3

Sequences and series

3.1 Kick off with CAS

1 Add the following sets of numbers:

a $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$

b $9 + -6 + 4 + -\frac{8}{3} + \frac{16}{9} + -\frac{32}{27}$

c $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$

2 Using CAS technology, find the template or command to simplify the following:

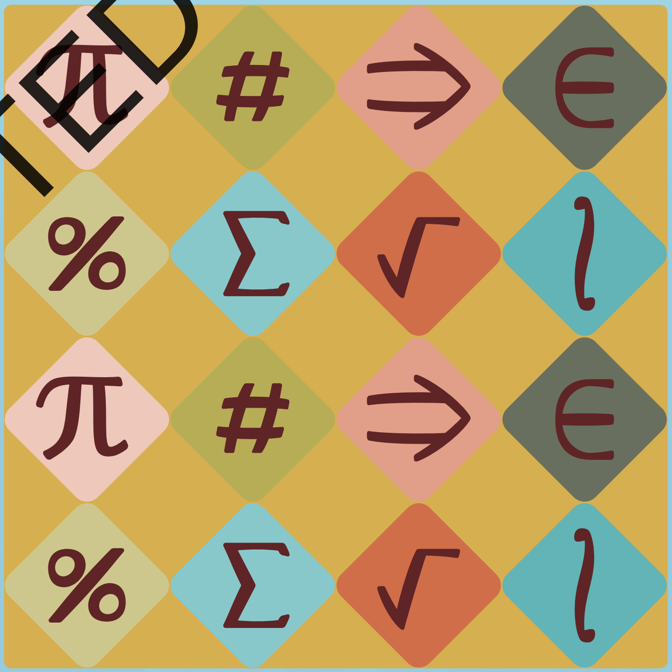
a $\sum_{n=1}^{10}(2n)$

b $\sum_{t=0}^{6} 9 \times \left(-\frac{2}{3}\right)^{t}$

c $\sum_{n=0}^{7}(2^{n})$

3 Compare the answers for questions 1 and 2, and explain how the sum (Σ) command was used to add these series of numbers.

4 Use the sum (Σ) command to add the following series:
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100$



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Describing sequences

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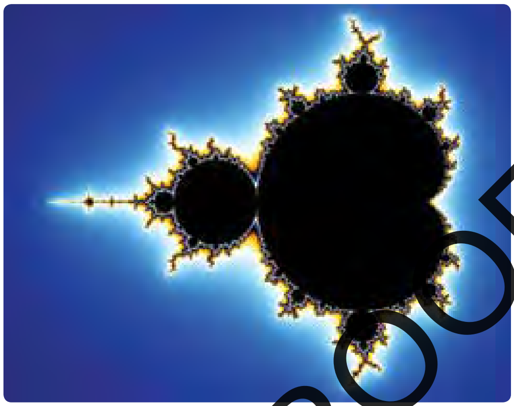


Sequences of numbers play an important part in our everyday life. For example, the following sequence:

$$2.25, 2.37, 2.58, 2.57, 2.63, \ldots$$

gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

$$10\,000, 9000, 8100, \ldots$$

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. The compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted; during the second minute, 9000, and so on. Can you predict the next number in the sequence? You're correct if you said 7290. Each successive term here is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They can be finite in number or infinite. Number sequences may follow an easily recognisable pattern or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting, to name but a few uses. The list is almost endless. The image above is a visual representation of a sequence of numbers called a Mandelbrot set.

Sequences that follow a pattern can be described in a number of different ways. They may be listed in sequential order; they may be described as a functional definition; or they may be described in an iterative definition.

Listing in sequential order

Consider the sequence of numbers t: $\{5, 7, 9, \ldots\}$. The numbers in sequential order are firstly 5 then 7 and 9, with the indication that there are more numbers to follow. The symbol t is the name of the sequence, and the first three terms in the sequence shown are $t_1 = 5$, $t_2 = 7$ and $t_3 = 9$. The fourth term, t_4, if the pattern were to continue, would be the number 11. In general, t_n is the nth term in the sequence. In this example, the next term is simply the previous term with 2 added to it, with the first term being the number 5.

Another possible sequence is t: $\{5, 10, 20, 40, \ldots\}$. In this case it appears that the next term is twice the previous term. The fifth term here, if the pattern continued, would be $t_5 = 80$. It can be difficult to determine whether or not a pattern exists in some sequences. Can you find the next term in the following sequence?

$$t: \{1, 1, 2, 3, 5, 8, \ldots\}$$

Here the next term is the sum of the previous *two terms*, hence the next term would be $5 + 8 = 13$, and so on. This sequence is called the Fibonacci sequence and is named after its discoverer Leonardo Fibonacci, a thirteenth century mathematician.

Here is another sequence; can you find the next term here?

$$t: \{7, 11, 16, 22, 29, \ldots\}$$

In this sequence the difference between successive terms increases by 1 for each pair. The first difference is 4, the next difference is 5 and so on. The sixth term is thus 37, which is 8 more than 29.

Functional definition

A functional definition of a sequence of numbers is expressed in the form: $t_n = f(n)$.

An example could be: $t_n = 2n - 7, n \in \{1, 2, 3, 4, \ldots\}$

Using this definition the nth term can be readily calculated. For this example $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value $n = 100$ into the expression for t_n.

Look at the following example:

$$d_n = 4.9n^2, n \in \{1, 2, 3, \ldots\}$$

For this example, in which the sequence is given the name d, $d_1 = 4.9 \times 1^2 = 4.9$, and $d_2 = 4.9 \times 2^2 = 19.6$. Listing the sequence would yield d: $\{4.9, 19.6, 44.1, 78.4, \ldots\}$. The 10th term would be $4.9 \times 10^2 = 490$.

Here is another example:

$$c_n = \cos(n\pi) + 1, n \in \{1, 2, 3, \ldots\}$$

Here the sequence would be c: $\{0, 2, 0, 2, \ldots\}$.

Recursive definition

A sequence can be generated by the repeated use of an instruction. This is known as **recursion**. Term n is represented by t_n; the next term after this one is represented by t_{n+1}, while the term before t_n is t_{n-1}. For these sequences, the first term must be stated.

Look at the following example:

$$t_{n+1} = 3t_n - 2; t_1 = 6.$$

The first term, t_1, is 6 (this is given in the definition), so the next term, t_2, is $3 \times 6 - 2 = 16$, and the following term is $3 \times 16 - 2 = 46$. In each and all cases, the next term is found by multiplying the previous term by 3 and then subtracting 2. We could write the sequence out as a table:

n	t_n	Comment
1	$t_1 = 6$	Given in the definition
2	$t_2 = 3t_1 - 2$ $= 3 \times 6 - 2$ $= 16$	Using t_1 to find the next term, t_2
3	$t_3 = 3t_2 - 2$ $= 3 \times 16 - 2$ $= 46$	Using t_2 to find the next term, t_3
4	$t_4 = 3t_3 - 2$ $= 3 \times 46 - 2$ $= 136$	Using t_3 to find the next term, t_4

An example of this sequence using notation found in a spreadsheet would be:

A1 = 6 (the first term is equal to 6)

A2 = 3 × A1 − 2 (the next term is 3 times the previous term minus 2).

You could then apply the **Fill Down** option in the **Edit** menu of the spreadsheet from cell A2 downwards to generate as many terms in the sequence as required. This would result in the next cell down being three times the previous cell, less 2. The recursive definition finds a natural use in a spreadsheet environment and consequently is used often. A drawback is that you cannot find the *n*th term directly as in the functional definition, but an advantage is that more complicated systems can be successfully modelled using recursive descriptions.

WORKED EXAMPLE 1

a **Find the next three terms in the sequence: $\left\{14, 7, \frac{7}{2}, \ldots\right\}$.**

b **Find the 4th, 8th and 12th terms in the following sequence: $e_n = n^2 - 3n$, $n \in \{1, 2, 3, \ldots\}$.**

c **Find the second, third and fifth terms for the following sequence: $k_{n+1} = 2k_n + 1$, $k_1 = -0.50$.**

THINK	WRITE
a In this example the sequence is listed and a simple pattern is evident. From inspection, the next term is half the previous term and so the sequence would be $14, 7, \frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.	**a** The next three terms are $\frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.
b **1** This is an example of a functional definition. The *n*th term of the sequence is found simply by substitution into the expression $e_n = n^2 - 3n$.	**b** $e_n = n^2 - 3n$
2 Find the fourth term by substituting $n = 4$.	$e_4 = 4^2 - 3 \times 4$ $= 4$
3 Find the eighth term by substituting $n = 8$.	$e_8 = 8^2 - 3 \times 8$ $= 40$
4 Find the 12th term by substituting $n = 12$.	$e_{12} = 12^2 - 3 \times 12$ $= 108$
c **1** This is an example of a recursive definition. We can find the second, third and fifth terms for the sequence $k_{n+1} = 2k_n + 1$, $k_1 = -0.50$ by recursion.	**c** $k_{n+1} = 2k_n + 1,$ $k_1 = -0.50$
2 Substitute $k_1 = -0.50$ into the formula to find k_2.	$k_2 = 2 \times -0.50 + 1$ $= 0$
3 Continue the process until the value of k_5 is found.	$k_3 = 2 \times 0 + 1$ $= 1$ $k_4 = 2 \times 1 + 1$ $= 3$ $k_5 = 2 \times 3 + 1$ $= 7$
4 Write the answer.	Thus $k_2 = 0$, $k_3 = 1$ and $k_5 = 7$.

Logistic equation

The **logistic equation** is a model of population growth. It gives the rule for determining the population in any year, based on the population in the previous year. Because we need the previous term in order to generate the next term of the sequence, the logistic equation is an example of a recursive definition. It is of the general form

$$t_{n+1} = at_n(1 - t_n)$$

where $0 < t_0 < 1$ and a is a constant.

Depending on the value of a, sequences generated by use of the logistic equation could be **convergent**, **divergent** or **oscillating**. A string of numbers that converges to (settles at) a certain fixed value is called a *convergent* sequence. A sequence t_n can converge to only one possible number, x, called the **limit** of the sequence. This can be written as $\lim\limits_{n\to\infty} t_n = x$. (In this context the symbol $\to$ is read as 'tends to' or 'approaches'.) A sequence whose terms grow further and further apart is called *divergent*. That is, a sequence is divergent if $t_n \to \infty$ or $t_n \to -\infty$ as $n \to \infty$. Finally, a sequence whose terms tend to fluctuate between two (or more) values is called *oscillating*. An oscillating sequence is neither convergent nor divergent.

WORKED EXAMPLE 2

Given that $a = 2$ and $t_0 = 0.7$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim\limits_{n\to\infty} t_n$ to determine the limit.

