Continuous probability distributions

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12.1 Kick off with CAS

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Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
12.2 Continuous random variables and probability functions

Continuous random variables

Discrete data is data that is finite or countable, such as the number of soft-centred chocolates in a box of soft- and hard-centred chocolates.

A continuous random variable assumes an uncountable or infinite number of possible outcomes between two values. That is, the variable can assume any value within a given range. For example, the birth weights of babies and the number of millimetres of rain that falls in a night are continuous random variables. In these examples, the measurements come from an interval of possible outcomes. If a newborn boy is weighed at 4.46 kilograms, that is just what the weight scale’s output said. In reality, he may have weighed 4.463279... kilograms. Therefore, a possible range of outcomes is valid, within an interval that depends on the precision of the scale.

Consider an Australian health study that was conducted. The study targeted young people aged 5 to 17 years old. They were asked to estimate the average number of hours of physical activity they participated in each week. The results of this study are shown in the following histogram.

![Histogram](image)

Remember, continuous data has no limit to the accuracy with which it is measured. In this case, for example, $0 \leq x < 1$ means from 0 seconds to 59 minutes and 59 seconds, and so on, because $x$ is not restricted to integer values. In the physical activity study, $x$ taking on a particular value is equivalent to $x$ taking on a value in an appropriate interval. For instance,

$$\Pr(X = 0.5) = \Pr(0 \leq X < 1)$$
$$\Pr(X = 1.5) = \Pr(1 \leq X < 2)$$

and so on. From the histogram,

$$\Pr(X = 2.5) = \Pr(2 \leq X < 3) = \frac{156}{(364 + 347 + 156 + 54 + 32 + 10 + 7)} = \frac{156}{970}$$

In another study, the nose lengths, $X$ millimetres, of 75 adults were measured. This data is continuous because the results are measurements. The result of the study is shown in the table and accompanying histogram.
It is possible to use the histogram to find the number of people who have a nose length of less than 47.5 mm.

\[
\Pr(\text{nose length is } < 47.5) = \frac{2 + 5 + 17 + 21}{75} = \frac{45}{75} = \frac{3}{5}
\]

It is worth noting that we cannot find the probability that a person has a nose length which is less than 45 mm, as this is not the end point of any interval. However, if we had a mathematical formula to approximate the shape of the graph, then the formula could give us the answer to this important question.

In the histogram, the midpoints at the top of each bar have been connected by line segments. If the class intervals were much smaller, say 1 mm or even less, these line segments would take on the appearance of a smooth curve. This smooth curve is of considerable importance for continuous random variables, because it represents the probability density function for the continuous data.

This problem for a continuous random variable can be addressed by using calculus.
For any continuous random variable, $X$, the probability density function is such that

$$\Pr(a < X < b) = \int_{a}^{b} f(x) \, dx$$

which is the area under the curve from $x = a$ to $x = b$.

A probability density function must satisfy the following conditions:

- $f(x) \geq 0$ for all $x \in [a, b]$
- $\int_{a}^{b} f(x) \, dx = 1$; this is absolutely critical.

Other properties are:

- $\Pr(X = x) = 0$, where $x \in [a, b]$
- $\Pr(a < X < b) = P(a \leq X < b) = \Pr(a < X \leq b) = \Pr(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$
- $\Pr(X < c) = \Pr(X \leq c) = \int_{a}^{c} f(x) \, dx$ when $x \in a, b$ and $a < c < b$.

**Probability density functions**

In theory, the domain of a continuous probability density function is $R$, so that

$$\int_{-\infty}^{\infty} f(x) \, dx = 1.$$  

However, if we must address the condition that

$$\int_{a}^{b} f(x) \, dx = 1,$$

then the function must be zero everywhere else.
Sketch the graph of each of the following functions and state whether each function is a probability density function.

\( f(x) = \begin{cases} 
2(x - 1), & 1 \leq x \leq 2 \\
0, & \text{elsewhere}
\end{cases} \)

\( f(x) = \begin{cases} 
0.5, & 2 \leq x \leq 4 \\
0, & \text{elsewhere}
\end{cases} \)

\( f(x) = \begin{cases} 
2e^{-x}, & 0 \leq x \leq 2 \\
0, & \text{elsewhere}
\end{cases} \)

THINK

a 1 Sketch the graph of \( f(x) = 2(x - 1) \) over the domain \( 1 \leq x \leq 2 \), giving an x-intercept of 1 and an end point of \((2, 2)\). Make sure to include the horizontal lines for \( y = 0 \) either side of this graph.

Note: This function is known as a triangular probability function because of its shape.

2 Inspect the graph to determine if the function is always positive or zero, that is, \( f(x) \geq 0 \) for all \( x \)-values.

3 Calculate the area of the shaded region to determine if \( \int_{1}^{2} 2(x - 1) \, dx = 1 \).

Method 1: Using the area of triangles

Area of shaded region = \( \frac{1}{2} \times \text{base} \times \text{height} \)

= \( \frac{1}{2} \times 1 \times 2 \)

= 1

Method 2: Using calculus

Area of shaded region = \( \int_{1}^{2} 2(x - 1) \, dx \)

= \( \int_{1}^{2} (2x - 2) \, dx \)

= \( [x^2 - 2x]_{1}^{2} \)

= \( (2^2 - 2(2)) - (1^2 - 2(1)) \)

= 0 - 1 + 2

= 1

4 Interpret the results.

Yes, \( f(x) \geq 0 \) for all \( x \)-values, and the area under the curve = 1. Therefore, this is a probability density function.
Sketch the graph of \( f(x) = 0.5 \) for \( 2 \leq x \leq 4 \). This gives a horizontal line, with end points of (2, 0.5) and (4, 0.5). Make sure to include the horizontal lines for \( y = 0 \) on either side of this graph.

Note: This function is known as a uniform or rectangular probability density function because of its rectangular shape.

 Inspector the graph to determine if the function is always positive or zero, that is, \( f(x) \geq 0 \) for all \( x \in [a, b] \).

Calculate the area of the shaded region to determine if \( \int_2^4 0.5\,dx = 1 \).

Interpret the results.

Sketch the graph of \( f(x) = 2e^{-x} \) for \( 0 \leq x \leq 2 \). End points will be (0, 2) and (2, \( e^{-2} \)). Make sure to include the horizontal lines for \( y = 0 \) on either side of this graph.

Yes, \( f(x) \geq 0 \) for all \( x \)-values.

Again, it is not necessary to use calculus to find the area.

Method 1:
Area of shaded region = length \( \times \) width
= \( 2 \times 0.5 \)
= 1

Method 2:
Area of shaded region = \( \int_2^4 0.5\,dx \)
= \( [0.5x]^4_2 \)
= 0.5(4) - 0.5(2)
= 2 - 1
= 1

\( f(x) \geq 0 \) for all values, and the area under the curve = 1. Therefore, this is a probability density function.
Given that the functions below are probability density functions, find the value of \( a \) in each function.

\[
a f(x) = \begin{cases} 
  a(x - 1)^2, & 0 \leq x \leq 4 \\
  0, & \text{elsewhere}
\end{cases}
\]

\[
b f(x) = \begin{cases} 
  ae^{-4x}, & x > 0 \\
  0, & \text{elsewhere}
\end{cases}
\]

**THINK**

1. As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

2. Remove \( a \) from the integral, as it is a constant.

3. Antidifferentiate and substitute in the terminals.

4. Solve for \( a \).

**WRITE**

\[
a \int_{0}^{4} f(x) \, dx = 1
\]

\[
a \int_{0}^{4} (x - 1)^2 \, dx = 1
\]

\[
a \int_{0}^{4} (x - 1)^2 \, dx = 1
\]

\[
a \left[ \frac{(x - 1)^3}{3} \right]_{0}^{4} = 1
\]

\[
a \left[ \frac{3^3}{3} - \frac{(-1)^3}{3} \right] = 1
\]

\[
a \left( 9 + \frac{1}{3} \right) = 1
\]

\[
a \times \frac{28}{3} = 1
\]

\[
a = \frac{3}{28}
\]
b 1 As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

\[ f(x) = \begin{cases} \frac{1}{4} e^{2x}, & 0 \leq x \leq \log_e 3 \\ 0, & \text{elsewhere} \end{cases} \]

\[ \int_{0}^{\log_e 3} \frac{1}{4} e^{2x} \, dx = 1 \]

2 Remove \( a \) from the integral, as it is a constant.

3 To evaluate an integral containing infinity as one of the terminals, we find the appropriate limit.

4 Antidifferentiate and substitute in the terminals.

5 Solve for \( a \). Remember that a number divided by an extremely large number is effectively zero, so \( \lim_{k \to \infty} \left( \frac{1}{e^{4k}} \right) = 0 \).

