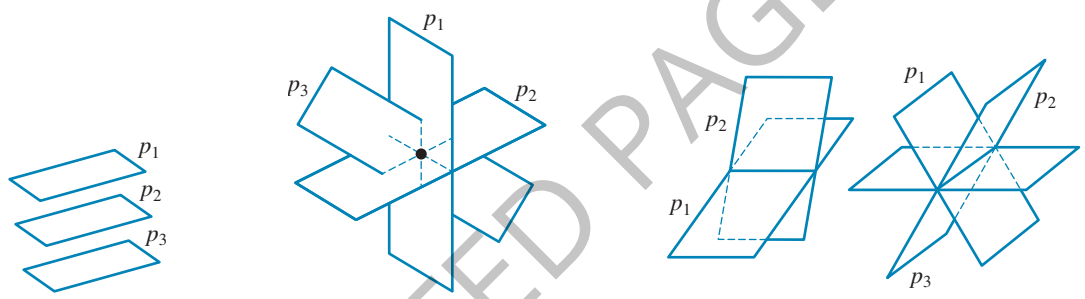
3 Solving systems of linear equations and the application of matrices

3.1 Overview

Systems of linear equations, and their applications, are closely related to the study of algebra, geometry, vectors and matrices. Often referred to as linear algebra, this branch of mathematics simplifies higher order systems of equations by translating the information into matrices. The equation of a plane was studied in the previous chapter using both a vector and an analytical geometry approach. If you were to consider three planes simultaneously, you would be modelling three equations in a system containing three variables (x, y, z). Systems in three dimensions may have zero, one or infinitely many solutions, the diagrams represent generalised geometric interpretations of each solution type.



Zero-solutions **One-solutions** **Infinitely many-solutions**

Matrices are used to find solutions across a wide range of disciplines including ranking players at competitive tournaments through dominance matrices, network flow analysis, code breaking (cryptology), polynomial curve fitting, input/output models in economics using Leontief matrices and the use of Leslie matrices in ecological studies. This incredible diversity demonstrates to power of matrices to solve problems.

Leslie matrices are primarily used in ecology as a powerful predictor of population growth. Combining the principles of both matrix multiplication and powers of matrices, estimates can be made on future population numbers based on the species fecundity (reproductive) and survival rates.

LEARNING SEQUENCE

3.1 Overview
3.2 Solving linear equations using matrix algebra
3.3 Solving a system of linear equations using Gaussian elimination
3.4 The three cases for solutions of systems of linear equations
3.5 Using technology for matrix calculations
3.6 Dominance and Leslie matrices
3.7 Applications of matrices
3.8 Review: exam practice

Fully worked solutions for this chapter are available in the resources section of your eBookPLUS at www.jacplus.com.au.

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3.2 Solving linear equations using matrix algebra

3.2.1 System of linear equations

Linear algebra is fundamental in the study of engineering, business, economics, network analysis, electrical circuits and vectors. This branch of mathematics explores **linear equations**, systems of linear equations and the various techniques that can be applied to solve them.

Recall from previous units that a linear equation in two-dimensions, x and y is given by,

$ax + by = c$, where a, b and c are real constants.

A linear equation in three-dimensions contains three variables x, y and z. It is commonly expressed in the form,

$ax + by + cz = d$, where a, b, c and d are real constants.

Linear equations contain variables to the first power. The variables are not products of other variables, nor do they contain roots of variables or are connected to logarithmic, exponential or trigonometric functions.

In general, a linear equation with n variables, is of the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots\ldots\ldots + a_nx_n = b$$

Where $a_1, a_2, \ldots, a_n$ and b are real constants. The constants a_n are the coefficients of x_n.

Other examples of linear and non-linear equations include:

Linear equation	*Non-linear equations*
$4x - 11y = 12$	$2yz + 4xy - z^2 = 12$
$2x - y + 7z = 14$	$\cos^2(x) + \ln(y) + z = 100$
$7x - \frac{121}{7}y + 3z = \pi$	$4\sqrt[5]{x} - \frac{2}{y} = -50$

A finite list of linear equations is referred to as a **system of linear equations**.

The general form of a system of m linear equations in n variables is given by:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots + a_{3n}x_n &= b_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots + a_{mn}x_n &= b_m
\end{aligned}$$

Where a_n are the coefficients, x_n are the variables and b is a constant term.