THINK	WRITE
1 Write the logistic equation, replacing a with its given value (that is, 2).	$t_{n+1} = at_n(1 - t_n)$ $= 2t_n(1 - t_n)$
2 To find t_1, substitute the value of t_0 (that is, 0.7) in place of t_n and evaluate.	$t_1 = 2t_0(1 - t_0)$ $= 2 \times 0.7 \times (1 - 0.7)$ $= 0.42$
3 To find the next term, t_2, substitute the value of t_1 (that is, 0.42) in place of t_n and evaluate.	$t_2 = 2t_1(1 - t_1)$ $= 2 \times 0.42 \times (1 - 0.42)$ $= 0.4872$
4 Continue the iterative process four more times, each time substituting the value of the previous term into the logistic equation to find the next term.	$t_3 = 2t_2(1 - t_2)$ $= 2 \times 0.4872 \times (1 - 0.4872)$ $= 0.499\,672\,3$ $t_4 = 2t_3(1 - t_3)$ $= 2 \times 0.499\,672\,3 \times (1 - 0.499\,672\,3)$ $= 0.499\,999\,8$ $t_5 = 2t_4(1 - t_4)$ $= 2 \times 0.499\,999\,8 \times (1 - 0.499\,999\,8)$ $= 0.5$ $t_6 = 2t_5(1 - t_5)$ $= 2 \times 0.5 \times (1 - 0.5) = 0.5$
5 The terms of the sequence are growing closer and closer to each other, finally settling at 0.5.	The sequence is convergent; the limit of the sequence is 0.5.

Note that instead of saying ‘the limit of the sequence is 0.5’ in the previous example, we could simply write $t_n \to 0.5$.

EXERCISE 3.2

Describing sequences

PRACTISE

1 **WE1** a Find the next three terms in the sequence: $\left\{3, \frac{3}{2}, \frac{3}{4}, \ldots\right\}$.

b Find the second, fourth and sixth terms in the following sequence: $t_n = 4 \times 3^{n-2}$, $n \in \{1, 2, 3, \ldots\}$.

c Find the second, third and fifth terms for the following sequence: $k_{n+1} = k_n + 2$, $k_1 = -5$.

2 a Find the next three terms in the sequence: $\{2, -5, 8, -11, 14, \ldots\}$.

b Find the 4th, 8th and 12th terms in the following sequence: $t_n = n^2 - n + 41$, $n \in \{1, 2, 3, \ldots\}$.

c Find the second, third and fourth terms for the following sequence: $k_{n+1} = -\left(k_n{}^2\right) - 2$, $k_1 = 3$.

3 **WE2** Given that $a = 0.8$ and $t_0 = 0.5$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim\limits_{n\to\infty} t_n$ to determine the limit.

4 Given that $a = 1.1$ and $t_0 = 0.9$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim\limits_{n\to\infty} t_n$ to determine the limit.

CONSOLIDATE

5 For each of the following sequences, write a rule for obtaining the next term in the sequence and hence evaluate the next three terms.

a $\{1, 4, 7, \ldots\}$

b $\{1, 0, -1, -2, \ldots\}$

c $\{1, 4, 16, 64, \ldots\}$

d $\{2, 5, 9, 14, 20, \ldots\}$

e $\{3, 4, 7, 11, 18, \ldots\}$

f $\{2a - 5b, a - 2b, b, -a + 4b, \ldots\}$

g $\{1, 0, -1, 0, 1, \ldots\}$

h $\{1.0, 1.1, 1.11, \ldots\}$

i $\{1024, -512, 256, -128, \ldots\}$

6 Find the first, fifth and tenth terms in the following sequences.

a $t_n = 2n - 5, n \in \{1, 2, 3, \ldots\}$

b $t_n = \dfrac{n}{n+1}, n \in \{1, 2, 3, \ldots\}$

c $t_n = 17 - 3.7n, n \in \{1, 2, 3, \ldots\}$

d $t_n = 5 \times \left(\dfrac{1}{2}\right)^n, n \in \{1, 2, 3, \ldots\}$

e $t_n = 5 \times \left(\dfrac{1}{2}\right)^{(3-n)}, n \in \{1, 2, 3, \ldots\}$

f $t_n = (-1)^n + n, n \in \{1, 2, 3, \ldots\}$

g $t_n = 3^n 2^{-n}, n \in \{1, 2, 3, \ldots\}$

h $t_n = a + (n-1)d, n \in \{1, 2, 3, \ldots\}$

i $t_n = ar^{n-1}, n \in \{1, 2, 3, \ldots\}$

7 Using technology, find the third, eighth and tenth terms in the following sequences.

a $t_{n+1} = -2t_n, t_1 = -3$

b $t_{n+1} = t_n - 7, t_1 = 14$

c $t_{n+1} = -t_n + 2, t_1 = 3$

d $t_{n+1} = t_n + (-1)^n t_n, t_1 = 3$

8 For the sequences in question 7, use technology to generate their graphs. Place the term number on the horizontal axis and the value of the term on the vertical axis.

9 Given the following values of a and t_0, use the logistic equation to generate a sequence of six terms. State whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, state its limit.

a $a = 0.4, t_0 = 0.6$
b $a = 1.9, t_0 = 0.4$
c $a = 2.1, t_0 = 0.5$
d $a = 2.5, t_0 = 0.3$
e $a = 3, t_0 = 0.2$
f $a = 3.4, t_0 = 0.7$
g $a = 4.2, t_0 = 0.1$
h $a = 4.5, t_0 = 0.8$

10 Study the pattern in each of the following sequences and where possible write the next two terms in the sequence, describing the pattern that you use.

a $5, 6, 8, 11, \ldots$
b $4, 9, 12, 13, 12, 9, \ldots$
c $9, 8, 9, 0, \ldots$
d $6, 12, 12, 6, 1\frac{1}{2}, \ldots$
e $5, 8, 13, 21, \ldots$
f $1, 3, 7, 15, \ldots$
g $1, 3, 2, 4, 3, \ldots$

11 Which of the following functional definitions could be used to describe the sequence $\{3, 1, -1, \ldots\}$?

A $t_n = n - 2, n \in \{1, 2, 3, \ldots\}$
B $t_n = 2n - 5, n \in \{1, 2, 3, \ldots\}$
C $t_n = 5n - 2, n \in \{1, 2, 3, \ldots\}$
D $t_n = 5 - 2n, n \in \{1, 2, 3, \ldots\}$
E $t_n = 2(5 - n), n \in \{1, 2, 3, \ldots\}$

12 Which of the following recursive definitions could be used to describe the sequence $\{20, -10, 5, \ldots\}$?

A $t_{n+1} = t_n - 30, t_1 = 20$
B $t_{n+1} = \frac{t_n}{2}, t_1 = -20$
C $t_{n+1} = t_n - \frac{t_n}{2}, t_1 = 20$
D $t_{n+1} = t_n - 10, t_1 = 20$
E $t_{n+1} = \frac{-t_n}{2}, \ t_1 = 20$

13 Which of the following sequences is generated by the definition $t_n = \frac{6n^2 - 12}{2}$, $n \in \{1, 2, 3, \ldots\}$?

A $\{-3, 6, 15, \ldots\}$
B $\{-3, 6, -12, \ldots\}$
C $\{-3, 6, 21, \ldots\}$
D $\{-3, 6, 12, \ldots\}$
E $\{-3, 6, 18, \ldots\}$

14 Write the iterative definition for each of the following sequences.

a $\{7, 5, 3, 1, -1, \ldots\}$
b $\{12, 6, 3, 1.5, \ldots\}$
c $\{12, 12.6, 13.2, \ldots\}$
d $\{2, 11, 56, 281, \ldots\}$
e $\{4, -12, 36, \ldots\}$
f $\{2, 4, 16, 256, \ldots\}$

MASTER

15 In the township of Grizabella, the population of stray cats in any given year is given as p_{n+1}. This can be calculated using the formula $p_{n+1} = 1.3p_n(1 - p_n)$, where p_n is the number of cats (in hundreds) in the preceding year. If in 2005 there were 28 stray cats in Grizabella township, calculate:

a the expected number of stray cats for 2006 and 2007
b the limiting number of stray cats that Grizabella township can sustain.

16 In the township of Macavity, the population of stray cats follows the logistic equation $p_{n+1} = 0.3p_n(1 - p_n)$ where p_{n+1} and p_n refer to the population (in hundreds) in any given year and in the preceding year respectively. In 2005, there were 62 stray cats in the township. By generating the sequence of numbers using the above equation, decide what will happen in the long run to the population size of stray cats. (That is, will the population of cats keep increasing, keep decreasing or settle at a particular value?)

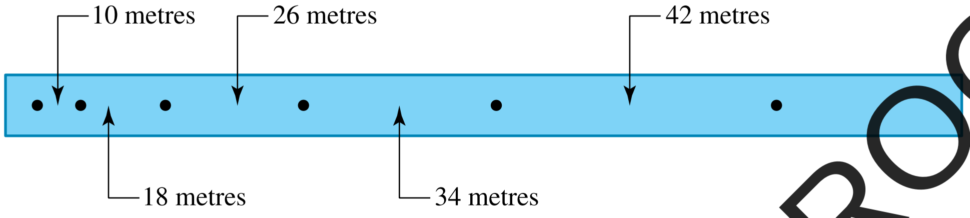
3.3 Arithmetic sequences

studyon

Units 1 & 2
AOS 2
Topic 2
Concept 2

Arithmetic sequences
Concept summary
Practice questions

At a racetrack a new prototype racing car unfortunately develops an oil leak. Each second, a drop of oil hits the road. The driver of the car puts her foot on the accelerator and the car increases speed at a steady rate as it hurtles down the straight. The diagram below shows the pattern of oil drops on the road with the distances between the drops labelled.



The sequence of distances travelled in metres each second is $\{10, 18, 26, 34, 42, \ldots\}$. The first term in the sequence, t_1, is 10, and as you can see, each subsequent term is 8 more than the previous term. This type of sequence is given a special name — an **arithmetic sequence**.

An arithmetic sequence is a sequence where there is a common difference between any two successive terms.

We can list the sequence in a table as shown in table A below. From this table we can see that it is possible to write a **functional definition** for the sequence in terms of the first term, 10, and the common difference, 8, and thus:

$$t_n = 2 + 8n, \; n \in \{1, 2, 3, \ldots\}$$

This can be rewritten as: $t_n = 10 + (n - 1) \times 8$.

We can readily get a general formula for the nth term of an arithmetic sequence whose first term is a and whose common difference is d (see table B).

Table A

n	t_n	t_n
1	$10 + 0 \times 8$	10
2	$10 + 1 \times 8$	18
3	$10 + 2 \times 8$	26
4	$10 + 3 \times 8$	34
n	$10 + (n - 1) \times 8$ $= 10 + 8n - 8$ $= 2 + 8n$	$2 + 8n$

Table B

n	t_n
1	$a + 0 \times d$
2	$a + 1 \times d$
3	$a + 2 \times d$
4	$a + 3 \times d$
n	$a + (n - 1) \times d$ $= (a - d) + dn$

In general, then:

The *n*th term of an arithmetic sequence is given by

$$\begin{aligned} t_n &= a + (n-1) \times d \\ &= (a-d) + nd, n \in \{1, 2, 3, \ldots\} \end{aligned}$$

where *a* is the first term and *d* is the common difference.