\[ a \left( 0 + \frac{1}{4} \right) = 1 \]

\[ a \frac{1}{4} = 1 \]

\[ a = 4 \]

---

### EXERCISE 12.2

**Continuous random variables and probability functions**

1 Sketch each of the following functions and determine whether each one is a probability density function.

**a** \( f(x) = \begin{cases} \frac{1}{4} e^{2x}, & 0 \leq x \leq \log_e 3 \\ 0, & \text{elsewhere} \end{cases} \)

**b** \( f(x) = \begin{cases} 0.25, & -2 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases} \)

2 Sketch each of the following functions and determine whether each one is a probability density function.

**a** \( f(x) = \begin{cases} \frac{1}{2} \cos(x), & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases} \)

**b** \( f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \frac{1}{2} \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases} \)
3. Given that the function is a probability density function, find the value of \( n \).

\[
  f(x) = \begin{cases} 
    n(x^3 - 1), & 1 \leq x \leq 3 \\
    0, & \text{elsewhere}
  \end{cases}
\]

4. Given that the function is a probability density function, find the value of \( a \).

\[
  f(x) = \begin{cases} 
    -ax, & -2 \leq x < 0 \\
    2ax, & 0 \leq x \leq 3 \\
    0, & \text{elsewhere}
  \end{cases}
\]

5. A small car-hire firm keeps note of the age and kilometres covered by each of the cars in their fleet. Generally, cars are no longer used once they have either covered 350 000 kilometres or are more than five years old. The following information describes the ages of the cars in their current fleet.

<table>
<thead>
<tr>
<th>Age</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 &lt; ( x ) \leq 1</td>
<td>10</td>
</tr>
<tr>
<td>1 &lt; ( x ) \leq 2</td>
<td>26</td>
</tr>
<tr>
<td>2 &lt; ( x ) \leq 3</td>
<td>28</td>
</tr>
<tr>
<td>3 &lt; ( x ) \leq 4</td>
<td>20</td>
</tr>
<tr>
<td>4 &lt; ( x ) \leq 5</td>
<td>11</td>
</tr>
<tr>
<td>5 &lt; ( x ) \leq 6</td>
<td>4</td>
</tr>
<tr>
<td>6 &lt; ( x ) \leq 7</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Determine:
   i. \( \Pr(\leftarrow X \leq 2 \right) \)
   ii. \( \Pr(X > 4) \).

b. Determine:
   i. \( \Pr(1 < X \leq 4) \)
   ii. \( \Pr(X > 1 \mid X \leq 4) \).

6. The battery life for batteries in television remote controls was investigated in a study.

<table>
<thead>
<tr>
<th>Hours of life</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 &lt; ( x ) \leq 15</td>
<td>15</td>
</tr>
<tr>
<td>15 &lt; ( x ) \leq 30</td>
<td>33</td>
</tr>
<tr>
<td>30 &lt; ( x ) \leq 45</td>
<td>23</td>
</tr>
<tr>
<td>45 &lt; ( x ) \leq 60</td>
<td>26</td>
</tr>
<tr>
<td>60 &lt; ( x ) \leq 75</td>
<td>3</td>
</tr>
</tbody>
</table>

a. How many remote control batteries were included in the study?

b. What is the probability that a battery will last more than 45 hours?

c. What is the probability that a battery will last between 15 and 60 hours?

d. A new battery producer is advocating that their batteries have a long life of 60+ hours. If it is known that this is just advertising hype because these batteries are no different from the batteries in the study, what is the probability that these new batteries will have a life of 60+ hours?
A number of experienced shot-putters were asked to aim for a line 10 metres away. After each of them put their shot, its distance from the 10-metre line was measured. All of the shots were on or between the 8- and 10-metre lines. The results of the measurements are shown, where \( X \) is the distance in metres from the 10-metre line.

<table>
<thead>
<tr>
<th>Metres</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 &lt; x ≤ 0.5</td>
<td>75</td>
</tr>
<tr>
<td>0.5 &lt; x ≤ 1</td>
<td>63</td>
</tr>
<tr>
<td>1 &lt; x ≤ 1.5</td>
<td>45</td>
</tr>
<tr>
<td>1.5 &lt; x ≤ 2</td>
<td>17</td>
</tr>
</tbody>
</table>

**a** How many shot-put throws were measured?

**b** Calculate:

i \( \Pr(X > 0.5) \)

ii \( \Pr(1 < X ≤ 2) \)

**c** A guest shot-putter is visiting the athletics club where the measurements are being conducted. His shot-putting ability is equivalent to the abilities of the club members. Find the probability that he puts the shot within 50 cm of the 10-metre line if it is known that he put the shot within 1 metre of the 10-metre line.

**8** Sketch each of the following functions and determine whether each function is a probability density function. Note: Use CAS where appropriate.

**a** \( f(x) = \begin{cases} \frac{1}{x} & -e ≤ x ≤ -1 \\ 0, & \text{elsewhere} \end{cases} \)

**b** \( f(x) = \begin{cases} \cos(x) + 1, & \frac{\pi}{4} ≤ x ≤ \frac{3\pi}{4} \\ 0, & \text{elsewhere} \end{cases} \)

**c** \( f(x) = \begin{cases} \frac{1}{2} \sin(x), & 0 ≤ x ≤ \pi \\ 0, & \text{elsewhere} \end{cases} \)

**d** \( f(x) = \begin{cases} \frac{1}{2\sqrt{x-1}}, & 1 < x ≤ 2 \\ 0, & \text{elsewhere} \end{cases} \)

**9** The rectangular function, \( f \), is defined by the rule

\[
f(x) = \begin{cases} c, & 0.25 < x < 1.65 \\ 0, & \text{elsewhere} \end{cases}
\]

Find the value of the constant \( c \), given that \( f \) is a probability density function.
10 The graph of a function, \( f \), is shown.

\[
\begin{array}{c}
\text{(0, z)} \\
\text{(-1, 0)} \\
\text{(5, 0)}
\end{array}
\]

If \( f \) is known to be a probability density function, show that the value of \( z \) is \( \frac{1}{3} \).

11 Find the value of the constant \( m \) in each of the following if each function is a probability density function.

a. \( f(x) = \begin{cases} 
    m(6 - 2x), & 0 \leq x \leq 2 \\
    0, & \text{elsewhere}
\end{cases} \)

b. \( f(x) = \begin{cases} 
    me^{-x^2}, & x \geq 0 \\
    0, & \text{elsewhere}
\end{cases} \)

c. \( f(x) = \begin{cases} 
    me^{2x}, & 0 \leq x \leq \log_e 3 \\
    0, & \text{elsewhere}
\end{cases} \)

12 Let \( X \) be a continuous random variable with the probability density function

\[
f(x) = \begin{cases} 
    x^2 + 2kx + 1, & 0 \leq x \leq 3 \\
    0, & \text{elsewhere}
\end{cases}
\]

Show that the value of \( k \) is \(-\frac{11}{9}\).

13 \( X \) is a continuous random variable such that

\[
f(x) = \begin{cases} 
    \frac{1}{2} \log_e \left( \frac{x}{2} \right), & 2 \leq x \leq a \\
    0, & \text{elsewhere}
\end{cases}
\]

and \( \int_2^a f(x) \, dx = 1 \). The graph of this function is shown.

Find the value of the constant \( a \).

14 \( X \) is a continuous random variable such that

\[
f(x) = \begin{cases} 
    -x, & -1 \leq x < 0 \\
    x, & 0 \leq x \leq a \\
    0, & \text{elsewhere}
\end{cases}
\]

where \( a \) is a constant.

\( Y \) is another continuous random variable such that

\[
f(y) = \begin{cases} 
    \frac{1}{y}, & 1 \leq y \leq e \\
    0, & \text{elsewhere}
\end{cases}
\]
a Sketch the graph of the function for $X$ and find $\int_{-1}^{a} f(x) \, dx$.

b Sketch the graph of the function for $Y$ and find $\int_{1}^{e} f(y) \, dy$.

c Find the value of the constant $a$ if $\int_{-1}^{a} f(x) \, dx = \int_{1}^{e} f(y) \, dy$.

15 $X$ is a continuous random variable such that

$$f(x) = \begin{cases} n \sin(3x) \cos(3x), & 0 < x < \frac{\pi}{12}, \\ 0, & \text{elsewhere} \end{cases}$$

If $f$ is known to be a probability density function, find the value of the constant, $n$.

16 A function $f$ is defined by the rule

$$f(x) = \begin{cases} \log_{e}(x), & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

a If $\int_{1}^{a} f(x) \, dx = 1$, find the value of the real constant $a$.

b Does this function define a probability density function?