For example, this system consists of two linear equations in two variables, x and y. As the number of equations $m = 2$ and the number of variables $n = 2$ are equal, this is referred to as a square system.

$$\begin{aligned}
55x + 38y &= 207 \\
17x - 6y &= -7
\end{aligned}$$

In three dimensions, an example of a square system is shown.

$$\begin{aligned}
4x + 2y - 7z &= -27 \\
5x - 5y + z &= 0 \\
-2x + 3y - 4z &= -16
\end{aligned}$$

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In a system of linear equations, the variables are called the unknowns. Consider the following system of linear equations, where x and y represent the price of two different items.

$$\begin{aligned} 120x + 10y &= 70 \\ 38x + 20y &= 39 \end{aligned}$$

The **unique solution** for this set of simultaneous equations is $x = \frac{1}{2}, y = 1$. These scalar values represent a sequence of numbers that satisfy every equation in the system as shown below.

$$\begin{aligned} LHS &= 120x + 10y \\ &= 120 \times \frac{1}{2} + 10 \times 1 \\ &= 70 \\ &= RHS \end{aligned} \qquad \begin{aligned} LHS &= 38x + 20y \\ &= 38 \times \frac{1}{2} + 20 \times 1 \\ &= 39 \\ &= RHS \end{aligned}$$

If there is only one possible solution, such as the example above, then it is known as a unique solution. The point of intersection of two lines is an example of a unique solution. The three cases for the solution of a linear system will be examined in detail later in this chapter. Until that point we will focus on systems with one unique solution.

Systems of linear equations have either zero, one or an infinite number of solutions. The set of all possible solutions is called the solution set.

WORKED EXAMPLE 1

Identify which of the following equations are non-linear:

a. $-3x - 2y + 6z = 1$ b. $3x - 4yz = 12$ c. $7x + 9y + z = -2$ d. $12x^2 + y - z = 14$

THINK	WRITE
a. This equation contains variables raised to the first power.	a. $-3x - 2y + 6z = 1$ is linear.
b. This equation contains a term that is product of two variables, yz.	b. $3x - 4yz = 12$ is non-linear.
c. This equation contains variables raised to the first power.	c. $7x + 9y + z = -2$ is linear.
d. This equation contains a variable, x, that is not risen to the first power.	d. $12x^2 + y - z = 14$ is non-linear

WORKED EXAMPLE 2

Given the following systems of linear equations:

a. $$\begin{aligned} -4x + y - 6z &= -7 \\ x + 5y - 8z &= 13 \end{aligned}$$

b. $$\begin{aligned} x + 3y - 4z &= -21 \\ 9z - 12z + 2z &= 41 \\ -6x + y - 2z &= -16 \end{aligned}$$

i. **State the number of m equations and n variables in the system.**
ii. **Justify whether the system is square or not.**
iii. **Determine if $x = 1, y = -2, z = 4$ is a valid solution for all equations in the system.**

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THINK	WRITE
a. i. The system contains two equations, each containing three variables.	**a.** $m = 2, n = 3$
ii. Square systems exist if $m = n$.	$m \neq n$, therefore the system is not a square system.
iii. Assess the validity of the solution $x = 1, y = -2, z = 4$ by substituting the corresponding values for x, y and z into each equation. The solution satisfies each equation if the LHS equals the RHS.	$-4x + y - 6z = -7$ $LHS = -4 \times 1 + -2 - 6 \times 4$ $= -4 - 2 - 24$ $= -30$ $\neq RHS$ $x + 5y - 8z = 13$ $LHS = 1 + 5 \times -2 - 8 \times 4$ $= 1 - 10 - 32$ $= -41$ $\neq RHS$ $(1, -2, 4)$ is not a valid solution as it doesn't satisfy all equations in the system.
b. i. The system contains three equations, each containing three variables.	**b.** $m = 3, n = 3$
ii. Square systems exist if $m = n$.	$m = n$, therefore it can be categorised as a square system.
iii. Assess the validity of the solution $x = 1, y = -2, z = 4$. Substitute the corresponding values for x, y and z into each equation. The solutions satisfies each equations if the LHS equals the RHS.	$x + 3y - 4z = -21$ $LHS = 1 + 3 \times -2 - 4 \times 4$ $= 1 - 6 - 16$ $= -21$ $= RHS$ $9x - 12y + 2z = 41$ $LHS = 9 \times 1 - 12 \times -2 + 2 \times 4$ $= 9 + 24 + 8$ $= 41$ $= RHS$ $-6x + y - 2z = -16$ $LHS = 6 \times 1 + -2 - 2 \times 4$ $= 6 - 2 - 8$ $= 16$ $= RHS$ $(1, -2, 4)$ is a valid solution as it satisfies all equations in the system.

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3.2.2 Solving systems of linear equations using inverse matrices

Review of inverse matrices and determinants

For every square matrix, the determinant, can be calculated.

For the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant can then be used to calculate the inverse of a matrix.

For the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse, A^{-1} is given by $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

When a matrix is multiplied by its inverse, the identity matrix is obtained. That is $AA^{-1} = I$, where, for a 2×2 matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Remember that an inverse does not exist for a **singular matrix** because the determinant equals zero. A non-singular matrix is often referred to as an **invertible matrix** because an inverse does exist.

Solving systems of linear equations using inverse matrices

In this section, the solution of a system of linear equations will be calculated by means of matrix algebra and inverses. Consider the system of equations $ax + by = e$ and $cx + dy = f$. The coefficients, variables and scalar quantities can be translated into matrix form.