If we consider three successive terms in an arithmetic sequence, namely x, y and z, then since $y - x =$ the common difference, d, and $z - y = d$, it follows that:

$$y - x = z - y \Rightarrow y = \frac{z + x}{2}$$

The middle term of any three consecutive terms in an arithmetic sequence is called an *arithmetic mean* and is the average of the outer two terms.

That is, $y = \dfrac{z + x}{2}$ for any 3 consecutive terms x, y, z of an arithmetic sequence.

WORKED EXAMPLE 3

State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the *n*th term for the sequence, t_n.

a t: {4, 9, 15, . . .} **b** t: {−2, 1, 4, . . .}

THINK	WRITE
a **1** To check that a sequence is arithmetic, see if a common difference exists.	**a** $9 - 4 = 5$ $15 - 9 = 6$
2 There is no common difference as $5 \neq 6$.	Since there is no common difference the sequence is not arithmetic.
b **1** To check that a sequence is arithmetic, see if a common difference exists.	**b** $1 - -2 = 3$ $4 - 1 = 3$
2 The common difference is 3.	The sequence is arithmetic with the common difference $d = 3$.
3 The next term in the sequence, t_4, can be found by adding 3 to the previous term, t_3.	$t_4 = t_3 + 3$ $= 4 + 3$ $= 7$
4 To find the functional definition, write the formula for the *n*th term of the arithmetic sequence.	$t_n = a + (n-1) \times d$ $= (a - d) + nd$
5 Identify the values of a and d.	$a = -2$ and $d = 3$
6 Substitute $a = -2$ and $d = 3$ into the formula and simplify.	$t_n = (-2 - 3) + n \times 3$ $t_n = 3n - 5$

Graphing an arithmetic sequence

- Since an arithmetic sequence involves adding or subtracting the same value repeatedly, the relationship between the terms is a linear one.
- This means that the graph of terms of an arithmetic sequence forms a straight line.

WORKED EXAMPLE 4

For the arithmetic sequence 2, 4, 6, 8, 10, . . .

a use CAS to draw up a table showing the term number with its value

b use CAS to graph the values in the table

c from your graph, determine the value of the tenth term in the sequence.

THINK

a Draw up a table using CAS to show the term number and value.

WRITE/DRAW

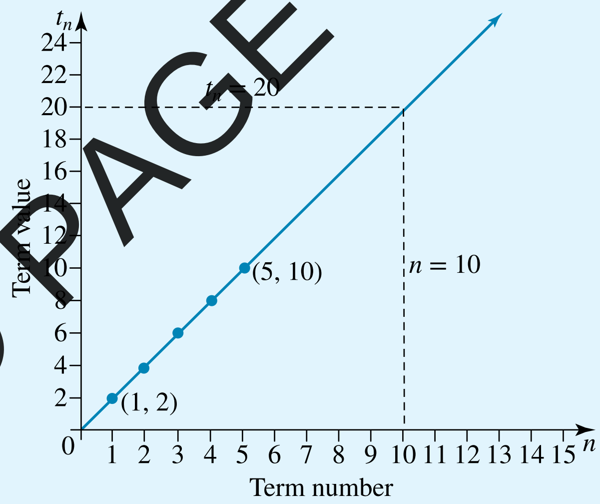
a

Term number	1	2	3	4	5
Term value	2	4	6	8	10

b The value of the term depends on the term number, so ‘value’ is graphed on the y-axis. Draw up a suitable scale on both axes, plot the points and join with a straight line. Note that, even though the graph is a straight line, values on the line have meaning only for integer values of the term number.

c 1 Find the term number 10 on the x-axis, and draw a vertical line from the x-axis to meet the straight line. Read the y-value of this point.

b



2 Write the answer.

c The tenth number in this sequence is 20.

WORKED EXAMPLE 5

Find the missing terms in this arithmetic sequence: {41, a, 55, b, . . .}.

THINK

1 The first three successive terms are 41, a, 55. Write the rule for the middle term of the three successive terms of an arithmetic sequence.

2 Identify the variables.

3 Substitute the values of x, y and z into the formula in step 1 and evaluate.

WRITE

For x, y, z: $y = \dfrac{x + z}{2}$

$x = 41;\ y = a;\ z = 55$

$$a = \frac{41 + 55}{2}$$
$$= 48$$

4 Find the common difference. (The second term is now known.)	$d = t_2 - t_1$ $= 48 - 41$ $= 7$
5 Find the value of b by adding the common difference to the preceding term.	$b = 55 + 7$ $= 62$
6 State your answer.	So $a = 48$, $b = 62$.

WORKED EXAMPLE 6 **Find the 16th and nth terms in an arithmetic sequence with the 4th term 15 and 8th term 37.**

THINK	WRITE
1 Write the formula for the nth term of the arithmetic sequence.	$t_n = a + (n - 1) \times d$
2 Substitute $n = 4$ and $t_4 = 15$ into the formula and label it equation [1].	t_4: $a + 3d = 15$ [1]
3 Substitute $n = 8$ and $t_8 = 37$ into the formula and label it equation [2].	t_8: $a + 7d = 37$ [2]
4 Solve the simultaneous equations: subtract equation [1] from equation [2] to eliminate a.	[2] − [1]: $a + 7d - a - 3d = 37 - 15$
5 Divide both sides by 4.	$4d = 22$ $d = \frac{22}{4}$ $= 5\frac{1}{2}$
6 Substitute $d = 5\frac{1}{2}$ into equation [1] and solve for a.	Substituting $d = 5\frac{1}{2}$ into [1]: $a + 3 \times 5\frac{1}{2} = 15$ $a = -1\frac{1}{2}$
7 To find the nth term of the arithmetic sequence, substitute the values of a and d into the general formula and simplify.	$t_n = -1\frac{1}{2} + (n - 1) \times 5\frac{1}{2}$ $= \frac{-3}{2} + (n - 1)\frac{11}{2}$ $= \frac{-3 + 11n - 11}{2}$ $= \frac{11n - 14}{2}$
8 To find the 16th term, substitute $n = 16$ into the formula established in the previous step and evaluate.	$t_n = \frac{11n - 14}{2}, n \in \{1, 2, 3, \ldots\}$ If $n = 16$, $t_{16} = \frac{11 \times 16 - 14}{2}$ $= 81$

EXERCISE 3.3 Arithmetic sequences

PRACTISE

1 **WE3** State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the nth term for the sequence, t_n.

a t: $\{3, 6, 12, \ldots\}$

b t: $\{-3, 0, 3, \ldots\}$

2 State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the nth term for the sequence, t_n.

a t: $\{4, 7, 11, \ldots\}$

b t: $\{-2, -6, -10, \ldots\}$

3 **WE4** For the arithmetic sequence 4, 8, 12, 16, 20, . . .

a use CAS technology to draw up a table showing the term number with its value
b use CAS technology to graph the values in the table
c from your graph, determine the value of the tenth term in the sequence.

4 For the arithmetic sequence 10, 13, 16, 19, 22, . . .

a use CAS technology to draw up a table showing the term number with its value
b use CAS technology to graph the values in the table
c from your graph, determine the value of the tenth term in the sequence.

5 **WE5** Find the missing terms in this arithmetic sequence: $\{16, m, 27, n\}$.

6 Find the missing terms in this arithmetic sequence: $\{33, x, 61, y\}$.

7 **WE6** Find the fourth term and nth term in the arithmetic sequence whose first term is 6 and whose seventh term is −10.

8 Find the eighth and nth terms in an arithmetic sequence with the third term 3.44 and the 12th term 5.42.

CONSOLIDATE

9 Find the term given in brackets for each of the following arithmetic sequences.

a $\{4, 9, 14, \ldots\}$, (t_{21})
b $\{-2, 10, 22, \ldots\}$, (t_{58})
c $\{-27, -12, 3, \ldots\}$ (t_{100})
d $\{2, -11, -24, \ldots\}$ (t_{2025})

10 Find the functional definition for the nth term of the arithmetic sequence:

a where the first term is 5 and the common difference is −3
b where the first term is 2.5 and the common difference is $\frac{1}{2}$
c where the first term is −3 and the common difference is 3
d where the first term is $2x$ and the common difference is $5x$.

11 Find the nth term in the arithmetic sequence where the first term is 6 and the third term is 10.

12 Find the nth term in the arithmetic sequence where the first term is 3 and the third term is 13.

13 Find the missing terms in the arithmetic sequence below.

$x - 3y$, ________________, $-3x + 5y$, ________________, . . .

14 If $t_{10} = 100$ and $t_{15} = 175$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

15 If $t_{10} = \frac{-1}{2}$ and $t_{13} = \frac{3}{4}$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

16 For the arithmetic sequence $\{22, m, n, 37, \ldots\}$, find the values for m and n.

17 The first three terms in an arithmetic sequence are 37, 32 and 27, and the kth term is -3. Find the value for k.

18 Find the value of x such that the following forms an arithmetic progression: $\ldots x, 3x + 4, 10x - 7 \ldots$

MASTER

19 For the following arithmetic sequences, find the recursive definition and use technology to generate the first 50 numbers in the sequence.

a $t_n: \{3, 7, 11, \ldots\}$

b $t_n: \{-3, 0, 3, \ldots\}$

c $t_n: \{-2, -6, -10, \ldots\}$

d $t_n: \left\{\frac{2}{7}, \frac{11}{14}, \frac{9}{7}, \ldots\right\}$

e $t_n: \left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, \ldots\right\}$

f $t_n: \left\{\frac{1}{4}, \frac{-3}{2}, \frac{-13}{4}, \ldots\right\}$

g $t_n: \{2\pi + 3, 4\pi + 1, 6\pi - 1, \ldots\}$

20 The ratio between the first term and the second term in an arithmetic sequence is $\frac{3}{4}$. The ratio between the second term and the third term is $\frac{4}{5}$.

a Calculate the ratio of the third term to the fourth term.

b Find the ratio of the nth and the nth + 1 term in the sequence.

3.4 Arithmetic series

Often we have a sequence of numbers and want to know their sum. For an example, we return to the oil drops on the racetrack from the start of the previous section on arithmetic sequences. The distance covered by the car each second illustrated the concept of an arithmetic sequence.

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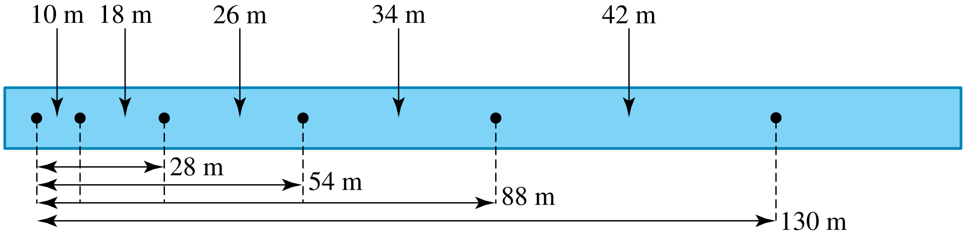
The total distance covered by the car is the sum of the individual distances covered each second. So after one second the car has travelled 10 m, after two seconds the car has travelled 10 + 18 m = 28 m, after three seconds the car has travelled a total distance of 10 + 18 + 26 m = 54 m, and so on.



