)

7 Find the dot product of \( u = 2i + 4j \) and \( v = i + 5j \).

8 Let \( u = 3i - 2j \) and \( v = -i - 3j \). Find the dot product of \( u \) and \( v \).

9 Let \( u = i + 6j \) and \( v = -2i - j \). Find \( u \cdot v \).

10 Find \( u \cdot v \), when \( u = -4i + 3j \) and \( v = 3i + 2j \).

11 Find the constant \( a \) if the vectors \( u = 2i + 7j \) and \( v = -7i - aj \) are perpendicular.

12 Find the constant \( a \), if the vectors \( v = ai + 3j \) and \( u = 6i - 2j \) are perpendicular.

13 Let \( u = 2i - 4j \). Find a vector parallel to \( u \) such that their dot product is 80.

14 Let \( u = 4i - 3j \). Find a vector parallel to \( u \) such that their dot product is 40.

15 Let \( u = 3i + j \) and \( v = 2i - 5j \). Find the angle between them to the nearest degree.

16 Find the angle, to the nearest degree, between the vectors \( i - 6j \) and \( 4i + 3j \).

17 Represent the following position vectors in the form \( xi + yj \).

\[
\begin{array}{c|c|c|c}
& (1, 3) & (2, 3) & \varepsilon \\
6 & 4 & 3 & b \\
5 & 4 & 3 & g \\
4 & 3 & 2 & \eta \\
3 & 2 & 1 & \iota \\
2 & 1 & 0 & \kappa \\
1 & 0 & -1 & \lambda \\
0 & -1 & -2 & \mu \\
-1 & -2 & -3 & \nu \\
-2 & -3 & -4 & \omega \\
-3 & -4 & -5 & \sigma \\
-4 & -5 & -6 & \tau \\
\end{array}
\]
18 The position of the points A, B and C is defined by: $\overrightarrow{OA} = 4\mathbf{i}, \overrightarrow{OB} = 10\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OC} = 4\mathbf{i} + 4\mathbf{j}$.

   a Find the vectors representing the three sides of the triangle ABC (that is, find in $\mathbf{i}, \mathbf{j}$ form the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$).
   b Calculate the magnitude of these three sides. Leave answers in exact form.
   c What type of triangle is ABC?

19 M, N and P are three points defined by: $\overrightarrow{OM} = -\mathbf{i} + \mathbf{j}$, $\overrightarrow{ON} = \mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{OP} = 5\mathbf{i} + 10\mathbf{j}$.

   a Find $\overrightarrow{MN}$ and $\overrightarrow{NP}$.
   b Show that $\overrightarrow{MN}$ and $\overrightarrow{NP}$ are parallel vectors.

20 $a = 4\mathbf{i} - 2\mathbf{j}$ and $b = -3\mathbf{i} + \mathbf{j}$.

   a Find $3a - 2b$ and $3a + 4b$.
   b Explain why $3a + 4b$ is parallel to the y-axis.

21 The magnitude of the vector $\sqrt{2}\mathbf{i} + 2\mathbf{j}$ is:

   A $\sqrt{2} + 2$
   B $2\sqrt{2}$
   C $\sqrt{6}$
   D $2$
   E $\sqrt{\sqrt{2} + 2}$

22 If $a = 3\mathbf{i} - 5\mathbf{j}$ and $b = -3\mathbf{i} - 2\mathbf{j}$, then $a - 2b$ equals:

   A $9\mathbf{i} - \mathbf{j}$
   B $9\mathbf{i} + \mathbf{j}$
   C $-3\mathbf{i} - \mathbf{j}$
   D $-3\mathbf{i} + \mathbf{j}$
   E $-4\mathbf{i} - 9\mathbf{j}$

23 The angle the vector $3\mathbf{i} - 4\mathbf{j}$ makes with the positive x-axis is nearest to:

   A $37^\circ$
   B $53^\circ$
   C $-53^\circ$
   D $-37^\circ$
   E $-127^\circ$

24 Find the vector $a + b$, which represents the planned shot of a pool player.

25 Vector $\mathbf{m} = 12\mathbf{i} + x\mathbf{j}$. The magnitude of $\mathbf{m}$ is 13. Find the value of $x$.

26 Find a vector perpendicular to $3\mathbf{i} - 6\mathbf{j}$.

27 Let $u = 4\mathbf{i} + 3\mathbf{j}$ and $v = \theta\mathbf{i} + 2\mathbf{j}$. Find $\cos(\theta)$, where $\theta$ is the angle between the two vectors.