Coefficient matrix $\rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Unknown/variable matrix (column vector) $\rightarrow \begin{bmatrix} x \\ y \end{bmatrix}$

Value/scalar matrix (column vector) $\rightarrow \begin{bmatrix} e \\ f \end{bmatrix}$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$, the system of equations $ax + by = e$ and $cx + dy = f$ can be written in the form of a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$AX = B$$

To solve matrix equations for X in this form, inverse matrices must be used. Recall that matrix division is not possible. Begin by multiplying both sides by A^{-1}, remembering that the order is important.

$$A^{-1}AX = A^{-1}B$$

Given that $A^{-1}A = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the matrix equation has a unique solution in the form,

$$IX = A^{-1}B$$
$$X = A^{-1}B$$

If A is a non-singular matrix, in a system of m linear equations with n variables, the unique solution is given by $X = A^{-1}B$.

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WORKED EXAMPLE 3

Solve the following system of linear equations algebraically by using an inverse matrix.

$$4x + y = 12$$
$$3x + 2y = 14$$

THINK	WRITE
1. Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.	$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$
2. Express matrices A, B and C in the form of a matrix equation.	$AX = B$ $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$
3. Multiply both sides from the left by A^{-1} to create a general matrix solution for X.	Assuming A is invertible, $A^{-1}AX = A^{-1}B$ $X = A^{-1}B$
4. Determine the inverse matrix, A^{-1}, using algebraic techniques. $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	$A^{-1} = \frac{1}{4 \times 2 - 1 \times 3}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$
5. Apply the rules for matrix multiplication. The product is a 2×1 matrix. Apply the rules for scalar multiplication.	$X = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 12 \\ 14 \end{bmatrix}$ $= \frac{1}{5}\begin{bmatrix} 24 + -14 \\ -36 + 56 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
6. Apply the rules for the equality of matrices to then state the solution.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $\therefore$ The solution is $x = 2, y = 4$ $(x, y) = (2, 4)$

Inverse matrices are not limited to 2×2 systems. For any square system, the inverse matrix can be found using technology. Once the inverse matrix has been calculated, the general formula $X = A^{-1}B$ can then be applied to calculate the solution.

WORKED EXAMPLE 4

Solve the following system of linear equations by using an inverse matrix that has been calculated using technology.

$$3x - 5y + z = 25$$
$$-2x - 7y - 10z = -23$$
$$-6x + 2y - 9z = -54$$

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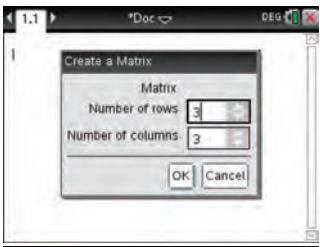
THINK	WRITE
1. Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.	$A = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$
2. Express matrices A, B and C in the form of a matrix equation.	$AX = B$ $\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$
3. Multiply both sides from the left by A^{-1}. A general matrix solution for X now exists.	Assuming A is invertible, $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$
4. Complete the matrix operations using appropriate technology.	$X = A^{-1}B$ $X = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1} \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$ $X = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$
5. Apply the rules for the equality of matrices to then state the solution.	$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ $\therefore$ The solution is $x = 2$, $y = -3$ and $z = 4$. $(x, y, z) = (2, -3, 4)$

TI | THINK **WRITE**

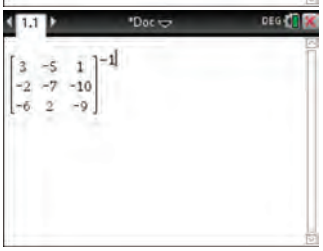
1. On a Calculator page, select

2. Complete the entry line as
3
3
Press the OK button.



3. Complete the entry line as

$$\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1}$$



CASIO | THINK **WRITE**

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.

2. Press the EXE button.
Complete the entry line as
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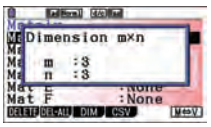
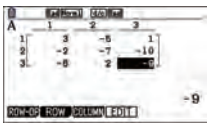
3
Press the EXE button.



3. Complete the entry line as

$$\begin{matrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{matrix}$$

Press the EXIT button when finished.



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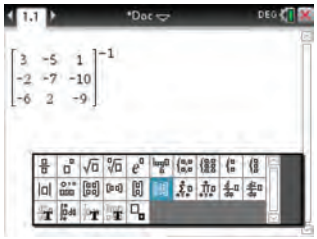
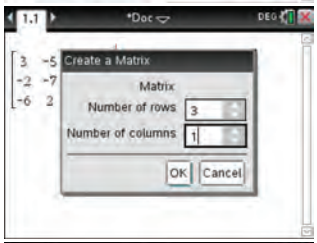
4. Press the button, followed by the button.