A series, S_n, is the sum of a sequence of n terms $t_1 + t_2 + t_3 + \ldots + t_n$.

Thus:

$$
\begin{aligned}
S_1 &= t_1 \\
S_2 &= t_1 + t_2 \\
S_3 &= t_1 + t_2 + t_3 \\
S_n &= t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n
\end{aligned}
$$

For an arithmetic sequence, the sum of the first n terms, S_n, can be written in two ways.

1. The first term in the arithmetic sequence is a, the common difference is d, and the last term — that is, the nth term — in the sequence is l.

$$
\begin{aligned}
S_n &= a + (a + d) + (a + 2d) + \ldots (a + (n - 3)d) + (a + (n - 2)d) + a + (n - 1)d \\
&= a + (a + d) + (a + 2d) + \ldots (l - 3d) + (l - 2d) + (l - d) + l \qquad [1]
\end{aligned}
$$

2. We can write the sum S_n in reverse order starting with the nth term and summing back to the first term a:

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a \qquad [2]$$

If we add equation [1] and equation [2] together and recognise that there are n terms, each of which equals $(a + l)$, we get:

$$
\begin{aligned}
2S_n &= (a + l) + (a + l) + \ldots n \text{ times} \\
&= n(a + l)
\end{aligned}
$$

and so: $$S_n = \frac{n}{2}(a + l)$$

or since l is the nth term, $l = a + (n - 1)d$, so $S_n = \frac{n}{2}[a + a + (n - 1)d]$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

The sum of the first n terms in an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a + l)$$

where a is the first term and l is the last term; or alternatively, since $l = a + (n - 1)d$, by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

where a is the first term and d is the common difference.

If we know the first term, a, the common difference, d, and the number of terms, n, that we wish to add together, we can calculate the sum directly without having to add up all the individual terms.

It is worthwhile also to note that $S_{n+1} = S_n + t_{n+1}$. This tells us that the next term in the series S_{n+1} is the present sum, S_n, plus the next term in the sequence, t_{n+1}. This result is useful in spreadsheets where one column gives the sequence and an adjacent column is used to give the series.

WORKED EXAMPLE 7 Find the sum of the first 20 terms in the sequence t_n: {12, 25, 38, . . .}.

THINK	WRITE
1 Write the formula for the sum of the first n terms in the arithmetic sequence.	$S_n = \frac{n}{2}(2a + (n - 1)d)$
2 Identify the variables.	$a = 12$, $d = 25 - 12 = 13$, $n = 20$
3 Substitute values of a, d and n into the formula and evaluate.	$S_{20} = \frac{20}{2}(2 \times 12 + 19 \times 13)$ $S_{20} = 2710$

EXERCISE 3.4 Arithmetic series

PRACTISE

1 **WE7** Find the sum of the first 20 terms in the sequence t_n: {1, 3, 5, . . .}

2 Find the sum of the first 50 terms in the sequence $t_n = 3n + 7$, $n \in \{1, 2, 3, \ldots\}$

CONSOLIDATE

3 a Find the sum of the first 50 positive integers.

b Find the sum of the first 100 positive integers.

4 a Find the sum of all the half-integers between 0 and 100.

Note: The sequence of half-integers is $\left\{\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, \ldots\right\}$.

b Compare your answer with that for question **3b**.

5 Find the sum of the first 12 terms of an arithmetic sequence in which the second term is 8 and 13th term is 41.

6 A sequence of numbers is defined by t_n: {15, 9, 3, −3, . . .}.

a Find the sum of the first 13, 16 and 19 terms in the sequence.

b Find the sum of all the terms between and including t_{10} and t_{15}.

7 A sequence of numbers is defined by $t_n = 2n - 7$, $n \in \{1, 2, 3, \ldots\}$. Find:

a the sum of the first 20 terms

b the sum of all the terms between and including t_{21} and t_{40}

c the average of the first 40 terms. *Hint:* You need to find the sum first.

8 Find the equation that gives the sum of the first n positive integers.

9 a Show that the sum of the first n odd integers is equal to the perfect square n^2.

b Show that the sum of the first n even integers is equal to $n^2 + n$.

10 A sequence is 5, 7, 9, 11, . . . How many consecutive terms need to be added to obtain 357?

11 Consider the sum of the first n integers. For what value of n will the sum first exceed 1000?

12 The first term in an arithmetic sequence is 5, and the sum of the first 20 terms is 1240. Find the common difference, d.

MASTER

13 The sum of the first four terms of an arithmetic sequence is 58, and the sum of the next four terms is twice that number. Find the sum of the following four terms.

14 The sum of a series is given by $S_n = 4n^2 + 3n$. Use the result that $t_{n+1} = S_{n+1} - S_n$ to prove that the sequence of numbers, t_n, whose series is $S_n = 4n^2 + 3n$ is arithmetic. Find both the functional and recursive equations for the sequence, t_n.

3.5 Geometric sequences

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Geometric sequences
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A farmer is breeding worms that he hopes to sell to local shire councils to decompose waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $\left(1 + \frac{10}{100}\right)$ or 1.1.

He starts off with 10 kg of worms. By the beginning of the second week he will expect $10 \times 1.1 = 11$ kg of worms, by the start of the third week he would expect $11 \times 1.1 = 10 \times (1.1)^2 = 12.1$ kg of worms, and so on. This is an example of a **geometric sequence**.

A geometric sequence is a sequence where each term is obtained by multiplying the preceding term by a certain constant factor.

The first term is 10, and the common factor is 1.10, which represents a 10% increase on the previous term. We can put the results of this example into a table.

n	t_n	t_n
1	$10\ (= 10 \times (1.1)^0)$	10
2	$10 \times (1.1)^1$	11
3	$10 \times (1.1)^2$	12.1
4	$10 \times (1.1)^3$	13.31
n	$10 \times (1.1)^{n-1}$	$10 \times (1.1)^{n-1}$

From this table we can see that

$$t_2 = 1.1 \times t_1,\ t_3 = 1.1 \times t_2$$

and so on. In general:

$$t_{n+1} = 1.1 \times t_n$$

The common factor or common ratio whose value is 1.1 for this example can be found by dividing any two successive terms: $\frac{t_{n+1}}{t_n}$.

A geometric sequence, t, can be written in terms of the first term, a, and the common ratio, r. Thus:

$$t\colon \{a, ar, ar^2, ar^3, \ldots, ar^{n-1}\}$$

The first term $t_1 = a$, the second term $t_2 = ar$, the third term $t_3 = ar^2$, and consequently the nth term, t_n is ar^{n-1}.

For a geometric sequence:

$$t_n = ar^{n-1}$$

where *a* is the first term and *r* the common ratio, given by

$$r = \frac{t_{n+1}}{t_n}.$$

If we consider three consecutive terms in a geometric sequence $\{x, y \text{ and } z\}$ then

$$\frac{y}{x} = r = \frac{z}{y}$$

where r is the common factor.

Thus the middle term, y, called the **geometric mean**, can be calculated in terms of the outer two terms, x and z.

For a geometric sequence $\{\ldots, x, y, z, \ldots\}$:

$$y^2 = xz$$

WORKED EXAMPLE 8

State whether the sequence t_n: {2, 6, 18, . . .} is geometric by finding the ratio of successive terms. If it is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

THINK	WRITE
1 Find the ratio $\frac{t_2}{t_1}$.	$\frac{t_2}{t_1} = \frac{6}{2}$ $= 3$
2 Find the ratio $\frac{t_3}{t_2}$.	$t_3 = \frac{18}{6}$ $= 3$
3 Compare the ratios and make your conclusion.	Since $\frac{t_2}{t_1} = \frac{t_3}{t_2} = 3$, the sequence is geometric with the common ratio $r = 3$.
4 Because the sequence is geometric, find the fourth term by multiplying the preceding (third) term by the common ratio.	$t_4 = t_3 \times r$ $= 18 \times 3$ $= 54$
5 Write the general formula for the nth term.	$t_n = ar^{n-1}$
6 Identify the values of a and r.	$a = 2; r = 3$
7 Substitute the values of a and r into the general formula.	$t_n = 2 \times 3^{n-1}$
8 Check the value for t_4.	$t_4 = 2 \times 3^{4-1} = 2 \times 27 = 54$

Graphs of geometric sequences

While the graph of an arithmetic sequence is a straight line, the graph of a geometric sequence is a curve for values of $r > 0$. (Different values of r produce graphs of different shapes.)

WORKED EXAMPLE 9

Consider the geometric sequence 2, 4, 8, 16, 32, . . .

a Using CAS, draw up a table showing the term number and its value.

b Using CAS, graph the entries in the table.

c Comment on the shape of the graph.

THINK

a Draw up a table showing the term number and its corresponding value.

b The value of the term depends on the term number, so ‘Term value’ is graphed on the *y*-axis. Draw a set of axes with suitable scales. Plot the points and join with a smooth curve.

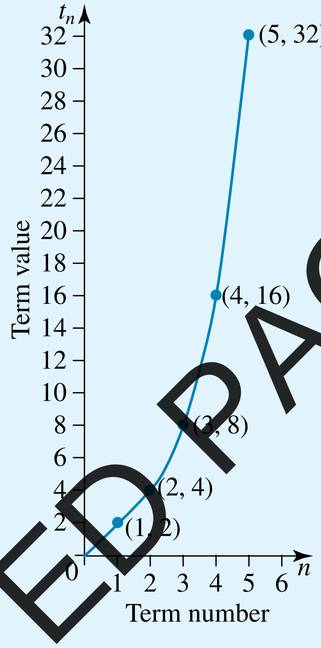
c Comment on the shape of the curve.

WRITE/DRAW

a

Term number	1	2	3	4	5
Term value	2	4	8	16	32

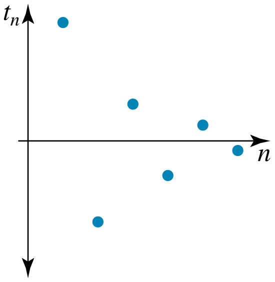
b



c The points lie on a smooth curve which increases rapidly. Because of this rapid increase in value, it would be difficult to use the graph to predict future values in the sequence with any accuracy.

Shape of graphs of geometric sequences

The shape of the graph of a geometric sequence depends on the value of *r*.



- When $r > 1$, the points lie on a curve, as shown in the graph in the previous worked example. Graphs of this kind are said to *diverge* (i.e. they move further and further away from the starting value).
- When $r < -1$, the points *oscillate* on either side of zero.
- When $-1 < r < 0$, the points oscillate and *converge* to a certain fixed number, as shown in the graph at right.
- When $0 < r < 1$, the points converge to a certain fixed number.