   Give answer in exact form.

28 Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ below. Their magnitudes are 7 and 8 respectively. Find $\mathbf{u} \cdot \mathbf{v}$.

8.6 Applications of vectors

Vectors have a wide range of applications, such as in orienteering, navigation, mechanics and engineering. Vectors are applied whenever quantities specified by both magnitude and direction are involved.
When solving problems involving vectors:
1. Draw a vector diagram depicting the situation described.
2. Use the appropriate skills to answer the question being asked.

**WORKED EXAMPLE 22**

A boat is being rowed straight across a river at a speed of 6 km/h. The river is flowing at 2 km/h. If \( \mathbf{i} \) is the unit vector in the direction that the river is flowing and \( \mathbf{j} \) is the unit vector in the direction straight across the river, represent the velocity of the boat in terms of \( \mathbf{i} \) and \( \mathbf{j} \). Hence, find the magnitude and direction of the velocity of the boat, correct to 1 decimal place.

**THINK**

1. Draw a set of axes with \( \mathbf{i} \) in the direction of the positive \( x \)-axis and \( \mathbf{j} \) in the direction of the positive \( y \)-axis.

2. Indicate the velocity vector of the boat, \( \mathbf{v} \), starting at \( O \) and finishing at the point (2, 6).

3. Represent the velocity of the boat in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

4. The magnitude of \( \mathbf{v} \) is \( \sqrt{2^2 + 6^2} \).

5. Evaluate the magnitude correct to 1 decimal place.

6. Draw a right-angled triangle with \( \mathbf{v} \) as the hypotenuse and \( \theta \) as the angle between \( \mathbf{v} \) and the \( \mathbf{i} \) direction.

7. Express \( \theta \) using the tangent ratio.

8. Evaluate \( \theta \) correct to 1 decimal place.

9. State the magnitude and direction of the velocity of the boat.

**DRAW/WRITE**

\[ \mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \]

\[ |\mathbf{v}| = \sqrt{2^2 + 6^2} \]

\[ = \sqrt{40} \]

\[ \approx 6.3 \]

\[ \tan (\theta) = \frac{6}{2} \]

\[ = 3 \]

\[ \theta = 71.6^\circ \]

The velocity of the boat has a magnitude of approximately 6.3 km/h and is directed at approximately 71.6° from the riverbank.

**Note:** The magnitude of velocity is referred to as *speed*. 
An aircraft is heading north with an airspeed of 500 km/h. A wind of 80 km/h is blowing from the south-west. Using \( \hat{i} \) and \( \hat{j} \) as unit vectors in the directions east and north respectively:

\( \text{a} \) represent the aircraft’s air velocity in terms of \( \hat{i} \) and \( \hat{j} \)

\( \text{b} \) represent the aircraft’s exact ground velocity, \( \vec{v} \), in terms of \( \hat{i} \) and \( \hat{j} \)

\( \text{c} \) find the direction in which the aircraft is heading and its ground speed.

**THINK**

\( \text{a} \) Express \( \vec{a} \) in terms of \( \hat{i} \) and \( \hat{j} \).

\( \text{b} \) Draw a set of axes with \( \hat{i} \) in the direction of the positive x-axis and \( \hat{j} \) in the direction of the positive y-axis.

\( \text{2} \) Indicate the vector representing the aircraft’s airspeed, \( \vec{a} \), starting at O and finishing at the point (0, 500).

\( \text{3} \) Indicate the vector representing the wind speed, \( \vec{w} \), by placing its tail at the head of the first vector, directed in a direction 45° from the north with a magnitude of 80, as the wind speed is 80 km/h from the south-west.