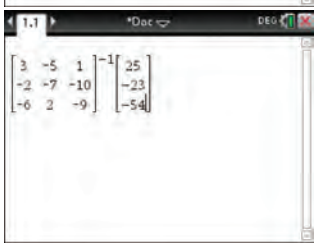


5. Complete the entry line as
3
1
Press the OK button.

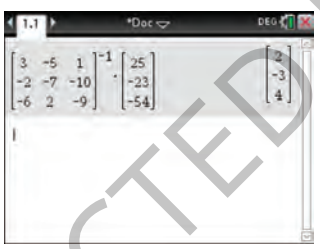


6. Complete the entry line as

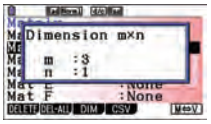
$$\begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$$



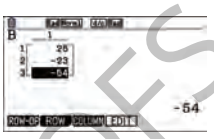
7. Press the ENTER button. The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.



4. Highlight the Mat B row and press the EXE button. Complete the entry line as
3
1
Press the EXE button



5. Complete the entry line as
25
−23
−54
Press the EXIT button twice when finished.



6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
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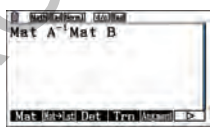
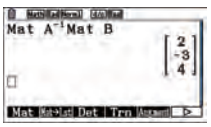
Mat
ALPHA
Log



7. Press the EXE button. The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.



3.2.3 Matrix algebra

When solving matrix equations both sides of the equation were multiplied by A^{-1} to yield the solution $X = A^{-1}B$. This process is an example of how a matrix equation can be solved through the application of matrix algebra. **Matrix algebra** shares many similarities with the algebra used to solve elementary linear equations, however not all.

For instance, the algebraic properties of a matrix equation and a simple linear equation are equivalent in the following examples.

1. The use of an additive inverse to solve linear system.

Matrix algebra	Algebra
$X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix}$ $X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix}$ $X = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$	$X + 2 = 3$ $X + 2 - 2 = 3 - 2$ $X = 1$

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2. The use of a scalar multiple to a solve linear system.

Matrix algebra	Algebra
$3X = \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $\frac{1}{3} \times 3X = \frac{1}{3} \times \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$	$3X = 12$ $\frac{1}{3} \times 3X = \frac{1}{3} \times 12$ $X = 4$

In contrast, the method for factorising is not consistent in both systems of algebra. Consider three matrices of equal dimension A, B and X.

The matrix expression, $XC + XD$, can be factorised as it follows the left distributive law of matrix algebra.

$$XC + XD = X(C + D)$$

A common misconception occurs when attempting to factorise a matrix expression such as $XC + DX$.

$$XC + DX \neq X(C + D)$$

In this case X is common to both terms, however it is incorrect to assume that $XC = CX$ and $DX = XD$. It is also important to note that the common factor, X, does not appear on the same side of both terms.

If the left distributive law of matrices was applied to $X(C + D)$ the resultant expression would be $XC + XD$.

The application of matrix algebra is vital when solving systems of linear equations. This is especially true when attempting to solve more complex equations such as:

$AX + 2BX - C = D$ or $3A(X + I) = C - D$.

SUMMARY OF MATRIX ALGEBRA LAWS

$\boldsymbol{A + B = B + A}$	**Commutative law for addition**
$\boldsymbol{(A + B) + C = A + (B + C)}$	**Associative law for addition**
$\boldsymbol{A + 0 = A}$	**Additive identity**
$\boldsymbol{A + (-A) = 0}$	**Additive inverse**
$\boldsymbol{(AB)C = A(BC)}$	**Associative law for multiplication**
$\boldsymbol{AI = A = IA}$	**Multiplicative identity**
$\boldsymbol{AA^{-1} = A^{-1}A = I}$	**Multiplicative inverse**
$\boldsymbol{A(B + C) = AB + AC}$	**Left distributive law**
$\boldsymbol{(B + C)A = BA + CA}$	**Right distributive law**

The following worked examples will apply the above laws to solve a range of matrix equations.

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WORKED EXAMPLE 5

Determine a general solution for the variable matrix X in the following matrix equations.

a. $XA - B = C$ **b.** $AX + BX = C$

THINK	WRITE
a. 1. Apply the additive inverse law to isolate XA by adding B to both sides of the equation.	**a.** $XA - B = C$ $XA - B + B = C + B$ $XA = C + B$
2. Apply the multiplicative inverse law by multiplying both sides by A^{-1} on the right.	Assuming A is invertible, $XAA^{-1} = (C + B)A^{-1}$
3. Simplify $AA^{-1} = I$	$XI = (C + B)A^{-1}$
4. State the general solution for X.	$\therefore X = (C + B)A^{-1}$
b. 1. The right distributive law can be applied to factorise the LHS as X appears on the right of both terms.	**b.** $AX + BX = C$ $(A + B)X = C$
2. Apply the multiplicative inverse law by multiplying both sides by $(A + B)^{-1}$ to the left.	Assuming $(A + B)$ is non-singular, $(A + B)^{-1}(A + B)X = (A + B)^{-1}C$
3. Simplify $(A + B)(A + B)^{-1} = I$	$IX = (A + B)^{-1}C$
4. State the general solution for X.	$\therefore X = (A + B)^{-1}C$