WORKED EXAMPLE 10 **Find the nth term and the tenth term in the geometric sequence where the first term is 3 and the third term is 12.**

THINK	WRITE
1 Write the general formula for the nth term in the geometric sequence.	$t_n = ar^{n-1}$
2 State the value of a (the first term in the sequence) and the value of the third term.	$a = 3;\ t_3 = 12$
3 Substitute all known values into the general formula.	$12 = 3 \times r^{3-1}$ $= 3 \times r^2$
4 Solve for r (note that there are two possible solutions).	$r^2 = \frac{12}{3}$ $= 4$ $r = \pm\sqrt{4}$ $= \pm 2$
5 Substitute the values of a and r into the general equation. Because there are two possible values for r, you must show both expressions for the nth term of the sequence.	So $t_n = 3 \times 2^{n-1}$, or $t_n = 3 \times (-2)^{n-1}$
6 Find the tenth term by substituting $n = 10$ into each of the two expressions for the nth term.	When $n = 10$, $t_{10} = 3 \times 2^{10-1}$(using $r = 2$) $= 3 \times 2^9$ $= 1536$ or $t_{10} = 3 \times (-2)^{10-1}$ (using $r = -2$) $= 3 \times (-2)^9 = -1536$

WORKED EXAMPLE 11 **The fifth term in a geometric sequence is 14 and the seventh term is 0.56. Find the common ratio, r, the first term, a, and the nth term for the sequence.**

THINK	WRITE
1 Write the general rule for the nth term of a geometric sequence.	$t_n = ar^{n-1}$
2 Use the information about the fifth term to form an equation. Label it [1].	When $n = 5$, $t_n = 14$ $14 = a \times r^{5-1}$ $14 = a \times r^4$ [1]
3 Similarly, use information about the seventh term to form an equation. Label it [2].	When $n = 7$, $t_n = 0.56$ $0.56 = a \times r^{7-1}$ $0.56 = a \times r^6$ [2]
4 Solve the equations simultaneously: divide equation [2] by equation [1] to eliminate a.	$\frac{[2]}{[1]}$ gives $\frac{ar^6}{ar^4} = \frac{0.56}{14}$

5 Solve for r.

$$r^2 = 0.04$$
$$r = \pm\sqrt{0.04}$$
$$= \pm 0.2$$

6 Because there are two solutions, we have to perform two sets of computations. Consider the positive value of r first. Substitute the value of r into either of the two equations, say equation [1], and solve for a.

If $r = 0.2$

Substitute r into [1]:

$$a \times (0.2)^4 = 14$$
$$0.0016a = 14$$
$$a = 14 \div 0.0016$$
$$= 8750$$

7 Substitute the values of r and a into the general equation to find the expression for the nth term.

The nth term is:

$$t_n = 8750 \times (0.2)^{n-1}$$

8 Now consider the negative value of r. Substitute the value of r into either of the two equations, say equation [1], and solve for a. (Note that the value of a is the same for both values of r.)

If $r = -0.2$

Substitute r into [1]:

$$a = (-0.2)^4 = 14$$
$$0.0016a = 14$$
$$a = 14 \div 0.0016$$
$$= 8750$$

9 Substitute the values of r and a into the general formula to find the second expression for the nth term of the sequence.

The nth term is:

$$t_n = 8750 \times (-0.2)^{n-1}$$

EXERCISE 3.5 Geometric sequences

PRACTISE

1 **WE8** State whether the sequence is geometric by finding the ratio of successive terms for t_n: $\{3, 6, 12, \ldots\}$. If the sequence is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

2 State whether the sequence is geometric by finding the ratio of successive terms for t_n: $\left\{-3, 1, \frac{-1}{3}, \ldots\right\}$. If the sequence is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

3 **WE9** Consider the geometric sequence 1, 3, 9, 27, 81, …

a Use CAS technology to draw up a table showing the term number and its value.

b Use CAS technology to graph the entries in the table.

c Comment on the shape of the graph.

4 Consider the geometric sequence −20, 10, −5, 2.5, −1.25, …

a Use CAS technology to draw up a table showing the term number and its value.

b Use CAS technology to graph the entries in the table.

c Comment on the shape of the graph.

5 **WE10** Find the nth term and the tenth term in the geometric sequence where the first term is 2 and the third term is 18.

6 Find the nth term and the tenth term in the geometric sequence where the first term is 1 and the third term is 4.

7 **WE11** The third term in a geometric sequence is 100 and the fifth term is 400. Find the common ratio, r, the first term, a, and the nth term for the sequence.

8 The fourth term in a geometric sequence is 48 and the eighth term is 768. Find the common ratio, r, the first term, a, and the nth term for the sequence.

CONSOLIDATE

9 For each of the following:

- i show that the sequence is geometric
- ii find the nth term and consequently the sixth and tenth terms.

a t: $\{5, 10, 20, \ldots\}$

b t: $\{2, 5, 12.5, \ldots\}$

c t: $\{1, -3, 9, \ldots\}$

d t: $\{2, -4, 8, \ldots\}$

e t: $\{2.3, 3.45, 5.175, \ldots\}$

f t: $\left\{\frac{1}{2}, 1, 2, \ldots\right\}$

g t: $\left\{\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \ldots\right\}$

h t: $\left\{\frac{3}{5}, \frac{-1}{5}, \frac{1}{15}, \ldots\right\}$

i t: $\{x, 3x^4, 9x^7, \ldots\}$

j t: $\left\{\frac{1}{x}, \frac{2}{x^2}, \frac{4}{x^3}, \ldots\right\}$

10 Find the nth term and the tenth term in the geometric sequence where:

a the first term is 5 and the fourth term is 40

b the first term is -1 and the second term is 2

c the first term is 9 and the third term is $\frac{1}{81}$. (Why are there two possible answers?)

11 Find the fourth term in the geometric sequence where the first term is 6 and the seventh term is $\frac{3}{32}$.

12 Find the nth term in the geometric sequence where the first term is 3 and the fourth term is $6\sqrt{2}$.

13 For the geometric sequence t: $\{3, m, n, 192, \ldots\}$, find the values for m and n.

14 Consider the geometric sequence t: $\{16, m, 81, n, \ldots\}$. Find the values of m and n, if it is known that both are positive numbers.

15 For the geometric sequence t: $\{a, 15, b, 0.0375, \ldots\}$, find the values of a and b, given that they are positive numbers.

16 If $t_2 = \frac{1}{2}$ and $t_5 = \frac{27}{16}$, find the first term, a, the common factor, r, and hence the nth term for the geometric sequence.

17 Find the value of x such that the following sequence forms a geometric progression: $x - 1, 3x + 4, 6x + 8$.

18 Insert three terms in between 8, _, _, _, $\frac{1}{32}$ such that the sequence of numbers is geometric.

MASTER

19 The difference between the first term and the second term in a geometric sequence is 6. The difference between the second term and the third term is 3.

a Calculate the difference between the third term and the fourth term.

b Find the nth term in the sequence.

20 The first two terms in a geometric sequence are 120 and 24, and the kth term is 0.0384. Find the value for k.

3.6 Geometric series

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When we add up or sum the terms in a sequence we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, . . .}, where the first term $t_1 = a = 2$ and the common ratio is 3, we can quickly calculate the first few terms in the series of this sequence.

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

In general the sum of the first n terms is:

$$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n$$

For a geometric sequence the first term is a, the second term is ar, the third term is ar^2 and so on up to the nth term, which is ar^{n-1}. Thus:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad [1]$$

If we multiply equation [1] by r we get:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad [2]$$

Note that on the right-hand side of equations [1] and [2] all but two terms are common, namely the first term in equation [1], a, and the last term in equation [2], ar^n. If we take the difference between equation [2] and equation [1] we get:

$$rS_n - S_n = ar^n - a \qquad [2] - [1]$$

$$\therefore \quad (r-1)S_n = a(r^n - 1)$$

$$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1 \qquad (r \text{ cannot equal } 1)$$

We now have an equation that allows us to calculate the sum of the first n terms of a geometric sequence.

The sum of the first n terms of a geometric sequence is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

where a is the first term of the sequence and r is the common ratio.

WORKED EXAMPLE 12

Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: {1, 4, 16, . . .}

b $t_n = 2(2)^{n-1}$, $n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = \frac{1}{4}t_n$, $t_1 = \frac{-1}{2}$

THINK	WRITE
a 1 Write the general formula for the sum of the first n terms of the geometric sequence.	a $S_n = \frac{a(r^n - 1)}{r - 1}$
2 Write the sequence.	t_n: {1, 4, 16, . . .}

3 Identify the variables: a is the first term; r can be established by finding the ratio; n is known from the question.

$$a = 1;\ r = \frac{4}{1} = 4;\ n = 5$$

4 Substitute the values of a, r and n into the formula and evaluate.

$$\begin{aligned} S_5 &= \frac{1(4^5 - 1)}{4 - 1} \\ &= \frac{1024 - 1}{3} \\ &= 341 \end{aligned}$$

b **1** Write the sequence.

b $t_n = 2(2)^{n-1},\ n \in \{1, 2, 3, \ldots\}$

2 Compare the given rule with the general formula for the nth term of the geometric sequence $t_n = ar^{n-1}$ and identify values of a and r; the value of n is known from the question.

$$a = 2;\ r = 2;\ n = 5$$

3 Substitute values of a, r and n into the general formula for the sum and evaluate.

$$\begin{aligned} S_5 &= \frac{2(2^5 - 1)}{2 - 1} \\ &= 62 \end{aligned}$$

c **1** Write the sequence.

c $t_{n+1} = \frac{1}{4}t_n,\ t_1 = \frac{-1}{2}$

2 This is an iterative formula, so the coefficient of t_n is r; $a = t_1$; n is known from the question.

$$r = \frac{1}{4};\ a = \frac{-1}{2};\ n = 5$$

3 Substitute values of a, r and n into the general formula for the sum and evaluate.

$$\begin{aligned} S_5 &= \frac{\frac{-1}{2}\left[\left(\frac{1}{4}\right)^5 - 1\right]}{\frac{1}{4} - 1} \\ &= \frac{\frac{-1}{2} \times \left(\frac{1}{1024} - 1\right)}{\frac{-3}{4}} \\ &= \frac{-341}{512} \end{aligned}$$

The infinite sum of a geometric sequence where $r < 1$

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Units 1 & 2
AOS 2
Topic 2
Concept 6

Infinite geometric series
Concept summary
Practice questions

When the constant ratio, r, is less than 1 and greater than −1, that is, $\{r: -1 < r < 1\}$, each successive term in the sequence gets closer to zero. This can readily be shown with the following two examples.

$$g: \{2, -1, \frac{1}{2}, \frac{-1}{4}, \ldots\} \text{ where } a = 2 \text{ and } r = \frac{-1}{2}$$

$$h: \{40, \frac{1}{2}, \frac{1}{160}, \ldots\} \text{ where } a = 40 \text{ and } r = \frac{1}{80}$$

In both the examples, successive terms approach zero as n increases. In the second case the approach is more rapid than in the first, and the first sequence alternates positive and negative. A simple investigation with a spreadsheet will quickly reveal that for geometric sequences with the size or magnitude of $r < 1$, the series eventually settles down to a near constant value. We say that the series converges to a value S_∞, which is the sum to infinity of all terms in the geometric sequence. We can find the value S_∞ by recognising that as $n \to \infty$ the term $r^n \to 0$, provided r is between -1 and 1. We write this technically as $-1 < r < 1$ or $|r| < 1$. The symbol $|r|$ means the magnitude or size of r. Using our equation for the sum of the first n terms:

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

Taking -1 as a common factor from the numerator and denominator:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

As $n \to \infty$, $r^n \to 0$ and hence $1 - r^n \to 1$. Thus the top line or numerator will equal a when $n \to \infty$:

$$S_\infty = \frac{a}{1 - r}; \quad |r| < 1$$

We now have an equation that allows us to calculate the sum to infinity, S_∞, of a geometric sequence.