\( \text{4} \) Represent the combined effect of the two speeds with a vector, \( \vec{v} \), using the triangle rule.

\( \text{5} \) Express \( \vec{w} \), exactly, in terms of \( \hat{i} \) and \( \hat{j} \) using basic trigonometry.

\( \text{6} \) Express the aircraft’s ground velocity, \( \vec{v} \), as the sum of \( \vec{a} \) and \( \vec{w} \).

\( \text{7} \) Express \( \vec{v} \) in terms of \( \hat{i} \) and \( \hat{j} \).

\( \text{c} \) 1 Indicate the angle between \( \vec{v} \) and the y-axis as \( \theta \).

\( \text{2} \) Use the tangent ratio to evaluate \( \theta \) to 1 decimal place. The length of the horizontal component of \( \vec{v} \) is 40√2. The length of the vertical component of \( \vec{v} \) is 500 + 40√2.

\( \text{3} \) Calculate the magnitude of \( \vec{v} \) correct to 1 decimal place.

\( \text{4} \) State the direction and magnitude of the ground speed of the aircraft.

**WRITE/DRAW**

\( \text{a} \ \vec{a} = 500\hat{j} \)

\( \text{b} \)

\( \vec{a} \)

\( \vec{w} = 80 \sin (45°)\hat{i} + 80 \cos (45°)\hat{j} \)

\( = 40\sqrt{2}\hat{i} + 40\sqrt{2}\hat{j} \)

\( \vec{v} = \vec{a} + \vec{w} \)

\( = 40\sqrt{2}\hat{i} + (500 + 40\sqrt{2})\hat{j} \)

\( \text{c} \)

\( \tan (\theta) = \frac{40\sqrt{2}}{500 + 40\sqrt{2}} \)

\( \approx 0.1016 \)

\( \theta = 5.8° \)

\( |\vec{v}| = \sqrt{(40\sqrt{2})^2 + (500 + 40\sqrt{2})^2} \)

\( \approx 559.4 \)

The aircraft is flying with a ground speed of approximately 559.4 km/h in a N5.8°E direction.
Statics

When the vector sum of the forces acting on a stationary particle is zero, the situation is said to be static and the particle will remain stationary. The particle is also said to be in equilibrium. In the case of two forces, we have the situation shown at right.

In the case of three forces, we have the situation shown in the diagram below left. Where the three forces are acting so that the particle is in equilibrium, the lines representing the forces can be rearranged into a triangle of forces (as in the diagram below right) because their vector sum is zero. Hence, problems can be solved using trigonometry (including the sine rule and cosine rule) and sometimes Pythagoras’ theorem.

Note: The three forces are still acting in the same direction and have the same magnitudes (or lengths) as they did in the ‘real’ situation.

WORKED EXAMPLE 24

Three forces are acting on the particle P as shown in the diagram below. Force A is vertically up and has a magnitude of 20 N (20 newtons); force B is horizontally to the right and has a magnitude of 40 N. If the particle is in equilibrium, find the magnitude of force C to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.

THINK
1. Draw the three forces as a triangle of forces.
2. Label the angle between forces A and C as \( \theta \).
3. Calculate \( |C| \) using Pythagoras’ theorem.
   \[ |C|^2 = |A|^2 + |B|^2 \]
   \[ = 20^2 + 40^2 \]
   \[ = 400 + 1600 \]
   \[ = 2000 \]
   \[ |C| = \sqrt{2000} \]
4. Evaluate \( |C| \) correct to 1 decimal place.
   \[ |C| = 44.7 \text{ newtons} \]
Geometric proofs

Vectors can also be used to prove a range of geometric theorems. From earlier in the chapter, you will remember that two vectors are equal if they are equal in magnitude, are parallel and point in the same direction. One important vector property that is useful in geometric proofs is that if $a = kb$, where $k \in R$ ($k \neq 0$), then the two vectors, $a$ and $b$ are parallel.