WORKED EXAMPLE 6

Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$ to determine a unique solution for each of the following linear systems.

a. $B + XA = C$ **b.** $AX - BX = C$

THINK	WRITE
a. 1. Apply the additive inverse law to isolate XA by adding $-B$ to both sides of the equation.	**a.** $B + XA = C$ $B + (-B) + XA = C + (-B)$
2. Simplify both sides.	$XA = C - B$
3. Apply the multiplicative inverse law by multiplying both sides by A^{-1} to the right.	Assuming A is non-singular. $XAA^{-1} = (C - B)A^{-1}$ $XI = (C - B)A^{-1}$
4. A general solution for X has been determined.	$X = (C - B)A^{-1}$
5. Recall the inverse matrix formula $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and apply it to determine $A^{-1} = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}^{-1}$.	$A^{-1} = \frac{1}{2 \times 3 - -6 \times -4}\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{-18}\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \frac{-1}{6} & \frac{-1}{3} \\ \frac{-2}{9} & \frac{-1}{9} \end{bmatrix}$

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6. Substitute for the given matrices C, B and A^{-1} into $X = (C - B)A^{-1}$ and then simplify the RHS by applying the rule for subtraction of matrices to simplify $(C - B)$.

$$X = \left(\begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}\right) \times \begin{bmatrix} \frac{-1}{6} & \frac{-1}{3} \\ \frac{-2}{9} & \frac{-1}{9} \end{bmatrix}$$

7. Apply the rules for matrix multiplication.

$$X = \begin{bmatrix} 4 & -10 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{-1}{6} & \frac{-1}{3} \\ \frac{-2}{9} & \frac{-1}{9} \end{bmatrix}$$

8. State the solution.

$$X = \begin{bmatrix} \frac{14}{9} & \frac{-2}{9} \\ \frac{5}{18} & \frac{8}{9} \end{bmatrix}$$

b. 1. The LHS can be factorised because X appears on the right of both terms.

b. $AX - BX = C$

$(A - B)X = C$

2. Apply the multiplicative inverse law by multiplying both sides by $(A - B)^{-1}$ to the left.

Assuming $(A - B)$ is invertible.

$(A - B)^{-1}(A - B)X = (A - B)^{-1}C$

3. Simplify $(A - B)^{-1}(A - B) = I$ by applying the multiplicative inverse law.

$IX = (A - B)^{-1}C$

$X = (A - B)^{-1}C$

4. Simplify $(A - B)$ by applying the matrix rule for subtraction.

$$A - B = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 \\ -9 & -1 \end{bmatrix}$$

5. Determine $(A - B)^{-1}$ by applying the rule for the inverse of a 2×2 matrix $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$(A - B)^{-1} = \frac{1}{3 \times -1 - -9 \times -9}\begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \frac{1}{-84}\begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix}$$

6. Substitute for the given matrices into $X = (C - B)A^{-1}$. Continue by applying the rules for matrix multiplication.

$X = (A - B)^{-1}C$

$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix} \times \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$$

7. State the solution.

$$X = \begin{bmatrix} -\frac{5}{28} & -\frac{13}{21} \\ -\frac{11}{28} & \frac{4}{7} \end{bmatrix}$$

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Exercise 3.2 Solving linear equations using matrix algebra

Technology free

1. WE1 Identify which of the following equations are non-linear.
   a. $4x - 5\sin(y) + z = 1$
   b. $8x + 9y + 11z = 98$
   c. $-8x + y = 7$
   d. $4x^2 - 8yz = 12$
2. WE2 Given the following systems of linear equations.
   a. $6x + 3y - 5z = 23$
      $-2x + y + 3z = -5$
   b. $x + 3y - 4z = 23$
      $9x - 12y + 2z = -62$
      $-6x + y - 2z = 23$
   i. State the number of m equations and n variables in the system.
   ii. Justify whether the system is square or not.
   iii. Determine if $x = -2, y = 3, z = -4$ is a valid solution set for all equations in the system.
3. WE3 Solve the following system of linear equations algebraically by using an inverse matrix.
   a. $5x + 2y = 26$
      $6x + 3y = 30$
   b. $-4x + 9y = 15$
      $x - 3y = -6$
   c. $5x - 4y = 14$
      $9x + 10y = 94$
   d. $-7x + 8y = 92$
      $6x - 3y = -48$

Technology active

4. WE4 Solve the following system of linear equations by using an inverse matrix.
   a. $2x + 2y - 8z = 38$
      $-5x - y - 7z = 40$
      $8x - 7y - 5z = 17$
   b. $x + 7y - 4z = 34$
      $-7x - 7y - 9z = -129$
      $7x + 5z = -60$
   c. $-x + 8y - 4z = -38$
      $-x - 5y + 4z = 17$
      $x + 3y - 5z = -13$
   d. $3x + 3y + 2z = -31$
      $-5x - 4y - z = 42$
      $3x + 3y - 4z = -19$
5. The points awarded at a school athletics carnival was determined by tallying the number of team members finishing in either 1st, 2nd, 3rd or 4th place in the events.