The sum to infinity, S_∞, of a geometric sequence is given by:

$$\boldsymbol{S_\infty = \frac{a}{1 - r}; \quad |r| < 1}$$

where a is the first term of the sequence and r is the common ratio whose magnitude is less than one.

WORKED EXAMPLE 13

a Find the sum to infinity for the sequence t_n: {10, 1, 0.1, . . .}.

b Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.

THINK	WRITE
a 1 Write the formula for the nth term of the geometric sequence.	**a** $t_n = ar^{n-1}$
2 From the question we know that the first term, a, is 10 and $r = 0.1$.	$a = 10, r = 0.1$
3 Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1 - r}; \; \lvert r \rvert < 1$
4 Substitute $a = 10$ and $r = 0.10$ into the formula and evaluate.	$S_\infty = \frac{10}{1 - 0.1}$ $S_\infty = \frac{10}{0.9} = \frac{100}{9}$ $= 11\frac{1}{9}$

b **1** Write the formula for the sum to infinity.	**b** $S_\infty = \dfrac{a}{1-r}; \lvert r \rvert < 1$
2 From the question we know that the infinite sum is equal to 10 and that the first term, *a*, is 6.	$a = 6; S_\infty = 10$
3 Substitute known values into the formula.	$10 = \dfrac{6}{1-r}$
4 Solve for *r*.	$10(1-r) = 6$ $10 - 10r = 6$ $r = 0.4$
5 Write the general formula for the *n*th term.	$t_n = ar^{n-1}$
6 To find the fourth term substitute $a = 6$, $n = 4$ and $r = 0.4$ into the formula and evaluate.	$t_4 = 6 \times (0.4)^3$ $= 0.384$

EXERCISE 3.6 Geometric series

PRACTISE

1 **WE12** Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: $\{1, 2, 4, \ldots\}$

b $t_n = 3(-2)^{n-1}, n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = 2t_n, t_1 = \dfrac{3}{2}$

2 Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: $\{1, 3, 9, \ldots\}$

b $t_n = -4(1.2)^{n-1},\ n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = \dfrac{1}{2}t_n, t_1 = \dfrac{-2}{3}$

3 **WE13** **a** Find the sum to infinity for the sequence t_n: $\left\{1, \dfrac{1}{2}, \dfrac{1}{4}, \ldots\right\}$.

a Find the fourth term in the geometric sequence whose first term is 4 and whose sum to infinity is 6.

4 **a** Find the sum to infinity for the sequence t_n: $\left\{1, \dfrac{2}{3}, \dfrac{4}{9}, \ldots\right\}$.

a Find the fourth term in the geometric sequence whose first term is 1 and whose sum to infinity is $\dfrac{3}{5}$.

CONSOLIDATE

5 Consider the following geometric sequences and find the terms indicated.

a The first term is 440 and the 12th term is 880. Find S_6.

b The fifth term is 1 and the eighth term is 8. Find S_1, S_{10}, S_{20}.

6 What minimum number of terms of the series $2 + 3 + 4\dfrac{1}{2} + \ldots$ must be taken to give a sum in excess of 100?

7 For the infinite geometric sequence $\left\{1, \dfrac{1}{4}, \dfrac{1}{16}, \ldots\right\}$, find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

8 A sequence of numbers is defined by $t_n = 3\left(\dfrac{1}{2}\right)^{n-1}, n \in \{1, 2, 3, \ldots\}$.

a Find the sum of the first 20 terms.

b Find the sum of all the terms between and including t_{21} and t_{40}.

c Find the sum to infinity, S_∞.

9 A sequence of numbers is defined by t_n: {9, −3, 1, . . .}.

a Find the sum of the first nine terms.

b Find the sum of all the terms between and including t_{10} and t_{15}.

c Find the sum to infinity, S_∞.

10 The first term of a geometric sequence is 5 and the fourth term is 0.078 125. Find the sum to infinity.

11 The sum of the first four terms of a geometric sequence is 30 and the sum to infinity is 32. Find the first three terms of the sequence if the common ratio is positive.

12 For the geometric sequence $\sqrt{5} + \sqrt{3}, \ \sqrt{5} - \sqrt{3}, \ldots,$ find the common ratio, r, and the sum of the infinite series, S_∞.

13 If $1 + 3x + 9x^2 + \ldots = \frac{2}{3}$, find the value of x.

14 If the common ratio for a geometric sequence is 0.99 and the sum to infinity is 100, what is the value of the first and second terms in the sequence?

MASTER

15 Show that $x^n - 1$ always has a factor $(x - 1)$ for $n \in \{1, 2, 3, \ldots\}$.

16 A student stands at one side of a road 10 metres wide, and walks halfway across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on.

a Will the student ever make it *past* the other side of the road?

b Does the width of the road affect your answer?

3.7 Applications of sequences and series

This section consists of a mixture of problems where the work covered in the first five exercises is applied to a variety of situations.

The following general guidelines can assist you in solving the problems.

1. Read the question carefully.
2. Decide whether the information suggests an arithmetic or geometric sequence. Check to see if there is a constant difference between successive terms or a constant ratio. If there is neither, look for a simple number pattern such as the difference between successive terms changing in a regular way.
3. Write the information from the problem using appropriate notation. For example, if you are told that the fifth term is 12, write $t_5 = 12$. If the sequence is arithmetic, you then have an equation to work with, namely: $a + 4d = 12$. If you know the sequence is geometric, then $ar^4 = 12$.
4. Define what you have to calculate and write an appropriate formula or formulas. For example, if you have to find the tenth number in a sequence that you know is geometric, you have an equation: $t_{10} = ar^9$. This can be calculated if a and r are known or can be established.
5. Use algebra to find what is required in the problem.

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Units 1 & 2
AOS 2
Topic 2
Concept 7

Applications of sequences and series
Concept summary
Practice questions

WORKED EXAMPLE 14

In 1970 the cost of 1 megabyte of computer memory was \$2025. In 1980 the cost for the same amount of memory had reduced to \$45, and by 1990 the cost had dropped to \$1.

a Assuming the pattern continues through the years, what was the cost of 1 megabyte of memory in the year 2000?

b How much memory, in megabytes, could you buy for \$10 in the year 2010 based on the trend?

THINK

WRITE

a 1 Present the given information in a table.

a

Year	1970	1980	1990	2000	2010
Cost (\$)	2025	45	1	?	?

2 Study the table. The information suggests a geometric sequence for the cost at each ten-year interval. Verify this by checking for a constant ratio between successive terms.

$45 \div 2025 = \frac{1}{45}$ and $1 \div 45 = \frac{1}{45}$, so the three terms form a geometric sequence with common ratio $r = \frac{1}{45}$.

3 To find the cost in the year 2000, find the fourth term in the sequence by multiplying the preceding (third) term by the common ratio.

$$t_4 = t_3 \times r$$

$$\begin{aligned} t_4 &= 1 \times \frac{1}{45} \\ &= \frac{1}{45} \\ &= 0.022\ldots \end{aligned}$$

4 Interpret the result and clearly answer the question.

In the year 2000 you would have paid about 2 cents for a megabyte of memory.

b 1 If the cost of 1 megabyte can be found in the year 2010, then the amount of memory purchased for \$10 can be determined. To find the predicted cost in the year 2010, the fifth term in the sequence needs to be determined.

b

$$\begin{aligned} t_5 &= t_4 \times r \\ &= \frac{1}{45} \times \frac{1}{45} \\ &= \frac{1}{2025} \text{ of a dollar per megabyte} \end{aligned}$$

2 Take the reciprocal of t_5 to get the amount of memory per dollar.

The amount of memory per dollar is 2025 megabytes.

3 Find the amount of memory that can be purchased for \$10.

So \$10 would buy 10×2025 = 20 250 megabytes.

WORKED EXAMPLE 15

Express the recurring decimal 0.131 313 13 . . . as a proper fraction.

THINK	WRITE
1 Express the given number as a geometric series.	$0.131\,313\ldots = 0.13 + 0.0013 + 0.000\,013\ldots$
2 State the values of a and r.	$a = 0.13$ and $r = \frac{0.0013}{0.13} = 0.01$

3 Find the sum to infinity, S_∞.
Write the formula for the sum to infinity.

$$S_\infty = \frac{a}{1 - r}$$

4 Substitute values of a and r into the formula and simplify.

$$S_\infty = \frac{0.13}{1 - 0.01}$$

$$S_\infty = \frac{0.13}{0.99}$$

5 Multiply both numerator and denominator by 100 to get rid of the decimal point.

$$S_\infty = \frac{13}{99}$$

EXERCISE 3.7 Applications of sequences and series

PRACTISE

1 **WE14** In 1970 the Smith family purchased a small house for $60 000. Over the following years, the value of their property rose steadily. In 1975 the value of the house was $69 000 and in 1980 it reached $79 350.

- a Assuming that the pattern continues through the years, find (to the nearest dollar) the value of the Smiths' house in i 1985 and ii 1995.
- b By what factor will the value of the house have increased by the year 2015, compared to the original value?

2 An accountant working with a company commenced on a salary of $58 000 and has received a $4200 increase each year.

- a How much did she earn in her 15th year of employment?
- b How much has she earned from the company altogether in those 15 years?

3 **WE15** Express the recurring decimal 0.1111 . . . as a proper fraction.

4 Express the recurring decimal 0.575 757 . . . as a proper fraction.

CONSOLIDATE

5 A chemist has been working with the same company for 15 years. He commenced on a salary of $28 000 and has received a 4% increase each year.

- a What type of sequence of numbers does his annual income follow?
- b How much did he earn in his 15th year of employment?
- c How much has he earned from the company altogether?
- d What was his increase in salary at the end of i his 1st and ii his 14th year of employment?

6 A biologist is growing a tissue culture in a Petri dish. The initial mass of the culture was 20 milligrams. By the end of the first day the culture had a mass of 28 milligrams.

- a Assuming that the daily growth is *arithmetic*, find the mass of the culture after the second, third, tenth and *n*th day.
- b On what day will the culture mass first exceed 200 milligrams if its growth is arithmetic?
- c Assuming that the daily growth is *geometric*, find the mass of the culture after the second, third, tenth and *n*th day.
- d On what day will the culture mass first exceed 200 milligrams if its growth is geometric?