Applications of vectors

1 WE22 A boat is being rowed straight across a river at a speed of 7 km/h. The river is flowing at 2.5 km/h. If $i$ is the unit vector in the direction that the river is flowing and $j$ is the unit vector in the direction straight across the river, represent the velocity of the boat in terms of $i$ and $j$. Hence, find the magnitude and direction of the velocity of the boat.
2 A boat is being rowed straight across a river at a speed of 10 km/h. The river is flowing at 3.4 km/h. Find the magnitude and direction of the velocity of the boat.

3 An aircraft is heading north with an airspeed of 650 km/h. A wind of 60 km/h is blowing from the south-west. Using \( \hat{i} \) and \( \hat{j} \) as unit vectors in the directions east and north respectively:
   a represent the aircraft’s airspeed
   b represent the aircraft’s ground speed in terms of \( \hat{i} \) and \( \hat{j} \)
   c find the direction in which the aircraft is heading and its ground speed.

4 An aircraft is heading south with an airspeed of 600 km/h. A wind of 50 km/h is blowing in a S30\(^\circ\)W direction. Find the direction in which the aircraft is heading and the ground speed.

5 Three coplanar forces are acting on the particle \( P \) as shown. Force \( A \) is vertically up and has magnitude of 16 N; force \( B \) is horizontally to the right and has a magnitude of 28 N. If the particle is in equilibrium, find the magnitude of force \( C \) to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.

6 Three coplanar forces are acting on the particle \( P \) as shown. Force \( A \) has a magnitude of 35 N, and force \( B \) has a magnitude of 40 N. If the particle is in equilibrium, find the magnitude of force \( C \) to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.

7 PQR is a triangle in which \( M \) is the midpoint of \( QR \). Prove that \( \overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PR} - \overrightarrow{QP}) \).

8 Prove that if the midpoints \( E, F, G \) and \( H \) of a rhombus \( ABCD \) are joined, then a parallelogram \( EFGH \) is formed. (Extension: Show that the parallelogram is, in fact, a rectangle.)

9 Forces of \( 3\hat{i} + 4\hat{j} \) and \( 2\hat{i} + 2\hat{j} \) act simultaneously on an object. Find the magnitude and direction of the resultant of the two forces.

10 Forces of \( 5\hat{i} - 4\hat{j} \), \( 3\hat{i} - \hat{j} \) and \( -2\hat{i} + 3\hat{j} \) act simultaneously on an object. Find the magnitude and direction of the resultant of the three forces.

11 A hiker is located at a position given by \( (8, 6) \), where the coordinates represent the distances in kilometres east and north of \( O \) respectively. If a campsite is at a position given by \( (3, 2) \), find the distance and direction of the hiker from the campsite.

12 A hiker is located at a position given by \( (-5, 3) \), where the coordinates represent the distances in kilometres east and north of \( O \) respectively. If a campsite is at a position given by \( (3, -2) \), find the distance and direction of the hiker from the campsite.
13 The position vectors for an arrow and a moving target are shown at right, where $t$ is the time in seconds since the target began to move, and $h$ is the height of the target. If the arrow is to hit the target, when must this happen and what must the value of $h$ be for this to occur?

14 Forces of $-2\hat{i} + 3\hat{j}$, $4\hat{i} - 5\hat{j}$, $x\hat{i} + \hat{j}$ and $3\hat{i} - y\hat{j}$ act on a particle that is in equilibrium. Find the values of $x$ and $y$.

15 A river flows through the jungle from west to east at a speed of 3 km/hr. An explorer wishes to cross the river by boat, and attempts this by travelling at 5 km/hr due north. Using $\hat{i}$ and $\hat{j}$ as unit vectors in the directions east and north respectively:
   a express the velocity of the river and the velocity of the boat in terms of $\hat{i}$ and $\hat{j}$
   b draw vectors represented by the velocity of the river and the velocity of the boat
   c calculate the magnitude of the resultant vector
   d find the bearing of the boat’s journey, correct to the nearest degree.

16 Boat A travels east at 20 km/hr, while Boat B travels south from the same point at 15 km/hr. Find the velocity of Boat A with respect to Boat B.