### 12.3 The continuous probability density function

As stated in section 12.2, if $X$ is a continuous random variable, then

$$\Pr(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx.$$ 

In other words, by finding the area between the curve of the continuous probability function, the $x$-axis, the line $x = a$ and the line $x = b$, providing $f(x) \geq 0$, then we are finding $\Pr(a \leq X \leq b)$. It is worth noting that because we are dealing with a continuous random variable, $\Pr(X = a) = 0$, and consequently:

$$\Pr(a \leq X \leq b) = \Pr(a < X \leq b) = \Pr(a \leq X < b) = \Pr(a < X < b)$$

Also,

$$\Pr(a \leq X \leq b) = \Pr(a \leq X \leq c) + \Pr(c < X \leq b), \text{ where } a < c < b.$$ 

This property is particularly helpful when the probability density function is a hybrid function and the required probability encompasses two functions.
A continuous random variable, $Y$, has a probability density function, $f$, defined by

$$f(y) = \begin{cases} 
-a y, & -3 \leq y \leq 0 \\
-a y, & 0 < y \leq 3 \\
0, & \text{elsewhere}
\end{cases}$$

where $a$ is a constant.

**THINK**

- **a** Sketch the graph of $f$.
- **b** Find the value of the constant, $a$.
- **c** Determine $\Pr(1 \leq Y \leq 3)$.
- **d** Determine $\Pr(Y < 2 \mid Y > -1)$

**WRITE/DRAW**

- **a** The hybrid function contains three sections.
  The first graph, $f(y) = -ay$, is a straight line with end points of $(0, 0)$ and $(-3, 3a)$. The second graph is also a straight line and has end points of $(0, 0)$ and $(3, 3a)$. Don’t forget to include the $f(y) = 0$ lines for $x > 3$ and $x < -3$.

- **b** Use the fact that $\int_{-3}^{3} f(y) dy = 1$ to solve for $a$.

**WORKED EXAMPLE 3**

**THINK**

- **a** The hybrid function contains three sections. $f(-3) = 3a$ and $f(3) = 3a$
  The first graph, $f(y) = -ay$, is a straight line with end points of $(0, 0)$ and $(-3, 3a)$. The second graph is also a straight line and has end points of $(0, 0)$ and $(3, 3a)$. Don’t forget to include the $f(y) = 0$ lines for $x > 3$ and $x < -3$.

- **b** $\int_{-3}^{3} f(y) dy = 1$

Using the area of a triangle, we find:

$$\frac{1}{2} \times 3 \times 3a + \frac{1}{2} \times 3 \times 3a = 1$$

$$\frac{9a}{2} + \frac{9a}{2} = 1$$

$$9a = 1$$

$$a = \frac{1}{9}$$

**WRITE/DRAW**

- **b** $\int_{-3}^{3} f(y) dy = 1$

**c** $\Pr(1 \leq Y \leq 3) = \int_{1}^{3} f(y) dy$

$$= \int_{1}^{3} \left( \frac{1}{9} y \right) dy$$

$$= \left[ \frac{1}{18} y^2 \right]_{1}^{3}$$

$$= \frac{1}{18} (3)^2 - \frac{1}{18} (1)^2$$

$$= \frac{8}{18} = \frac{4}{9}$$

**Note:** The method of finding the area of a trapezium could also be used.
d 1 State the rule for the conditional probability.

d Pr(Y < 2 | Y > −1) = \frac{Pr(Y < 2 \cap Y > -1)}{Pr(Y > -1)}

= \frac{Pr(-1 < Y < 2)}{Pr(Y > -1)}

2 Find Pr(−1 < Y < 2). As the interval is across two functions, the interval needs to be split.

Pr(−1 < Y < 2) = Pr(−1 < Y < 0) + Pr(0 ≤ Y < 2)

3 To find the probabilities we need to find the areas under the curve.

4 Antidifferentiate and evaluate after substituting the terminals.

5 Find Pr(Y > −1). As the interval is across two functions, the interval needs to be split.

Pr(Y > −1) = Pr(−1 < Y < 0) + Pr(0 ≤ Y ≤ 3)

6 To find the probabilities we need to find the areas under the curve. As Pr(0 ≤ Y ≤ 3) covers exactly half the area under the curve, Pr(0 ≤ Y ≤ 3) = \frac{1}{2}.

(The entire area under the curve is always 1 for a probability density function.)

7 Antidifferentiate and evaluate after substituting the terminals.

8 Now substitute into the formula to find

Pr(Y < 2 | Y > −1) = \frac{Pr(-1 < Y < 2)}{Pr(Y > -1)}.
**The continuous probability density function**

1. The continuous random variable $Z$ has a probability density function given by
   
   \[
   f(z) = \begin{cases} 
   -z + 1, & 0 \leq z < 1 \\
   z - 1, & 1 \leq z \leq 2 \\
   0, & \text{elsewhere}.
   \end{cases}
   \]

   **a** Sketch the graph of $f$.
   **b** Find $\Pr(Z < 0.75)$.
   **c** Find $\Pr(Z > 0.5)$.

2. The continuous random variable $X$ has a probability density function given by
   
   \[
   f(x) = \begin{cases} 
   4x^3, & 0 \leq x \leq a \\
   0, & \text{elsewhere}
   \end{cases}
   \]

   where $a$ is a constant.

   **a** Find the value of the constant $a$.
   **b** Sketch the graph of $f$.
   **c** Find $\Pr(0.5 \leq X \leq 1)$.

3. Let $X$ be a continuous random variable with a probability density function defined by
   
   \[
   f(x) = \begin{cases} 
   \frac{1}{2} \sin(x), & 0 \leq x \leq \pi \\
   0, & \text{elsewhere}
   \end{cases}
   \]

   **a** Sketch the graph of $f$.
   **b** Find $\Pr\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)$.
   **c** Find $\Pr(X > \frac{\pi}{4} \mid X < \frac{3\pi}{4})$.

4. A probability density function is defined by the rule
   
   \[
   f(x) = \begin{cases} 
   k(2 + x), & -2 \leq x < 0 \\
   k(2 - x), & 0 \leq x \leq 2 \\
   0, & \text{elsewhere}
   \end{cases}
   \]

   where $X$ is a continuous random variable and $k$ is a constant.

   **a** Sketch the graph of $f$.
   **b** Show that the value of $k$ is $\frac{1}{4}$.
   **c** Find $\Pr(-1 \leq X \leq 1)$.
   **d** Find $\Pr(X \geq -1 \mid X \leq 1)$.
5 The amount of petrol sold daily by a busy service station is a uniformly distributed probability density function. A minimum of 18,000 litres and a maximum of 30,000 litres are sold on any given day. The graph of the function is shown.

a Find the value of the constant $k$.

b Find the probability that between 20,000 and 25,000 litres of petrol are sold on a given day.

c Find the probability that as much as 26,000 litres of petrol were sold on a particular day, given that it was known that at least 22,000 litres were sold.

6 The continuous random variable $X$ has a uniform rectangular probability density function defined by

$$ f(x) = \begin{cases} \frac{1}{5}, & 1 \leq x \leq 6 \\ 0, & \text{elsewhere} \end{cases} $$

a Sketch the graph of $f$.

b Determine $\Pr(2 \leq X \leq 5)$.

7 The continuous random variable $Z$ has a probability density function defined by

$$ f(z) = \begin{cases} \frac{1}{2z}, & 1 \leq z \leq e^2 \\ 0, & \text{elsewhere} \end{cases} $$

a Sketch the graph of $f$ and shade the area that represents $\int_{1}^{e^2} f(z)\,dz$.

b Find $\int_{1}^{e^2} f(z)\,dz$. Explain your result.

The continuous random variable $U$ has a probability function defined by

$$ f(u) = \begin{cases} e^{4u}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases} $$

c Sketch the graph of $f$ and shade the area that represents $\int_{0}^{a} f(u)\,du$, where $a$ is a constant.

d Find the exact value of the constant $a$ if $\int_{1}^{e^2} f(z)\,dz$ is equal to $\int_{0}^{a} f(u)\,du$.

8 The continuous random variable $Z$ has a probability density function defined by

$$ f(z) = \begin{cases} \frac{1}{2} \cos(z), & \frac{\pi}{2} \leq z \leq \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases} $$
a Sketch the graph of \( f \) and verify that \( y = f(z) \) is a probability density function.

b Find \( \Pr \left( \frac{-\pi}{6} \leq Z \leq \frac{\pi}{4} \right) \).