7 Logs of wood can be stacked so that there is one more log on each descending layer than on the previous layer. The top row has 6 logs and there are 20 rows.

a How many logs are in the stack altogether?

b The logs are to be separated into two equal piles. They are separated by removing logs from the top of the pile. How many rows down will workers take away before they remove half the stack?

8 Kind-hearted Kate has 200 movie tickets to give away to people at the shopping centre. She gives the first person one ticket, the next person two tickets, the third person four tickets and so on following a geometric progression until she can no longer give the nth person 2^{n-1} tickets. How many tickets did the last lucky person receive? How many tickets did Kate have left?

9 The King of Persia, so the story goes, offered Xanadu any reward to secure the safety of his kingdom. As his reward, Xanadu requested a chessboard with one grain of rice on the first square, two grains on the second, four on the third and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 grams, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

10 As legend has it, the King of Constantinople offered Xanadu's cousin Yittrius any reward to secure the safety of his city. This Yittrius accepted: she requested a chessboard with one grain of rice on the first square, three grains of rice on the second square, five grains of rice on the third square and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 grams, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

11 A hiker walks 36 km on the first day and $\frac{2}{3}$ that distance on the second. Every day thereafter she walks $\frac{2}{3}$ of the distance she walked on the day before. Will the hiker cover the distance of 100 km to complete the walk? If so, on what day will she complete the walk?

12 Find the fraction equivalent of the following recurring decimal numbers by writing the decimal number as a sum of infinite terms.

a 0.333 333 333 . . .

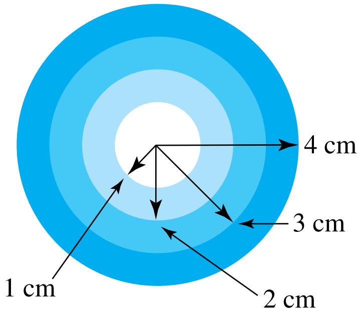
b 2.343 434 . . .

c 3.142 142 142 . . .

d 21.2121 . . .

e 16.666 . . .

13 A circular board is divided into a series of concentric circles of radius 1 cm, 2 cm, 3 cm and 4 cm as shown at right.



a Find the areas of each of the successive shaded regions and show that they form an arithmetic progression.

b A dart is fired at the board at random and hits the board. What is the probability of striking each of the four regions of the board?
(*Note:* The probability of striking a region = area of region ÷ total area.)

14 A bullet is fired vertically up into the air. In the first second it has an average speed of 180 m/s; that is, it travels 180 m up into the air during the first second. Each second its average speed diminishes by 12 m/s. Thus during the 2nd second the bullet has an average speed only 168 m/s and accordingly travels 168 m further up into the air.

a Find an equation for the average speed of the bullet for the nth second that it is in the air.

b Find the time when the average speed of the bullet is equal to zero.

c Find the maximum height of the bullet above where it was fired.

MASTER

15 Coffee cools according to Newton's Law of Cooling, in which the temperature of coffee *above* room temperature drops by a constant fraction each unit of time. The table below shows the temperature of a cup of coffee in a room at 20 °C each minute after it was made.

Time (min)	Temp. (°C)
1	80.0
2	74.0
3	68.6

Remember to subtract the room temperature from the temperature of the coffee before you do your calculations.

The person who made the coffee will drink it only if it has a temperature in excess of 50 °C. What is the minimum time after the cup of coffee has been made before it becomes undrinkable?

16 Two arithmetic sequences, t_n and u_n, are multiplied together. That is, each term is multiplied by the other to form a new term.

$$t_n = 2n - 3, n \in \{1, 2, 3, \ldots\} \text{ and}$$
$$u_n = 3n, n \in \{1, 2, 3, \ldots\}$$

Show that the new sequence of numbers $t_1 \times u_1$, $t_2 \times u_2$, $t_3 \times u_3$, . . . is an arithmetic series and hence find the arithmetic sequence for that new series. (*Hint:* For a sequence a_n with a series A_n, $a_n = A_n - A_{n-1}$.)

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3.8 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

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Units 1 & 2 **Sequences and series**

Sit topic test

3 Answers

EXERCISE 3.2

1 a $\frac{3}{8}, \frac{3}{16}, \frac{3}{32}$

b 4, 36, 324

c −3, −1, 3

2 a −17, 20, −23

b 53, 97, 173

c −11, −123, −15 131

3 0.2, 0.128, 0.089, 0.065, 0.049, 0.037; the sequence converges to 0.

4 0.099, 0.0981, 0.0973, 0.0967, 0.096, 0.0955; the sequence converges to 0.0909.

5 a Add 3 (to the previous term); 10, 13, 16.

b Subtract 1 (from the previous term); −3, −4, −5.

c Multiply by 4; 256, 1024, 4096.

d The difference between the terms increases by 1 for each pair; 27, 35, 44.

e Add the preceding two terms; 29, 47, 76.

f Add $3b - a$; $-2a + 7b$, $-3a + 10b$, $-4a + 13b$.

g Many possible answers — assume the sequence repeats; 0, −1, 0.

h Append 1 to the decimal expansion of the preceding term; 1.111, 1.1111, 1.111 11.

i Divide by −2; 64, −32, 16.

6 a −3, 5, 15

b $\frac{1}{2}, \frac{5}{6}, \frac{10}{11}$

c 13.3, −1.5, −20

d $\frac{5}{2}, \frac{5}{32}, \frac{5}{1024}$

e $\frac{5}{4}$, 20, 640

f 0, 4, 11

g $\frac{3}{2}, \frac{243}{32}, \frac{59\,049}{1024}$

h a, $a + 4d$, $a + 9d$

i a, ar^4, ar^9

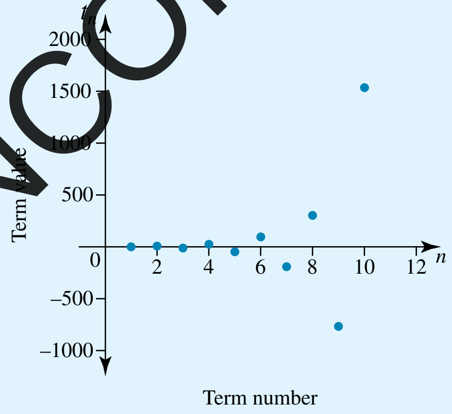
7 a −12, 384, 1536

b 0, −35, −49

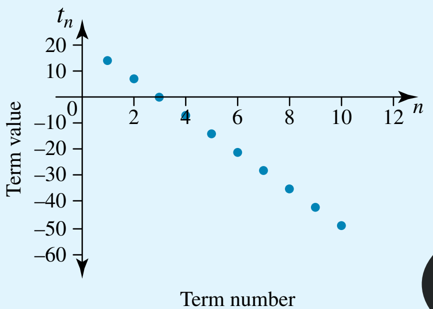
c 3, −1, −1

d 0, 0, 0

8 a



b



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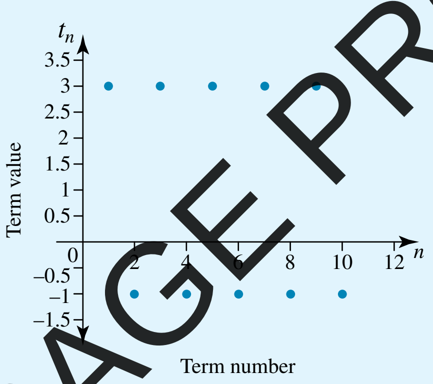
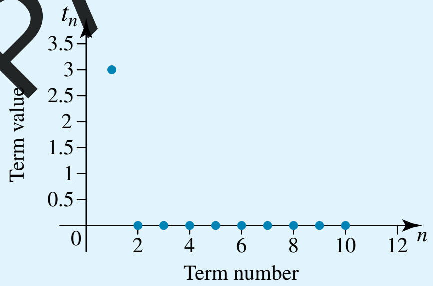


d



9 a 0.096, 0.0347, 0.0134, 0.0053, 0.0021, 0.0008; the sequence converges to 0.

b 0.456, 0.471 321, 0.473 437, 0.473 659, 0.473 682, 0.473 684; the sequence converges to $\frac{9}{19}$.

c 0.525, 0.523 687 5, 0.523 821 7, 0.523 808 3, 0.523 809 6, 0.523 809 5; the sequence converges to $\frac{11}{21}$.

d 0.525, 0.623, 0.587, 0.606, 0.597, 0.602; the sequence converges to $\frac{3}{5}$.

e 0.48, 0.749, 0.564, 0.738, 0.581, 0.73; oscillating

f 0.714, 0.694, 0.722, 0.683, 0.736, 0.66; oscillating

g 0.378, 0.987, 0.052, 0.207, 0.689, 0.901; divergent

h 0.72, 0.907, 0.379, 1.059, −0.281, −1.619; divergent

10 a 15, 20; the difference between subsequent terms increases by 1.

b There are many possible answers. A possible pattern is the addition of 5, then 3, then 1, then −1. The next two terms are 4, −3. Here the difference between successive terms follows an arithmetic sequence.

c Many possible answers as there is no obvious pattern. It could be the start of a telephone number.

d Each successive term is multiplied by an increasing factor of $\frac{1}{2}$, starting with $\left(\frac{1}{2}\right)^{-1} = 2$, then $\left(\frac{1}{2}\right)^{0} = 1$, and then $\left(\frac{1}{2}\right)^{1}$ followed by $\frac{1}{4}; \frac{3}{16}, \frac{3}{256}$.

e 34, 55; each subsequent term is the sum of the preceding two terms.

f 31, 63; terms are 1 less than powers of 2.

g 5, 4; add 2 to find the next term, then subtract 1 to find the subsequent term and repeat.

11 D **12** E **13** C

14 a $t_{n+1} = t_n - 2,\ t_1 = 7$

b $t_{n+1} = t_n \div 2,\ t_1 = 12$

c $t_{n+1} = t_n + 0.6,\ t_1 = 12$

d $t_{n+1} = t_n \times 5 + 1,\ t_1 = 2$

e $t_{n+1} = -3t_n,\ t_1 = 4$

f $t_{n+1} = (t_n)^2,\ t_1 = 2$

15 a 26 and 25 **b** 23 cats

16 The population size will rapidly decrease and by 2009, the stray cat population will be gone. (Happily, they were all taken in by good and loving households.)