17 A river flows west–east at 5 m/s. A swimmer, in still water, can swim 3 m/s and tries to swim directly across the river from south to north.
   a Draw a vector diagram to illustrate this situation.
   b Find the resultant speed of the swimmer correct to 1 decimal point.
   c Find the bearing of the swimmer correct to the nearest degree.
   d If it took the swimmer 2 minutes to reach the opposite bank, how wide is the river?
   e How far downstream would the swimmer be carried?

18 In the drawing at right, ABC is a triangle. Point D is along the line BC such that $BD = \frac{1}{3}BC$. The vectors $\vec{q}$, $\vec{r}$ and $\vec{t}$ are as shown in the diagram. Prove that: $\vec{t} = \frac{1}{3}(2\vec{q} + \vec{r})$.

19 A bushwalker starts walking at 8.00 am from a campsite at $(-4, 8)$, where the coordinates represent the distances in kilometres east and north of O respectively. After 1 hour she is at $(-2, 6.5)$. Take $\hat{i}$ and $\hat{j}$ as unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$.
   a Write, in terms of $\hat{i}$ and $\hat{j}$, her position at the start and after 1 hour.
   b Calculate the distance travelled in 1 hour.
c She then continues at the same rate and in the same direction. What is her position vector after:
   i 2 hour
   ii 3 hours?
d Show that her position \( t \) hours after 8.00 am is given by:
   \[
   \mathbf{r}_1 = (-4 + 2t)\mathbf{i} + (8 - 1.5t)\mathbf{j}
   \]
Another bushwalker commences walking from his campsite, also at 8.00 am. His position is given by:
   \[
   \mathbf{r}_2 = (7.4 - 1.8t)\mathbf{i} + (2 + 0.5t)\mathbf{j}
   \]
e What are the coordinates of this bushwalker’s campsite?
f What is his position after 2 hours of walking?
g By equating \( i \) and \( j \) components, show that the two bushwalker’s meet.
h Find the distance from each campsite that each bushwalker has travelled when they meet.

20 The \( i, j \) system may be extended to three dimensions with a unit vector \( k \) in the \( z \)-direction.
Take \( i, j \) and \( k \) as unit vectors in the directions east, north and vertically up respectively.

Frank travels 2 km in a direction N30°E from \( O \) to a point \( A \). He then climbs a 100 m high cliff.
a Write the vector \( \mathbf{OA} \) in \( i, j \) form.
b Calculate how far Frank has travelled to the north of his starting point.
c If \( T \) represents the top of the cliff, write down the vectors \( \mathbf{AT} \) and \( \mathbf{OT} \) using \( i, j, k \) components.
d Calculate the magnitude of \( \mathbf{OT} \).
The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic. The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods.
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology.
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions. A summary of the key points covered in this topic is also available as a digital document.

**REVIEW QUESTIONS**

Download the Review questions document from the links found in the Resources section of your eBookPLUS.
EXERCISE 8.2
1 a (2, 2)
b (1, -2)
c (-4, -3)
d (-1, -4)
e (-6, -2)

2 a
\[
\begin{bmatrix}
-5 \\
2
\end{bmatrix}
\]
b
\[
\begin{bmatrix}
-1 \\
2
\end{bmatrix}
\]
c
\[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]
d
\[
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]
e
\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

3 (-2, 7)

4 a (5, -7)
b (-1, 0)
c (2, -1)
d (4, -10)

5

6

7 a = b
\[\mathbf{c} = \mathbf{c}\]

8 a = d = c
\[\mathbf{b} = \mathbf{c}\]

9

10

11 a \(q = (1, 1)\)
b \(b = (0, 3)\)
c \(\mathbf{c} = (-1, 2)\)
d \(d = (-1, 2)\)
e \(\mathbf{e} = (-1, -1)\)
f \(f = (2, 4)\)
g \(g = (1, 0)\)
h \(h = (1, 1)\)

12 a
\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
b
\[
\begin{bmatrix}
3 \\
-1
\end{bmatrix}
\]
c
\[
\begin{bmatrix}
2 \\
2
\end{bmatrix}
\]
d
\[
\begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]
e
\[
\begin{bmatrix}
2 \\
4
\end{bmatrix}
\]
f
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
g
\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
h