9 The continuous random variable \( U \) has a probability density function defined by
\[
f(u) = \begin{cases} 
1 - \frac{1}{4} (2u - 3u^2), & 0 \leq u \leq a \\
0, & \text{elsewhere}
\end{cases}
\]
where \( a \) is a constant. Find:

a the value of the constant \( a \)
b \( \Pr(U < 0.75) \)
c \( \Pr(0.1 < U < 0.5) \)
d \( \Pr(U = 0.8) \).

10 The continuous random variable \( X \) has a probability density function defined by
\[
f(x) = \begin{cases} 
\frac{3}{8} x^2, & 0 \leq z \leq 2 \\
0, & \text{elsewhere}
\end{cases}
\]
Find:

a \( P(X > 1.2) \)
b \( P(X > 1 \mid X > 0.5) \), correct to 4 decimal places
c the value of \( n \) such that \( P(X \leq n) = 0.75 \).

11 The continuous random variable \( Z \) has a probability density function defined by
\[
f(z) = \begin{cases} 
e^{-z^3}, & 0 \leq z \leq a \\
0, & \text{elsewhere}
\end{cases}
\]
where \( a \) is a constant. Find:

a the value of the constant \( a \) such that \( \int_{0}^{a} f(z) dz = 1 \)
b \( \Pr(0 < Z < 0.7) \), correct to 4 decimal places
c \( \Pr(Z < 0.7 \mid Z > 0.2) \), correct to 4 decimal places
d the value of \( a \), correct to 2 decimal places, such that \( \Pr(Z \leq a) = 0.54 \).

12 The continuous random variable \( X \) has a probability density function given as
\[
f(x) = \begin{cases} 
3e^{-3x}, & x \geq 0 \\
0, & \text{elsewhere}
\end{cases}
\]
a Sketch the graph of \( f \).
b Find \( \Pr(0 \leq X \leq 1) \), correct to 4 decimal places.
c Find \( \Pr(X > 2) \), correct to 4 decimal places.

13 The continuous random variable \( X \) has a probability density function defined by
\[
f(x) = \begin{cases} 
\log_e (x^2), & x \geq 1 \\
0, & \text{elsewhere}
\end{cases}
\]
Find, correct to 4 decimal places:

a the value of the constant \( a \) if \( \int_{1}^{a} f(x) dx = 1 \)
b \( \Pr(1.25 \leq X \leq 2) \).
The graph of the probability function

\[ f(z) = \frac{1}{\pi(z^2 + 1)} \]

is shown.

a Find, correct to 4 decimal places, \( \Pr(-0.25 < Z < 0.25) \).

Suppose another probability density function is defined as

\[ f(x) = \begin{cases} \frac{1}{x^2 + 1}, & -a \leq x \leq a \\ 0, & \text{elsewhere} \end{cases} \]

b Find the value of the constant \( a \).

12.4 Measures of centre and spread

The commonly used measures of central tendency and spread in statistics are the mean, median, variance, standard deviation and range. These same measurements are appropriate for continuous probability functions.

Measures of central tendency

The mean

Remember that for a discrete random variable,

\[ E(X) = \mu = \sum_{x=1}^{n} x \Pr(X = x) \]

This definition can also be applied to a continuous random variable.

We define \( E(X) = \mu = \int_{-\infty}^{\infty} xf(x)dx \).

If \( f(x) = 0 \) everywhere except for \( x \in [a, b] \), where the function is defined, then

\[ E(X) = \mu = \int_{a}^{b} xf(x)dx \].
Consider the continuous random variable, $X$, which has a probability density function defined by

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

For this function,

$$E(X) = \mu = \int_{0}^{1} x f(x) \, dx$$

$$= \int_{0}^{1} x \left( x^2 \right) \, dx$$

$$= \int_{0}^{1} x^3 \, dx$$

$$= \left[ \frac{x^4}{4} \right]_{0}^{1}$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

Similarly, if the continuous random variable $X$ has a probability density function of

$$f(x) = \begin{cases} 7e^{-7x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

then

$$E(X) = \mu = \int_{0}^{\infty} x f(x) \, dx$$

$$= \lim_{k \to \infty} \int_{0}^{k} 7xe^{-7x} \, dx$$

$$= 0.1429$$

where CAS technology is required to determine the integral.

The mean of a function of $X$ is similarly found.

The function of $X$, $g(x)$, has a mean defined by:

$$E(g(x)) = \mu = \int_{-\infty}^{\infty} g(x) f(x) \, dx.$$ 

So if we again consider

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$
then

\[
E(X^2) = \int_0^1 x^2 f(x) \, dx
\]

\[
= \int_0^1 x^4 \, dx
\]

\[
= \left[ \frac{x^5}{5} \right]_0^1
\]

\[
= \frac{1^5}{5} - 0
\]

\[
= \frac{1}{5}
\]

This definition is important when we investigate the variance of a continuous random variable.

**Median and percentiles**

The median is also known as the 50th percentile, \(Q_2\), the halfway mark or the middle value of the distribution.

For a continuous random variable, \(X\), defined by the probability function \(f\), the median can be found by solving

\[
\int_{-\infty}^{m} f(x) \, dx = 0.5.
\]

Other percentiles, which are frequently calculated, are the 25th percentile or lower quartile, \(Q_1\), and the 75th percentile or upper quartile, \(Q_3\).

The interquartile range is calculated as:

\[
IQR = Q_3 - Q_1
\]

Consider a continuous random variable, \(X\), that has a probability density function of

\[
f(x) = \begin{cases} 
0.21e^{2x-x^2}, & -3 \leq x \leq 5 \\
0, & \text{elsewhere} 
\end{cases}
\]

To find the median, \(m\), we solve for \(m\) as follows:

\[
\int_{-3}^{m} 0.21e^{2x-x^2} \, dx = 0.5
\]

The area under the curve is equated to 0.5, giving half of the total area and hence the 50th percentile. Solving via CAS, the result is that \(m = 0.9897 \approx 1\).

This can be seen on a graph as follows.
Consider the continuous random variable $X$, which has a probability density function of

$$f(x) = \begin{cases} \frac{x^3}{4}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

The median is given by $\Pr(0 \leq x \leq m) = 0.5$:

$$\int_{0}^{m} \frac{x^3}{4} \, dx = 0.5$$

$$\left[ \frac{x^4}{16} \right]_{0}^{m} = \frac{1}{2}$$

$$\frac{m^4}{16} - 0 = \frac{1}{2}$$

$$m^4 = 8$$

$$m = \pm \sqrt[4]{8}$$

$$m = 1.6818 \ (0 \leq m \leq 2)$$

To find the lower quartile, we make the area under the curve equal to 0.25. Thus the lower quartile is given by $\Pr(0 \leq x \leq a) = 0.25$:

$$\int_{0}^{a} \frac{x^3}{4} \, dx = 0.25$$

$$\left[ \frac{x^4}{16} \right]_{0}^{a} = \frac{1}{4}$$

$$\frac{a^4}{16} - 0 = \frac{1}{4}$$

$$a^4 = 4$$

$$a = \pm \sqrt[4]{4}$$

$$a = Q_1 = 1.4142 \ (0 \leq a \leq m)$$

Similarly, to find the upper quartile, we make the area under the curve equal to 0.75. Thus the upper quartile is given by $\Pr(0 \leq x \leq n) = 0.75$:

$$\int_{0}^{n} \frac{x^3}{4} \, dx = 0.75$$

$$\left[ \frac{x^4}{16} \right]_{0}^{n} = \frac{3}{4}$$

$$\frac{n^4}{16} - 0 = \frac{3}{4}$$

$$n^4 = 12$$

$$n = \pm \sqrt[4]{12}$$

$$n = Q_3 = 1.8612 \ (m \leq x \leq 2)$$
So the **interquartile range** is given by \( Q_3 - Q_1 = 1.8612 - 1.4142 = 0.4470 \).

These values are shown on the following graph.

---

**WORKED EXAMPLE 4**

A continuous random variable, \( Y \), has a probability density function, \( f \), defined by

\[
f(y) = \begin{cases} 
ky, & 0 \leq y \leq 1 \\
0, & \text{elsewhere}
\end{cases}
\]

where \( k \) is a constant.

a Sketch the graph of \( f \).
b Find the value of the constant \( k \).
c Find:
   i the mean of \( Y \)
   ii the median of \( Y \).
d Find the interquartile range of \( Y \).