Team	1st	2nd	3rd	4th	Total Points
Blue	5	6	2	1	105
Green	3	2	8	2	90
Yellow	8	2	2	2	110
Red	10	2	3	4	141

Determine the points awarded for each of the respective placings using an inverse matrix.

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6. Every month a food stall operates at the local market. Four food items are sold, shown in the table below. Create a matrix equation and then use technology to calculate the individual cost per food item.

	Number of items sold				
Month	***Lolly bag***	***Pizza***	***Meat Pie***	***Hamburger***	**Total sales**
January	12	61	12	11	\$565
February	17	36	17	13	\$432
March	23	25	23	8	\$361
April	38	51	9	9	\$494

7. **MC** Identify which of the following matrix operations exhibit the right distributive law.
   A. $A + O = A$ **B.** $P(Q + R) = PQ + PR$
   C. $(T + U)V = TV + UV$ **D.** $C^{-1}C = CC^{-1} = I$
8. **MC** Determine which of the following statements is false. It is assumed that all the matrices are square and of the same dimension.
   A. $AI = IA$ **B.** $CX - XD = X(C - D)$
   C. $TX - XT = -XT + TX$ **D.** $XM + XN = X(M + N)$
9. Construct an expression for X from the matrix equation $EX + X + F = G$.
10. **WE5** Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).
    a. $AXB = C$ **b.** $XA = B$ **c.** $AXB = I$ **d.** $AX + X = C$
11. Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).
    a. $AX + BX + C = D$ **b.** $2X + 3AX = B$ **c.** $AX + 3BX = C$ **d.** $7A(X - I) = B + C$
12. **WE6** Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.
    a. $XA = B$ **b.** $B + XA = C$ **c.** $AX - BX = C$ **d.** $CXD = 2I$
13. Use $A = \begin{bmatrix} 3 & -1 \\ 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -8 \\ -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 7 & 5 \\ -1 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -5 \\ 3 & 4 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.
    a. $CX = B$ **b.** $ACX = 4I$ **c.** $CX - 5DX = B$ **d.** $3A(X + I) = C - D$
14. Assuming that A is a square invertible matrix, use matrix algebra to justify the mathematical statement below.
$$(A + I)(A - I) = A^2 - I$$
15. Solve the following system of nonlinear equations for the unknown angles α and β, where $0 \le \alpha \le \frac{\pi}{2}$ and $0 \le \beta \le \frac{\pi}{2}$.
$$2\sin\alpha + 3\cos\beta = \frac{3\sqrt{2} + 2}{2}$$
$$2\sin\alpha + 3\cos\beta = \frac{1 - \sqrt{2}}{2}$$

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3.3 Solving a system of linear equations using Gaussian elimination

3.3.1 Gaussian elimination

A variety of methods can be used to solve a system of simultaneous equations, such as those applied in the previous section. In elementary mathematics, simultaneous equations are often solved using the substitution or elimination method. Both techniques, and especially the later, are applicable when solving matrix equations using the **Gaussian elimination** method.

Consider a set of two simultaneous linear equations in two variables, x and y.

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

As we saw earlier, this system can be translated into the following matrices,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} B = \begin{bmatrix} u \\ v \end{bmatrix} \text{and } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

where A is the coefficient matrix, B is the value matrix and X is the variable or unknown matrix.

We can combine matrices A and B as $\left[\begin{array}{cc|c} a & b & u \\ c & d & v \end{array}\right]$ where this is referred to as an augmented matrix, denoted by $[\mathrm{A}\,|\mathrm{B}\,]$.

Consider the system below.

System	Coefficient matrix	Value matrix	Augmented matrix
$5x - 2y = 8$ $3x + 4y = 10$	$\begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 10 \end{bmatrix}$	$\left[\begin{array}{cc\|c} 5 & -2 & 8 \\ 3 & 4 & 10 \end{array}\right]$

The purpose of Gaussian elimination is to solve a system of linear equations using elementary row operation proceedures which correspond to the operations used when solving equations by elimination.