13 a

b

\[
\begin{bmatrix}
5 \\
0
\end{bmatrix}
\]
c
\[
\begin{bmatrix}
-3 \\
0
\end{bmatrix}
\]
14 a

17 \[ \begin{bmatrix} 4 \\ 3 \end{bmatrix} \]

18 a

Note: Vector can start at any point. Started at (0, 0) and (–2, 1) to draw 2 different line segments for the vector \[ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \].
EXERCISE 8.3

1 a  

b  

c  

d  

e  

f  

2 a  

b  

c  

d  

e  

f  

3 a  

b  

4 a  

b  

5 a  

b  

6 a  

b  

7 a  

b  

c  

d  

e  

8 a  

b  

c  

d  

e  

9 Teacher to check student proofs.

10

11  

or any two vectors equal in length but opposite in direction

12  

or any similar result

13  

or any similar result

14 a  

b  

c  

d  

e  

f  

15 a  

b  

c  

d  

e  

f  

16 Teacher to check student proofs.

17

18 B

19 D

20 a  

b  

c  

d  

e  

21 a  

b  

c  

d  

e  

d  

e  

d  

Teacher to check student proofs.
e $\ell - s$
f $s + \ell - r$
g $r + s + \ell$
h $-s - \ell$

22 a $a + b$
b $a + b + c$
c $a - b$
d $a + b$
e $b + c$

EXERCISE 8.4

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Direction</th>
</tr>
</thead>
</table>
a  $2\sqrt{10}$  | 18.4°           |
b  $\sqrt{17}$   | $-14.0°$        |
c  $2\sqrt{5}$   | 63.4°           |
d  $\sqrt{2}$    | 45°             |

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Direction</th>
</tr>
</thead>
</table>
a  $\sqrt{3}$   | 153.4°          |
b  $\sqrt{17}$   | 104.0°          |
c  1           | Parallel to x-axis |
d  $2\sqrt{2}$   | $-135°$         |

3 a $1, \sqrt{3}$
b $-3\sqrt{3}/2$, 1.5
c 5, $-5\sqrt{3}$
d $-1, -\sqrt{3}$
e $10\sqrt{3}, 10\sqrt{2}$
f 0, 4
g 12, 0

4 a 0, $-30$
b $-5\sqrt{3}, -5$
c $10\sqrt{3}, 10$
d 50, 0
e 37.6, 13.7
f 0, $-98$
g 12074, 3235

5 a 29
b 13.5 km

6 a 14 km
b 14 km

7 a 1
b 1
c $\sqrt{2}$
d 1

e $\sqrt{3}$
f $\sqrt{2}$
g $\sqrt{2}$
h 1
i $\sqrt{3}$
j $\sqrt{3}$

8 $\sqrt{x^2 + y^2}$, angle $\theta$ from the positive direction of the x-axis where $\tan(180° - \theta) = \frac{-y}{x}$

9 a 5, 53°
b 5, $-53°$

10 19.3, $-23$

11 a i $6\sqrt{2}$
ii 45°
b i $\sqrt{65}$
ii 119.7°
c i 4.9
ii $-134.2°$
d i $50\sqrt{11}$
ii $-1.8°$

12 a $\sqrt{34}$
b $-59°$
c 149°

13 C

14 $-50, -50\sqrt{3}$

15 10 km, N36.9°E

16 $-61, 109$

17 a 14.86 km, N16.6°E
b 22.91 units, S40.9°E
c 2 units in direction of $q$

18 a 16.97 km, SW and 16.97 km, NW
b 35 km in direction of $q$ and 5 km in direction of $q$
c 64.03 units, 338.7° and 64.03 units, 261.3°

EXERCISE 8.5

1 a

b

c
2 a $5, 36.9^\circ$
   b $5, -36.9^\circ$
   c $2\sqrt{2}, 45^\circ$
   d $\sqrt{2}, -45^\circ$
   e $\sqrt{17}, 14^\circ$
   f $5, 0^\circ$
   g $6, -90^\circ$
   h $2, 180^\circ$
   i $10, -143.1^\circ$
   j $13, -112.6^\circ$