**THINK**

a The graph \( f(y) = ky \) is a straight line with end points at \((0, 0)\) and \((1, k)\). Remember to include the lines \( f(y) = 0 \) for \( y > 1 \) and \( y < 0 \).
\[ b \text{ Solve } \int_{0}^{1} ky \, dy = 1 \text{ to find the value of } k. \]

\[ k \int_{0}^{1} y \, dy = 1 \]
\[ k \left[ \frac{y^2}{2} \right]_{0}^{1} = 1 \]
\[ \frac{k(1)^2}{2} - 0 = 1 \]
\[ k = 2 \]

Using the area of a triangle also enables you to find the value of \( k \).
\[ \frac{1}{2} \times 1 \times k = 1 \]
\[ k = 2 \]

\[ c \text{ i 1 State the rule for the mean.} \]
\[ c \text{ i } \mu = \int_{0}^{1} y(2y) \, dy \]
\[ = \int_{0}^{1} 2y^2 \, dy \]
\[ = \left[ \frac{2y^3}{3} \right]_{0}^{1} \]
\[ = \frac{2(1)^3}{3} - 0 \]
\[ = \frac{2}{3} \]

\[ 2 \text{ Antidifferentiate and simplify.} \]

\[ c \text{ i 1 State the rule for the median.} \]
\[ c \text{ i } \int_{0}^{m} f(y) \, dy = 0.5 \]
\[ \int_{0}^{m} 2y \, dy = 0.5 \]
\[ \frac{[y^2]}{2} \bigg|_{0}^{m} = 0.5 \]
\[ m^2 - 0 = 0.5 \]
\[ m = \pm \sqrt{0.5} \]
\[ m = \frac{1}{\sqrt{2}} \quad (0 < m < 1) \]

\[ ii \text{ 1 State the rule for the median.} \]
\[ ii \int_{0}^{m} f(y) \, dy = 0.5 \]
\[ \int_{0}^{m} 2y \, dy = 0.5 \]
\[ \frac{[y^2]}{2} \bigg|_{0}^{m} = 0.5 \]
\[ m^2 - 0 = 0.5 \]
\[ m = \pm \sqrt{0.5} \]
\[ m = \frac{1}{\sqrt{2}} \quad (0 < m < 1) \]

\[ 2 \text{ Antidifferentiate and solve for } m. \text{ Note that } m \text{ must be a value within the domain of the function, so within } 0 \leq y \leq 1. \]
Measures of spread

Variance, standard deviation and range

The variance and standard deviation are important measures of spread in statistics. From previous calculations for discrete probability functions, we know that

\[ \text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \]

and

\[ \text{SD}(X) = \sqrt{\text{Var}(X)} \]

For continuous probability functions,

\[
\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x) \, dx
\]
\[\begin{align*}
&= \int_{-\infty}^{\infty} x^2 f(x) \, dx - \int_{-\infty}^{\infty} 2xf(x) \mu \, dx + \int_{-\infty}^{\infty} \mu^2 f(x) \, dx \\
&= E(X^2) - 2\mu \int_{-\infty}^{\infty} x f(x) \, dx + \mu^2 \int_{-\infty}^{\infty} f(x) \, dx \\
&= E(X^2) - 2\mu \times E(X) + \mu^2 \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2 \\
&= E(X^2) - [E(X)]^2
\end{align*}\]

Two important facts were used in this proof: \(\int_{-\infty}^{\infty} f(x) \, dx = 1\) and \(\int_{-\infty}^{\infty} xf(x) \, dx = \mu = E(X)\).

Substituting this result into \(SD(X) = \sqrt{Var(X)}\) gives us

\[SD(X) = \sqrt{E(X^2) - [E(X)]^2}\]

The range is calculated as the highest value minus the lowest value, so for the probability density function given by \(f(x) = \begin{cases} 
\frac{1}{5}, & 1 \leq x \leq 6 \\
0, & \text{elsewhere}
\end{cases}\), the highest possible \(x\)-value is 6 and the lowest is 1. Therefore, the range for this function is \(6 - 1 = 5\).

**WORKED EXAMPLE 5**

For a continuous random variable, \(X\), with a probability density function, \(f\), defined by

\[f(x) = \begin{cases} 
\frac{1}{2}x + 2, & -4 \leq x \leq -2 \\
0, & \text{elsewhere}
\end{cases}\]

find:

- **a** the mean
- **b** the median
- **c** the variance
- **d** the standard deviation, correct to 4 decimal places.

**THINK**

**WRITE**

\[\begin{align*}
\text{a} & \quad \text{State the rule for the mean and simplify.} \\
\text{b} & \quad \text{Use the definition of the median for continuous distributions.} \\
\text{c} & \quad \text{Use the definition of the variance for continuous distributions.} \\
\text{d} & \quad \text{Use the definition of the standard deviation for continuous distributions.}
\end{align*}\]
2 Antidifferentiate and evaluate.

\[ \int_a^b \frac{1}{6}x^3 + x^2 \, dx = \left[ \frac{1}{6}x^3 + x^2 \right]_a^b \]

\[ = \left( \frac{1}{6}(-2)^3 + (-2)^2 \right) - \left( \frac{1}{6}(-4)^3 + (-4)^2 \right) \]

\[ = \frac{4}{3} + 4 + \frac{32}{3} - 16 \]

\[ = -\frac{22}{3} \]

\[ a \]

1 State the rule for the median.

\[ \int_a^b \frac{1}{2}x + 2 \, dx = 0.5 \]

\[ \int_a^b \frac{1}{2}m + 2 \, dx = 0.5 \]

2 Antidifferentiate and solve for \( m \).

The quadratic formula is needed as the quadratic equation formed cannot be factorised.

Alternatively, use CAS to solve for \( m \).

\[ \left( \frac{1}{4}m^2 + 2m \right) - \left( \frac{1}{4}(-4)^2 + 2(-4) \right) = 0.5 \]

\[ \frac{1}{4}m^2 + 2m + 4 = 0.5 \]

\[ m^2 + 8m + 16 = 2 \]

\[ m^2 + 8m + 14 = 0 \]

So \( m = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(14)}}{2(1)} \)

\[ m = \frac{-8 \pm \sqrt{8}}{2} \]

\[ = -4 \pm \sqrt{2} \]

\[ \therefore m = -4 + \sqrt{2} \text{ as } m \in [-4, 2] \]

The median is \(-4 + \sqrt{2}\).

3 Write the answer.

The median is \(-4 + \sqrt{2}\).

1 Write the rule for variance.

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 \]

2 Find \( E(X^2) \) first.

\[ E(X^2) = \int_a^b x^2 f(x) \, dx \]

\[ = \int_a^b x^2 \left( \frac{1}{2}x + 2 \right) \, dx \]

\[ = \left[ \frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_a^b \]

\[ = \left[ \frac{1}{8}(-2)^4 + \frac{2}{3}(-2)^3 \right] - \left[ \frac{1}{8}(-4)^4 + \frac{2}{3}(-4)^3 \right] \]

\[ = 2 - \frac{16}{3} - 32 + \frac{128}{3} \]

\[ = -30 + \frac{112}{3} \]

\[ = \frac{22}{3} \]
3 Substitute $E(X)$ and $E(X^2)$ into the rule for variance. 

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{22}{3} - \left( \frac{-8}{3} \right)^2$$

$$= \frac{22}{3} - \frac{64}{9}$$

$$= \frac{66}{9} - \frac{64}{9}$$

$$= \frac{2}{9}$$

\[d\] 1 Write the rule for standard deviation. 
\[d\] $SD(X) = \sqrt{\text{Var}(X)}$

$$= \sqrt{\frac{2}{9}}$$

$$= 0.4714$$

### Exercise 12.4

#### Measures of centre and spread

1. **WE4** The continuous random variable $Z$ has a probability density function of

$$f(z) = \begin{cases} 
\frac{1}{\sqrt{z}}, & 1 \leq z \leq a \\
0, & \text{elsewhere}
\end{cases}$$

where $a$ is a constant.

a. Find the value of the constant $a$.

b. Find:

i. the mean of $Z$

ii. the median of $Z$.

2. **WE4** The continuous random variable, $Y$, has a probability density function of

$$f(y) = \begin{cases} 
\sqrt{y}, & 0 \leq y \leq a \\
0, & \text{elsewhere}
\end{cases}$$

where $a$ is a constant. Find, correct to 4 decimal places:

a. the value of the constant $a$

b. $E(Y)$

c. the median value of $Y$.

3. **WE5** For the continuous random variable $Z$, the probability density function is

$$f(z) = \begin{cases} 
2 \log_e (2z), & 1 \leq z \leq \frac{e}{2} \\
0, & \text{elsewhere}
\end{cases}$$

Find the mean, median, variance and standard deviation correct to 4 decimal places.

4. The function

$$f(x) = \begin{cases} 
3e^{-3x}, & x \geq 0 \\
0, & \text{elsewhere}
\end{cases}$$

defines the probability density function for the continuous random variable, $X$.

Find the mean, median, variance and standard deviation of $X$. 