EXERCISE 3.3

1 a Not arithmetic

b Arithmetic, difference = 3; $t_4 = 6,\ t_n = -6 + 3n$

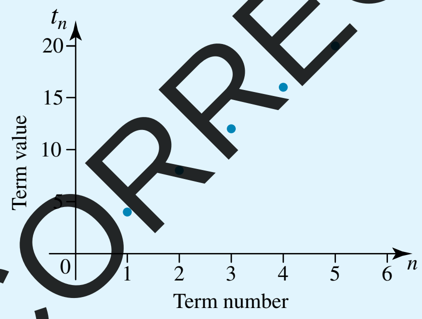
2 a Not arithmetic

b Arithmetic, difference = −4; $t_4 = -14,\ t_n = 2 - 4n$

3 a

Term number	1	2	3	4	5
Term value	4	8	12	16	20

b

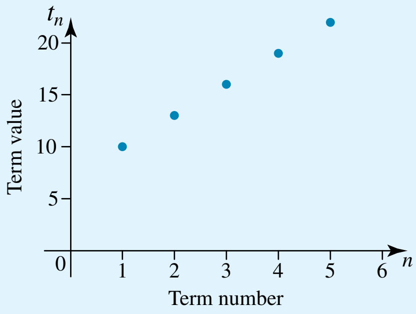


c $t_{10} = 40$

4 a

Term number	1	2	3	4	5
Term value	10	13	16	19	22

b



c $t_{10} = 37$

5 $m = 21.5,\ n = 32.5$

6 $x = 47,\ y = 75$

7 $-2, \frac{26}{3} - \frac{8}{3}n$

8 4.54, $t_n = 2.78 + 0.22n$

9 a 104 **b** 682

c 1458 **d** −26 310

10 a $t_n = 8 - 3n,\ n = 1, 2, 3, \ldots$

b $t_n = 2 + \frac{n}{2},\ n = 1, 2, 3, \ldots$

c $t_n = -6 + 3n,\ n = 1, 2, 3, \ldots$

d $t_n = -3x + 5nx,\ n = 1, 2, 3, \ldots$

11 $t_n = 4 + 2n,\ n = 1, 2, 3, \ldots$

12 $5n - 2$

13 $-x + y,\ -5x + 9y$

14 −35; 15; $15n - 50$

15 $-4\frac{1}{4}; \frac{5}{12}; -4\frac{2}{3} + \frac{5}{12}n$

16 $m = 27,\ n = 32$

17 9 **18** 3

19 a $t_{n+1} = t_n + 4;\ t_1 = 3$

b $t_{n+1} = t_n + 3;\ t_1 = -3$

c $t_{n+1} = t_n - 4;\ t_1 = -2$

d $t_{n+1} = t_n + \frac{1}{2};\ t_1 = \frac{2}{7}$

e $t_{n+1} = t_n + \frac{3}{4};\ t_1 = \frac{3}{4}$

f $t_{n+1} = t_n - \frac{7}{4};\ t_1 = \frac{1}{4}$

g $t_{n+1} = t_n + 2\pi - 2;\ t_1 = 2\pi + 3$

20 a $\frac{5}{6}$ **b** $\dfrac{n+2}{n+3}$

EXERCISE 3.4

1 400 **2** 4175

3 a 1275 **b** 5050

4 a 5000

b Each of the 100 terms is $\frac{1}{2}$ less than its corresponding term in question **3**. There are 100 terms, so the answer to this question is 50 less than in question **3b**.

5 258

6 a −273, −480, −741

b −324

7 a 280

b 1080

c 34

8 $\dfrac{n(n+1)}{2}$

9 a, b Various answers

10 17

11 45

12 6

13 174

14 The iterative equation is $t_{n+1} = t_n + 8$, $t_1 = 7$.
The functional equation is $t_n = 8n - 1$, $n = 1, 2, 3, \ldots$

EXERCISE 3.5

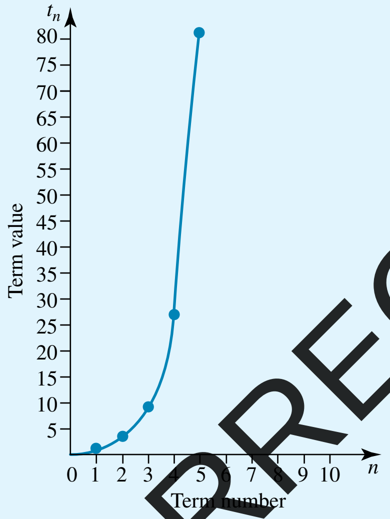
1 Geometric, ratio = 2; $t_4 = 24$; $t_n = 3 \times 2^{n-1}$

2 Geometric, ratio $= \frac{-1}{3}$; $t_4 = \frac{1}{9}$; $t_n = (-3)^{2-n}$

3 a

Term number	1	2	3	4	5
Term value	1	3	9	27	81

b

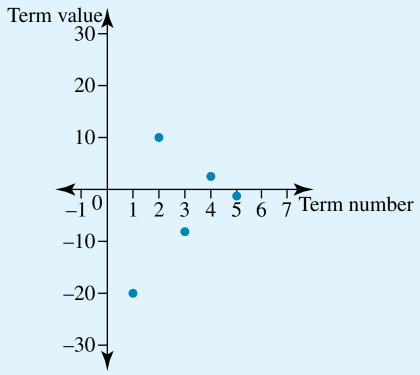


c The points lie on a smooth curve which increases rapidly. Because of this rapid increase in value, it would be difficult to use the graph to predict future values in the sequence with any accuracy.

4 a

Term number	1	2	3	4	5
Term value	−20	10	−5	2.5	−1.25

b



c The points are oscillating and converge on a point.

5 There are two possible answers because the ratio could be −3 or 3. The nth term is $t_n = 2 \times 3^{n-1}$ or $t_n = 2 \times (-3)^{n-1}$, $t_{10} = \pm 39\,366$.

6 There are two possible answers because the ratio could be −2 or 2. The nth term is $t_n = 2^{n-1}$ or $t_n = (-2)^{n-1}$, $t_{10} = \pm 512$.

7 $t_1 = 25$, $r = \pm 2$, $t_n = 25 \times 2^{n-1}$ or $t_n = 25 \times (-2)^{n-1}$

8 $a = \pm 6$, $r = \pm 2$, $t_n = \pm 6 \times (\pm 2)^{n-1}$

9 a $t_n = 5 \times 2^{n-1}$, $t_6 = 160$, $t_{10} = 2560$

b $t_n = 2 \times 2.5^{n-1}$, $t_6 = 195.31$, $t_{10} = 7629.39$

c $t_n = 1 \times (-3)^{n-1}$, $t_6 = -243$, $t_{10} = -19\,683$

d $t_n = 2 \times (-2)^{n-1}$, $t_6 = -64$, $t_{10} = -1024$

e $t_n = 2.3 \times (1.5)^{n-1}$, $t_6 = 17.47$, $t_{10} = 88.42$

f $t_n = \frac{1}{2} \times 2^{n-1}$, $t_6 = 16$, $t_{10} = 256$

g $t_n = \frac{1}{3} \times \left(\frac{1}{4}\right)^{n-1}$, $t_6 = \frac{1}{3072}$, $t_{10} = \frac{1}{786\,432}$

h $t_n = \frac{3}{5} \times \left(\frac{-1}{3}\right)^{n-1}$, $t_6 = \frac{-1}{405}$, $t_{10} = \frac{-1}{32\,805}$

i $t_n = x \times (3x^3)^{n-1}$, $t_6 = 243x^{16}$, $t_{10} = 19\,683x^{28}$

j $t_n = \dfrac{1}{x} \times \left(\dfrac{2}{x}\right)^{n-1}$, $t_6 = \dfrac{32}{x^6}$, $t_{10} = \dfrac{512}{x^{10}}$

10 a The nth term is $t_n = 5 \times 2^{n-1}$, $t_{10} = 2560$.

b The nth term is $t_n = -1 \times (-2)^{n-1}$, $t_{10} = 512$.

c There are two possible answers because the ratio could be $\frac{-1}{27}$ or $\frac{1}{27}$. The nth term is $t_n = 3^{5-3n}$ or $t_n = (-3)^{5-3n}$, $t_{10} = \pm 3^{-25}$.

11 $\frac{\pm 3}{4}$

12 $3 \times 2^{\frac{(n-1)}{2}}$

13 $m = 12$, $n = 48$

14 $m = 36$, $n = \frac{729}{4}$

15 $a = 300$, $b = 0.75$

16 $t_1 = \frac{1}{3}$, $r = \frac{3}{2}$, $t_n = 3^{n-2}2^{1-n}$

17 -6

18 $2, \frac{1}{2}, \frac{1}{8}$, or $-2, \frac{1}{2}, \frac{-1}{8}$

19 a $\frac{3}{2}$ **b** $\frac{24}{2^n}$

20 $k = 6$

EXERCISE 3.6

1 a 31

b 33

c 46.5

2 a 121

b -29.8

c $\frac{-31}{24}$

3 a 2 **b** $\frac{4}{27}$

4 a 3 **b** $\frac{-8}{27}$

5 a 3108 **b** $\frac{1}{16}, 63\frac{15}{16}, 66\,535\frac{15}{16}$

6 9

7 $\frac{4}{3}$; 75%, 18.75%, 4.6875%

8 a $6\left[1 - \left(\frac{1}{2}\right)^{20}\right] = 5.999\,994\,278$

b 5.722×10^{-6}

c 6

9 a $6\frac{3}{4}\left[1 - \left(\frac{-1}{3}\right)^{9}\right] = 6.750\,343$

b -3.425×10^{-4}

c $6\frac{3}{4}$

10 $6\frac{2}{3}$

11 16, 8, 4

12 $4 - \sqrt{15}, \dfrac{(\sqrt{3} + \sqrt{5})}{(\sqrt{15} - 3)} = \dfrac{(4\sqrt{3} + 3\sqrt{5})}{3}$

13 $\frac{-1}{6}$

14 1, 0.99

15 Check with your teacher.

16 a Mathematically, the student will never make it past the other side of the road. After each attempt, the distance remaining is halved, and this result is the extra distance walked at the next attempt. Thus the distance travelled across the road approaches but never reaches 10 metres.

b As shown in part **a**, the extra distance travelled at each attempt is equal to half the remaining distance from the previous attempt. Given that there will always be an amount remaining to travel, only half this amount can be achieved on the next attempt, regardless of the width of the road.

EXERCISE 3.7

1 a i \$91 253

ii \$120 681

b 3.518 times

2 a \$116 800 **b** \$1 311 000

3 $\frac{1}{9}$

4 $\frac{57}{99}$

5 a Geometric

b \$48 487

c \$560 660

d i \$1120

ii \$1865

6 a 36 mg, 44 mg, 100 mg, $20 + 8n$ mg

b 23rd day

c 39 mg, 55 mg, 579 mg, $28 \times (1.4)^{n-1}$ or 20×1.4^n

d Seventh day

7 a 310

b The workers must remove 12 full rows and 17 logs from the 13th row.

8 The last person received 64 tickets and Kate had 73 left.

9 a 1.8×10^{19} grains of rice

b 1.8×10^{15} kg

10 a 4096 grains of rice

b 0.41 kg

11 Yes, seventh day

12 a $\frac{1}{3}$ **b** $\frac{232}{99}$

c $\frac{3139}{999}$ **d** $\frac{700}{33}$

e $\frac{50}{3}$

13 a $\pi, 3\pi, 5\pi, 7\pi$ — arithmetic progression with $a = \pi$ and $d = 2\pi$

b $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$

14 a $192 - 12n$ m/s

b During the 16th second

c 1440 m

15 After 7 minutes the coffee has cooled to below 50 °C.

16 The sequence for the arithmetic series $t_n u_n$ is $12n - 15$, $n \in \{1, 2, 3, \ldots\}$