Row operation		Example notation
1.	Multiply a row (equation) by a constant.	If the third row was multiplied by -2 it would be denoted as $-2 \times R_3$
2.	Add and subtract one row (equation) to or from another.	If row 2 was multiplied by 4 and the result is then added to row 3 it would be denoted as $4R_2 + R_3$
3.	Swap rows (equations).	If we interchange rows one and two it is denoted as $R_1 \leftrightarrow R_2$

The objective of these proceedures is to prduce a vector with ones in the leading diagonal and zeros under the leading diagonal. This is known as **row-echelon form**. The following matrices are in row-echelon form.

$$\begin{bmatrix} 1 & 6 & 2 & 19 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

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Once row-echelon form is achieved, the solution can be found by a using an algebraic method referred to as back substitution. The following worked example demonstrates this approach.

WORKED EXAMPLE 7

Solve the following system of linear equation using Gaussian elimination and back substitution.

$$\begin{aligned} 4x + 8y &= 24 \\ -3x + y &= 17 \end{aligned}$$

THINK	WRITE
1. Translate the linear equations to matrix form.	$A = \begin{bmatrix} 4 & 8 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 24 \\ 17 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$
2. Express A and B in augmented matrix form.	$[A\,\vert B] = \left[\begin{array}{cc\vert c} 4 & 8 & 24 \\ -3 & 1 & 17 \end{array}\right]$
3. Begin the process of elementary row operations to reduce the augmented matrix to row-echelon form. Divide the first row by four.	$\left[\begin{array}{cc\vert c} 1 & 2 & 6 \\ -3 & 1 & 17 \end{array}\right]_{\frac{1}{4}R_1}$
4. Continue to produce row-echeon form so $a_{21} = 0$. Add $3R_1$ to R_2.	$\left[\begin{array}{cc\vert c} 1 & 2 & 6 \\ 0 & 7 & 35 \end{array}\right]_{R_2+3R_1 \to R_2}$
5. Multiply R_2 by $\frac{1}{7}$ so $a_{22} = 1$. The second row has now reached the desired form.	$\left[\begin{array}{cc\vert c} 1 & 2 & 6 \\ 0 & 1 & 5 \end{array}\right]_{\frac{1}{7}R_1}$
6. State the solution for second row.	$\therefore\ y = 5.$
7. Rewrite the first row of the augmented matrix in terms of x and y. Substitute $y = 5$ into row one to then solve for x.	Use back-substitution to find a solution. $\begin{aligned} x + 2y &= 6 \\ x + 2(5) &= 6 \\ x + 10 &= 6 \\ x + 10 - 10 &= 6 - 10 \\ x &= -4 \end{aligned}$
8. State the unique solution for the system of linear equations.	The unique solution for the system of linear equations is $(x, y) = (-4, 5)$.

In the previous worked example the coefficient matrix was transformed into row-echelon form. To determine the solution, the process of back substitution was applied.

Another approach to determine a solution using Gaussian elimination is the **reduced row-echelon form**. In this method, the solution set (x, y) is attained by continuing elementary row operations until all elements above the leading diagonal are zero. In this form, the augmented matrix will have I on the left-hand side and the solution will be on the right-hand side.

$$[A\,\vert B] \;\rightarrow\; [I\,\vert X]$$

The solution from the previous worked example was $x = -4, y = 5$. The reduced row echelon matrix for this system would be $\left[\begin{array}{cc\vert c} 1 & 0 & -4 \\ 0 & 1 & 5 \end{array}\right]$. The following matrices are in reduced row-echelon form.

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 11 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$$

$x = -8, y = -3, z = 11 \qquad w = -2, x = 4, y = -5, z = 10$

WORKED EXAMPLE 8

Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{aligned} 5x + 2y - 2z &= 2 \\ 2x + 5y + 3z &= -8 \\ x + 3y - 4z &= -11 \end{aligned}$$

THINK	WRITE
1. Translate the linear equations to matrix form.	$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & 3 \\ 1 & 3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -8 \\ -11 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
2. Express A and B in augmented matrix form.	$[A \mid B] = \left[\begin{array}{ccc\|c} 5 & 2 & -2 & 2 \\ 2 & 5 & 3 & -8 \\ 1 & 3 & -4 & -11 \end{array}\right]$
3. The coefficient for x in row 3 is 1. Interchange rows 1 and 3.	$= \left[\begin{array}{ccc\|c} 1 & 3 & -4 & -11 \\ 2 & 5 & 3 & -8 \\ 5 & 2 & -2 & 2 \end{array}\right]_{R_1 \leftrightarrow R_3}$
4. Multiples of row 1 can be added/subtracted to row 2 and 3 so that $a_{21} = 0$ and $a_{31} = 0$.	$= \left[\begin{array}{ccc\|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & -13 & 18 & 57 \end{array}\right]_{\substack{R_2 - 2R_1 \to R_2 \\ R_3 - 5R_1 \to R_3}}$
5. Subtract $13 \times R_2$ from R_3 so that $a_{32} = 0$.	$= \left[\begin{array}{ccc\|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & 0 & -125 & -125 \end{array}\right]_{R_3 - 13 \times R_2}$
6. Complete two operations. First multiply R_2 by -1 so $a_{22} = 1$. Follow this by multiplying R_2 by $\frac{-1}{125}$ so $a_{33} = 1$.	$= \left[\begin{array}{ccc\|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -11 & -14 \\ 0 & 0 & 1 & 1 \end{array}\right]_{\substack{-1 \times R_2 \to R_2 \\ \frac{-1}{125} \times R_3 \to R_3}}$
7. State the solution for the third row.	$\therefore z = 1.$
8. Rewrite the second row of the augmented matrix in terms of x and y. Substitute in $z = 1$ to then solve for y.	Use back-substitution to find a solution. $y - 11z = -14$ (From R_2) $y - 11(1) = -14$ $y - 11 + 11 = -14 + 11$ $y = -3$