5 Let $X$ be a continuous random variable with a probability density function of
\[ f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases} \]

a Prove that $f$ is a probability density function.
b Find $E(X)$.
c Find the median value of $f$.

6 The time in minutes that an individual must wait in line to be served at the local bank branch is defined by
\[ f(t) = 2e^{-2t}, t \geq 0 \]
where $T$ is a continuous random variable.

a What is the mean waiting time for a customer in the queue, correct to 1 decimal place?
b Calculate the standard deviation for the waiting time in the queue, correct to 1 decimal place.
c Determine the median waiting time in the queue, correct to 2 decimal places.

7 The continuous random variable $Y$ has a probability density function defined by
\[ f(y) = \begin{cases} \frac{y^2}{3}, & 0 \leq y \leq \sqrt[3]{9} \\ 0, & \text{elsewhere} \end{cases} \]
Find, correct to 4 decimal places:
a the expected value of $Y$
b the median value of $Y$
c the lower and upper quartiles of $Y$
d the inter-quartile range of $Y$.

8 The continuous random variable $Z$ has a probability density function defined by
\[ f(z) = \begin{cases} \frac{a}{z}, & 1 \leq z \leq 8 \\ 0, & \text{elsewhere} \end{cases} \]
where $a$ is a constant.
a Find the value, correct to four decimal places, of the constant $a$.
b Find $E(Z)$ correct to 4 decimal places.
c Find $\text{Var}(Z)$ and $\text{SD}(Z)$.
d Determine the inter-quartile range for $Z$.
e Determine the range for $Z$.

9 $X$ is a continuous random variable. The graph of the probability density function
\[ f(x) = \frac{1}{\pi}(\sin(2x) + 1) \quad \text{for} \quad 0 \leq x \leq \pi \]
is shown.
a Show that $f(x)$ is a probability density function.
b Calculate $E(X)$ correct to 4 decimal places.
c Calculate, correct to 4 decimal places:
i $\text{Var}(X)$
ii $\text{SD}(X)$.
d Find the median value of $f$ correct to 4 decimal places.
10 The continuous random variable \( X \) has a probability density function defined by
\[
f(x) = \begin{cases} 
ax - bx^2, & 0 \leq x \leq 2 \\
0, & \text{elsewhere}
\end{cases}
\]
Find the values of the constants \( a \) and \( b \) if \( E(X) = 1 \).

11 The continuous random variable, \( Z \), has a probability density function of
\[
f(z) = \begin{cases} 
\frac{3}{z^2}, & 1 \leq z \leq a \\
0, & \text{elsewhere}
\end{cases}
\]
where \( a \) is a constant.
   a) Show that the value of \( a \) is \( \frac{3}{2} \).
   b) Find the mean value and variance of \( f \) correct to 4 decimal places.
   c) Find the median and interquartile range of \( f \).

12 a) Find the derivative of \( \sqrt{4 - x^2} \).
   b) Hence, find the mean value of the probability density function defined by
\[
f(x) = \begin{cases} 
\frac{3}{\pi \sqrt{4 - x^2}}, & 0 \leq x \leq \sqrt{3} \\
0, & \text{elsewhere}
\end{cases}
\]

13 Consider the continuous random variable \( X \) with a probability density function of
\[
f(x) = \begin{cases} 
h(2 - x), & 0 \leq x \leq 2 \\
h(x - 2), & 2 < x \leq 4 \\
0, & \text{elsewhere}
\end{cases}
\]
where \( h \) is a constant.
   a) Find the value of the constant \( h \).
   b) Find \( E(X) \).
   c) Find \( \text{Var}(X) \).

14 Consider the continuous random variable \( X \) with a probability density function of
\[
f(x) = \begin{cases} 
k, & a \leq x \leq b \\
0, & \text{elsewhere}
\end{cases}
\]
where \( a, b \) and \( k \) are positive constants.
   a) Sketch the graph of the function \( f \).
   b) Show that \( k = \frac{1}{b - a} \).
   c) Find \( E(X) \) in terms of \( a \) and \( b \).
   d) Find \( \text{Var}(X) \) in terms of \( a \) and \( b \).

15 The continuous random variable \( Y \) has a probability density function
\[
f(y) = \begin{cases} 
0.2 \log_2 \left( \frac{y}{2} \right), & 2 \leq y \leq 7.934 \\
0, & \text{elsewhere}
\end{cases}
\]
   a) Verify that \( f \) is a probability density function.
   b) Find \( E(Y) \) correct to 4 decimal places.
   c) Find \( \text{Var}(Y) \) and \( \text{SD}(Y) \) correct to 4 decimal places.
   d) Find the median value of \( Y \) correct to 4 decimal places.
   e) State the range.
The continuous random variable $Z$ has a probability density function

$$f(z) = \begin{cases} \sqrt{z - 1}, & 1 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

a Find the value of the constant $a$ correct to 4 decimal places.

b Determine, correct to 4 decimal places:
   i $E(Z)$
   ii $E(Z^2)$
   iii $\text{Var}(Z)$
   iv $\text{SD}(Z)$

### 12.5 Linear transformations

Sometimes it is necessary to apply transformations to a continuous random variable. A transformation is a change that is applied to the random variable. The change may consist of one or more operations that may involve adding or subtracting a constant or multiplying or dividing the variable by a constant.

Suppose a linear transformation is applied to the continuous random variable $X$ to create a new continuous random variable, $Y$. For instance

$$Y = aX + b$$

It can be shown that $E(Y) = E(aX + b) = aE(X) + b$

and $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X)$.

First let us show that $E(Y) = E(aX + b) = aE(X) + b$.

Since $E(X) = \int_{-\infty}^{\infty} xf(x) \, dx$,

then $E(aX + b) = \int_{-\infty}^{\infty} (ax + b)f(x) \, dx$.

Using the distributive law, it can be shown that this is equal to

$$E(aX + b) = \int_{-\infty}^{\infty} axf(x) \, dx + \int_{-\infty}^{\infty} bf(x) \, dx$$

$$= a \int_{-\infty}^{\infty} xf(x) \, dx + b \int_{-\infty}^{\infty} f(x) \, dx$$

But $E(X) = \int_{-\infty}^{\infty} xf(x) \, dx$, so

$$E(aX + b) = aE(X) + b \int_{-\infty}^{\infty} f(x) \, dx.$$ 

Also, $\int_{-\infty}^{\infty} f(x) \, dx = 1$, so

$$E(aX + b) = aE(X) + b.$$ 

Also note that $E(aX) = aE(X)$ and $E(b) = b$.

Now let us show that $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X)$. 

Since \( \text{Var}(X) = E(X^2) - [E(X)]^2 \),
then
\[
\text{Var}(aX + b) = E((aX + b)^2) - [E(aX + b)]^2
\]
\[
= \int_{-\infty}^{\infty} (ax + b)^2 f(x) \, dx - (aE(X) + b)^2
\]
\[
= \int_{-\infty}^{\infty} (a^2 x^2 + 2abx + b^2) f(x) \, dx - [a^2 E(X)^2 + 2abE(X) + b^2]
\]
Using the distributive law to separate the first integral, we have
\[
\text{Var}(aX + b) = \int_{-\infty}^{\infty} a^2 x^2 f(x) \, dx + 2ab \int_{-\infty}^{\infty} x f(x) \, dx + b^2 \int_{-\infty}^{\infty} f(x) \, dx - a^2 [E(X)]^2
\]
\[
- 2abE(X) - b^2
\]
But \( E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \), \( E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) for a probability density function. Thus,
\[
\text{Var}(aX + b) = a^2 E(X^2) + 2abE(X) + b^2 - a^2 [E(X)]^2 - 2abE(X) - b^2
\]
\[
= a^2 E(X^2) - a^2 [E(X)]^2
\]
\[
= a^2 (E(X^2) - [E(X)]^2)
\]
\[
= a^2 \text{Var}(X)
\]

Thus,
\[
E(aX + b) = aE(X) + b
\]
and
\[
\text{Var}(aX + b) = a^2 \text{Var}(X).
\]

**Worked Example 6**

A continuous random variable, \( X \), has a mean of 3 and a variance of 2. Find:

- a \( E(2X + 1) \)
- b \( \text{Var}(2X + 1) \)
- c \( E(X^2) \)
- d \( E(3X^2) \)
- e \( E(X^2 - 5) \).

**Think**

- a Use \( E(aX + b) = aE(X) + b \) to find \( E(2X + 1) \).

- b Use \( \text{Var}(aX + b) = a^2 \text{Var}(X) \) to find \( \text{Var}(2X + 1) \).