3 a $9i + 6j$
   b $4i + j$
   c $3i + 4j$
   d $2i - 2j$
   e $4i - j$
   f $2i$
   g $11i + 2j$
   h $-8j$

4 a $\sqrt{13}$
   b $\sqrt{29}$
   c $3\sqrt{10}$
   d $\sqrt{17}$

5 a $2i + 4j$
   b $3i + j$
   c $4i + 6j$
   d $2i$
   e $6i$
   f $3j$
   g $-4i + 6j$
   h $-2i + 3j$

6 a $4i + 6j$
   b $2i$
   c $6j$
   d $3j$
   e $-4i + 6j$
   f $-2i + 3j$

7 22
   8 1
   9 8
   10 6
   11 -2
   12 1
   13 $4i + 8j$
   14 $\frac{64}{7}i - \frac{48}{7}j$
   15 87°
   16 117°

17 a $2i + 3j$
   b $i + 3j$
   c $-4i + 3j$
   d $-6i - 2j$
   e $-2i - 3j$
   f $i - 3j$
   g $6i - 4j$
   h $5i + j$

18 a $6i + 2j, 4j, -6i + 2j$
   b $2\sqrt{10}, 4, 2\sqrt{10}$
   c Isosceles

19 $\overline{MN}, 2i + 3j, \overline{NP}, 4i + 6j$

20 a $18i - 8j, -2j$
   b $i$

21 C
22 A
23 C
24 $3.9i + 2.7j$
25 \( x = \pm 5 \)
26 \( 6i + 3j \) or \( -6i - 3j \)
27 \( \frac{2\sqrt{3}}{25} \)
28 \( -36 \)

**EXERCISE 8.6**
1. \( 2.5i + 7j; 7.43 \text{ km/h; } 70.3^\circ \text{ from the river bank} \)
2. \( 10.6 \text{ km/h; } 71.2^\circ \text{ from the river bank} \)
3. a \( 650j \)
   b \( 30\sqrt{2}i + (650 + 30\sqrt{2})j \)
   c \( N3.5^\circ E, 693.7 \text{ km/h} \)
4. \( S2.2^\circ W, 643.8 \text{ km/h} \)
5. \( 32.3 \text{ N, } 60.3^\circ \text{ from the vertical} \)
6. \( 37.8 \text{ N, } 53.4^\circ \text{ from the vertical} \)
7. Teacher to check student proofs.
8. Teacher to check student proofs.
9. 7.8 units, \( 50.2^\circ \text{ from the positive } x\text{-axis} \)
10. 6.3 units, \( -18.4^\circ \text{ from the positive } x\text{-axis} \)
11. \( 6.4 \text{ km, } N51.3^\circ E \text{ or } 51.3^\circ T \)
12. \( 9.4 \text{ km, } N58^\circ W \text{ or } 302^\circ T \)
13. \( 1 \text{ s, } 2 \text{ m} \)
14. \( x = -5, y = -1 \)
15. a \( 3i, 5j \)
   b \( (3, 5) \)
16. \( 25 \text{ km/hr on a bearing of } N53^\circ E \)
17. a \( \frac{\sqrt{34}}{} \)
   b \( 5.8 \text{ m/s} \)
   c \( N59^\circ E \)
   d \( 360 \text{ metres} \)
   e \( 600 \text{ metres} \)
18. Teacher to check student proofs.
19. a \( -4i + 8j, -2i + 6.5j \)
   b \( 2.5 \text{ km} \)
   c i \( 5j \)
   ii \( 2i + 3.5j \)
   e \( (7.4, 2) \)
   f \( (3.8, 3) \)
   h \( \text{First } 7.5 \text{ km, second } 5.6 \text{ km} \)
20. a \( i + \sqrt{3}j \)
   b \( \sqrt{3} \text{ km} \)
   c \( 0.1k, i + \sqrt{3}j + 0.1k \)
   d \( 2.002 \)