**Write**

- a \[
E(2X + 1) = 2E(X) + 1
= 2(3) + 1
= 7
\]

- b \[
\text{Var}(2X + 1) = 2^2 \text{Var}(X)
= 4 \times 2
= 8
\]
c Use \( \text{Var}(X) = E(X^2) - [E(X)]^2 \) to find \( E(X^2) \).

d Use \( E(aX^2) = aE(X^2) \) to find \( E(3X^2) \).

e Use \( E(aX^2 + b) = aE(X^2) + b \) to find \( E(X^2 - 5) \).

It may also be necessary to find the expected value and variance before using the facts
that \( E(aX + b) = aE(X) + b \) and \( \text{Var}(aX + b) = a^2\text{Var}(X) \).

The graph of the probability density function for the continuous random
variable \( X \) is shown. The rule for the probability density function is given by
\[
f(x) = \begin{cases} 
3kx, & 0 \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}
\]
where \( k \) is a constant.

a Find the value of the constant \( k \).

b Calculate \( E(X) \) and \( \text{Var}(X) \).

c Find \( E(3X - 1) \) and \( \text{Var}(3X - 1) \).

d Find \( E(2X^2 + 3) \).

THINK

a Solve \( \int_{0}^{1} kx \, dx = 1 \) to find \( k \), or alternatively use the

\[
\int_{0}^{1} 3kx \, dx = 1 \\
\left[ \frac{3kx^2}{2} \right]_{0}^{1} = 1 \\
\frac{3k(1)^2}{2} - 0 = 1 \\
k = \frac{2}{3}
\]

WRITE

a Method 1:

b Method 2:
\[
\frac{1}{2} \times 1 \times 3k = 1 \\
\frac{3k}{2} = 1 \\
3k = 2 \\
k = \frac{2}{3}
\]
b 1 Write the rule for the mean.

\[ E(X) = \int_{0}^{1} xf(x)dx \]
\[ = \int_{0}^{1} (x \times 2x)dx \]
\[ = \int_{0}^{1} (2x^2)dx \]
\[ = \left[ \frac{2}{3}x^3 \right]_0^1 \]
\[ = \frac{2}{3}(1)^3 - 0 \]
\[ = \frac{2}{3} \]

2 Antidifferentiate and evaluate.

\[ \frac{2}{3} \cdot 1 - 0 = \frac{2}{3} \]

3 Write the rule for the variance.

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 \]

4 Find \( E(X^2) \).

\[ E(X^2) = \int_{0}^{1} x^2f(x)dx \]
\[ = \int_{0}^{1} 2x^3dx \]
\[ = \left[ \frac{1}{2}x^4 \right]_0^1 \]
\[ = \frac{1}{2}(1)^4 - 0 \]
\[ = \frac{1}{2} \]

5 Substitute the appropriate values into the variance formula.

\[ \text{Var}(X) = E(X^2) - [E(X)]^2 \]
\[ = \frac{1}{2} - \left( \frac{2}{3} \right)^2 \]
\[ = \frac{1}{2} - \frac{4}{9} \]
\[ = \frac{9}{18} - \frac{8}{18} \]
\[ = \frac{1}{18} \]

b \[ E(3X - 1) = 3E(X) - 1 \]
\[ = 3\left( \frac{2}{3} \right) - 1 \]
\[ = 2 - 1 \]
\[ = 1 \]

c \[ \text{Var}(3X - 1) = 3^2\text{Var}(X) \]
\[ = 9\left( \frac{1}{18} \right) \]
\[ = \frac{1}{2} \]

d \[ E(2X^2 + 3) = 2E(X^2) + 3 \]
\[ = 2\left( \frac{1}{2} \right) + 3 \]
\[ = 4 \]
Linear transformations

1. **WEB** If the continuous random variable $Y$ has a mean of 4 and a variance of 3, find:
   a. $E(2Y - 3)$
   b. $\text{Var}(2Y - 3)$
   c. $E(Y^2)$
   d. $E(Y(Y - 1))$

2. Two continuous random variables, $X$ and $Y$, are related such that $Y = aX + 5$ where $a$ is a positive integer and $E(aX + 5) = \text{Var}(aX + 5)$. The mean of $X$ is 9 and the variance of $X$ is 2.
   a. Find the value of the constant $a$.
   b. Find $E(Y)$ and $\text{Var}(Y)$.

3. **WEB** The continuous random variable $X$ has a probability density function defined by
   \[
   f(x) = \begin{cases} 
   -kx, & -2 \leq x \leq 0 \\
   kx, & 0 < x \leq 2 \\
   0, & \text{elsewhere}
   \end{cases}
   \]
   where $k$ is a constant. The graph of the function is shown.
   a. Find the value of the constant $k$.
   b. Determine $E(X)$ and $\text{Var}(X)$.
   c. Find $E(5X + 3)$ and $\text{Var}(5X + 3)$.
   d. Find $E((3X - 2)^2)$.

4. The continuous random variable $X$ has a probability density function defined by
   \[
   f(x) = \begin{cases} 
   -\cos(x), & \frac{\pi}{2} \leq x \leq \pi \\
   0, & \text{elsewhere}
   \end{cases}
   \]
   a. Sketch the graph of $f$ and verify that it is a probability density function.
   b. Calculate $E(X)$ and $\text{Var}(X)$.
   c. Calculate $E(3X + 1)$ and $\text{Var}(3X + 1)$.
   d. Calculate $E((2X - 1)(3X - 2))$.

5. For a continuous random variable $Z$, where $E(Z) = 5$ and $\text{Var}(Z) = 2$, find:
   a. $E(3Z - 2)$
   b. $\text{Var}(3Z - 2)$
   c. $E(Z^2)$
   d. $E\left(\frac{1}{3}Z^2 - 1\right)$.

6. The mean of the continuous random variable $Y$ is known to be 3.5 and its standard deviation is 1.2. Find:
   a. $E(2 - Y)$
   b. $E\left(\frac{Y}{2}\right)$
   c. $\text{Var}(Y)$
   d. $\text{Var}(2 - Y)$
   e. $\text{Var}\left(\frac{Y}{2}\right)$.

7. The length of time it takes for an electric kettle to come to the boil is a continuous random variable with a mean of 1.5 minutes and a standard deviation of 1.1 minutes.
   If each time the kettle is brought to the boil is an independent event and the kettle is boiled five times a day, find the mean and standard deviation of the total time taken for the kettle to boil during a day.
8 The probability density function for the continuous random variable $X$ is

$$f(x) = \begin{cases} mx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

where $m$ is a constant. Find:

- the value of the constant $m$
- $E(X)$ and $Var(X)$
- $E(5 - 2X)$ and $Var(5 - 2X)$.

9 The continuous random variable $Z$ has a probability density function given by

$$f(z) = \begin{cases} \frac{2}{z + 1}, & 0 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant. Calculate, correct to 4 decimal places:

- the value of the constant $a$
- the mean and variance of $Z$
- $E(3Z + 1)$ and $Var(3Z + 1)$.

10 The continuous random variable $X$ is transformed so that $Y = aX + 3$ where $a$ is a positive integer. If $E(X) = 5$ and $Var(X) = 2$, find the value of the constant $a$, given that $E(Y) = Var(Y)$. Then calculate both $E(Y)$ and $Var(Y)$ to verify this statement.

11 The continuous random variable $Y$ is transformed so that $Z = aY - 3$ where $a$ is a positive integer. If $E(Y) = 4$ and $Var(Y) = 1$, find the value(s) of the constant $a$, given that $E(Z) = Var(Z)$. Then calculate both $E(Z)$ and $Var(Z)$ to verify this statement.

12 The continuous random variable $Z$ has a probability density function given by

$$f(z) = \begin{cases} \frac{3}{\sqrt{z}}, & 1 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

- Find the value of the constant $a$.
- Calculate the mean and variance of $Z$ correct to 4 decimal places.
- Find, correct to 4 decimal places:
  - $E(4 - 3Z)$
  - $Var(4 - 3Z)$.

13 The daily rainfall, $X$ mm, in a particular Australian town has a probability density function defined by

$$f(x) = \begin{cases} \frac{x}{k\pi} \sin\left(\frac{x}{3}\right), & 0 \leq x \leq 3\pi \\ 0, & \text{elsewhere} \end{cases}$$

where $k$ is a constant.

- Find the value of the constant $k$.
- What is the expected daily rainfall, correct to 2 decimal places?
- During the winter the daily rainfall is better approximated by $W = 2X - 1$.
  - What is the expected daily rainfall during winter, correct to 2 decimal places?
14. The mass, $Y$ kilograms, of flour sold in bags labelled as 1 kilogram is known to have a probability density function given by

$$f(y) = \begin{cases} 
k(2y + 1), & 0.9 \leq y \leq 1.25 \\
0, & \text{elsewhere} \end{cases}$$

where $k$ is a constant.

a. Find the value of the constant $k$.

b. Find the expected mass of a bag of flour, correct to 3 decimal places.

c. On a particular day, the machinery packaging the bags of flour needed to be recalibrated and produced a batch which had a mass of $Z$ kilograms, where the probability density function for $Z$ was given by $Z = 0.75Y + 0.45$. What was the expected mass of a bag of flour for this particular batch, correct to 3 decimal places?

15. The continuous random variable $Z$ has a probability density function defined by

$$f(z) = \begin{cases} 
\frac{5 \log_e(z)}{\sqrt{z}}, & 1 \leq z \leq a \\
0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant. Determine, correct to 4 decimal places:

a. the value of the constant $a$

b. $E(Z)$ and $\text{Var}(Z)$

c. $E(3 - 2Z)$ and $\text{Var}(3 - 2Z)$.

16. A continuous random variable, $X$, is transformed so that $Y = aX + 1$, where $a$ is a positive constant. If $E(X) = 2$ and $\text{Var}(X) = 7$, find the value of the constant $a$, given $E(Y) = \text{Var}(Y)$. Then calculate both $E(Y)$ and $\text{Var}(Y)$ to verify this statement. Give your answers correct to 4 decimal places.
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The Review contains:
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
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- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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12 Answers

EXERCISE 12.2

1 a

This is a probability density function as the area is 1 unit².

b

This is a probability density function as the area is 1 unit².

2 a

This is a probability density function as the area is 1 unit².

b

This is not a probability density function as the area is 1.2929 units².

3 \( n = \frac{1}{18} \)

4 \( a = \frac{1}{11} \)

5 a

i 9/25

ii 4/25

b

i 37/50

ii 37/42

6 a

\( \frac{100}{200} \)

b

\( \frac{29}{100} \)

c

\( \frac{41}{50} \)

d

\( \frac{3}{100} \)

7 a

\( \frac{2}{8} \)

b

i \( \frac{31}{100} \)

ii \( \frac{37}{50} \)

c

\( \frac{21}{50} \)

8 a

This is a probability density function as the area is 1 unit².

b

This is not a probability density function as the area is \( \pi \) units².

c

This is a probability density function as the area is 1 unit².
This is a probability density function as the area is 1 unit².

9 \( c = \frac{5}{7} \)

10 \( \int_{-1}^{5} f(z)dz = 1 \)

\( \frac{1}{3} \times 6 \times z = 1 \)

\( z = \frac{1}{2} \)

11 a \( m = \frac{1}{8} \)

b \( m = 2 \)

c \( m = \frac{1}{4} \)

12 \( \int_{0}^{3} (x^2 + 2kx + 1)dx = 1 \)

\( \left[ \frac{1}{3}x^3 + kx^2 + x \right]_{0}^{3} = 1 \)

\( \left( \frac{1}{3}(3)^3 + k(3)^2 + 3 \right) - 0 = 1 \)

\( 9 + 9k + 3 = 1 \)

\( 9k + 12 = 1 \)

\( 9k = -11 \)

\( k = \frac{11}{9} \)

13 \( a = 2e \)

14 a \( f(x) = x \)

b \( f(x) = x^a \)

c \( f(x) = x \)

15 \( a = 1 \)

16 a = \( e \). As \( f(x) \geq 0 \) and \( \int_{-1}^{1} f(z)dz = 1 \), this is a probability density function.

EXERCISE 12.3

1 a \( f(x) = e^{-x} \)

b \( f(x) = -x + 1 \)

c \( f(x) = x - 1 \)

2 a \( a = 1 \)

b \( f(x) = 4x^3 \)

c \( f(x) = e^{-x} \)

3 a \( y = \frac{1}{2} \sin(x) \)
b \[ \frac{\sqrt{2}}{2} \]
\[ 2\sqrt{2} - 2 \]

c

\[ f(x) = k(2 + x) \]
\[ f(x) = k(2 - x) \]

4 a

b \[ A = \frac{1}{2}bh \]
\[ 1 = \frac{1}{2} \times 4 \times 2k \]
\[ 1 = 4k \]
\[ k = \frac{1}{4} \]

c \[ \frac{5}{7} \]

d \[ \frac{6}{7} \]

5 a \[ \frac{1}{12} \]

b \[ \frac{5}{12} \]

c \[ \frac{1}{2} \]

6 a

\[ f(x) = \frac{1}{5} \]
\[ f(x) = \frac{1}{2} \]
\[ f(x) = \frac{1}{2} \]
\[ f(x) = \frac{1}{2} \]

7 a

\[ f(z) = \frac{1}{2z} \]
\[ f(z) = \frac{1}{2z} \]
\[ f(z) = \frac{1}{2z} \]
\[ f(z) = \frac{1}{2z} \]

8 a

\[ f(z) = \frac{1}{2} \cos(z) \]

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(z) \, dz = \frac{1}{2} \sin(z) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \]
\[ = \frac{1}{2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin\left(-\frac{\pi}{2}\right) \]
\[ = \frac{1}{2} + \frac{1}{2} \]
\[ = 1 \]

This is a probability density function as the area under the curve is 1 and \( f(z) \geq 0 \) for all values of \( z \).

9 a \[ a = 1 \]

b \[ \frac{183}{256} \]

c \[ 0.371 \]

d \[ 0 \]

10 a \[ \frac{98}{125} \]

b \[ \frac{8}{9} \]

c \[ 6^i \]

11 a \[ a = 3 \log_e\left(\frac{3}{2}\right) \]

b \[ 0.6243 \]

c \[ 0.5342 \]

d \[ 0.60 \]
12 a \( f(x) = 3e^{-3x} \)

b 0.9502
c 0.0025

13 a \( a = 2.1555 \) b 0.7147

14 a 0.1560 b \( a = \tan \left( \frac{1}{2} \right) \approx 0.5463 \)

EXERCISE 12.4

1 a \( a = \frac{9}{4} \)
b i \( \frac{19}{12} \) ii 1.5625 or \( \frac{25}{16} \)

2 a 1.3104 b 0.7863
c Median = 0.8255

3 \( E(Z) = 0.7305, m = 1.3010, \text{Var}(z) = 0.3424, \) SD(Z) = 0.5851

4 \( E(X) = \frac{1}{3}, m = 0.2310, \text{Var}(X) = \frac{1}{9}, \text{SD}(X) = \frac{1}{3} \)

5 a \( \int_{0}^{1} \frac{1}{2\sqrt{x}} \, dx = \left[ \frac{1}{2} \sqrt{x} \right]_{0}^{1} = \frac{1}{2} \approx 0.1 \)

As \( f(x) \geq 0 \) for all \( x \)-values, and the area under the curve is 1, \( f(x) \) is a probability density function.
b \( \frac{1}{3} \)
c \( m = 0.25 \)

6 a 0.5 min b 0.5 min c \( m = 0.35 \) min

7 a 1.5601 b \( m = 1.6510 \)
c \( Q_1 = 1.3104, Q_3 = 1.8899 \)
d 0.5795

8 a \( a = 0.4809 \)
b 3.3663
c \( \text{VAR}(Z) = 3.8195, \text{SD}(Z) = 1.9571 \)
d 3.0751
e 7

9 a \( \int_{0}^{\frac{1}{\pi}} (\sin(2x) + 1) \, dx = \frac{1}{\pi} \int_{0}^{x} (\sin(2x) + 1) \, dx = \frac{1}{\pi} \left[ -\frac{1}{2} \cos(2x) + x \right]_{0}^{\frac{1}{\pi}} = \frac{1}{\pi} \left( \left( -\frac{1}{2} \cos(2\cdot\frac{1}{\pi}) + \frac{1}{\pi} \right) - \left( -\frac{1}{2} \cos(0) + 0 \right) \right) = \frac{1}{\pi} \left( -\frac{1}{2} + \pi + \frac{1}{2} \right) = 1 \)
As \( f(x) \geq 0 \) for all \( x \)-values, and the area under the curve is 1, \( f(x) \) is a probability density function.
b 1.0708
c i 0.5725 ii 0.7566 d \( m = 0.9291 \)

10 a \( \frac{3}{2}, b = \frac{3}{2} \)

11 a \( \int_{1}^{\pi} \frac{3}{x^2} \, dx = 1 \)
(\( \left[ -3x^{-1} \right]_{1}^{\pi} = 1 \))

12 a \( \frac{x}{\sqrt{4 - x^2}} \) b \( \frac{3}{\pi} \)

13 a \( h = \frac{1}{4} \) b 2 c 2
\[
\int \frac{a}{b} \, dx = 1
\]
\[
\int \frac{a}{b} \, dx = 1
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