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9. Rewrite the first row of the augmented matrix in terms of x, y and z. Substitute $y = -3$ and $z = 1$ to solve for x.

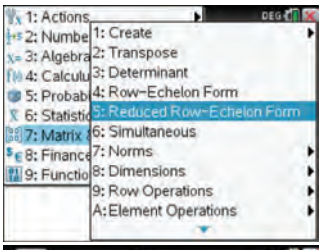
$$x + 3y - 4z = -11 \qquad \text{(From } R_1\text{)}$$
$$x + 3(-3) - 4(1) = -11$$
$$x - 9 - 4 = -11$$
$$x - 9 - 4 + 9 + 4 = -11 + 9 + 4$$
$$x = 2$$

10. State the unique solution for the system of linear equations.

The unique solution for the system of linear equations is $x = 2, y = -3$ and $z = 1$.
$\therefore (x, y, z) = (2, -3, 1)$

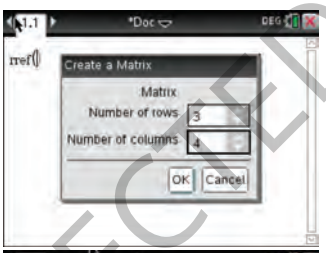
TI | THINK — **WRITE**

1. On a Calculator page, select
   MENU
   7: Matrix and Vector
   5: Reduced Row-Echelon Form



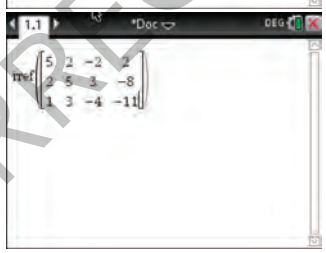
2. On a calculator screen, select

3. Complete the entry line as
   3
   4
   Press the OK button.



4. Complete the entry line as

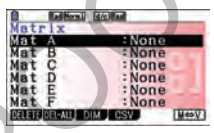
5	2	−2	2
2	5	3	−8
1	3	−4	−11



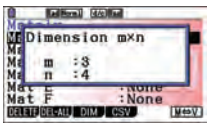
5. Press the ENTER button. The answer appears on the screen.

CASIO | THINK — **WRITE**

1. On a Run-Matrix screen, select MAT/VCT
   To switch between the vector and matrix screens, press the M ⇔ V button.



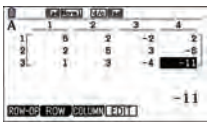
2. Press the EXE button.
   Complete the entry line as
   3
   4
   Press the EXE button.



3. Complete the entry line as

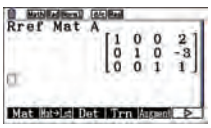
5	2	−2	2
2	5	3	−8
1	3	−4	−11

   Press the EXIT button twice when finished.



4. On a Run-Matrix screen, select OPTN,
   MAT/VCT
   ▷
   Rref
   ▷ (x3)
   Mat
   SHIFT
   $X.\theta$, T

5. Press the EXE button.
   The answer appears on the screen.



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Units 3 & 4 > Area 2 > Sequence 2 > Concept 2

Solving a system of linear equations using Gaussian elimination Summary screen and practice questions

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Exercise 3.3 Solving a system of linear equations using Gaussian elimination

Technology free

1. **MC** A matrix in row echelon form is defined by which description?
   - **A.** Contains ones in the leading diagonal and ones under the leading diagonal.
   - **B.** Contains ones in the leading diagonal and zeros under the leading diagonal. In addition, a row consisting of only zeros may exist as the bottom row of the matrix.
   - **C.** Contains ones in the leading diagonal and zeros in every other cell.
   - **D.** Contains ones in every cell.

2. **MC** Identify which of the following matrices is reduced row echelon form.

   A. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -7 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 5 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \end{bmatrix}$ **D.** $\begin{bmatrix} 1 & 9 & 4 \\ 0 & 1 & 7 \end{bmatrix}$

3. **MC** The augmented matrix $\left[\begin{array}{ccc|c} 2 & 7 & -4 & 5 \\ 1 & 2 & 3 & 8 \\ 4 & 8 & -12 & -20 \end{array}\right]$ changes to $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 7 & -4 & 5 \\ 1 & 2 & -3 & -5 \end{array}\right]$. Identify which row operations were completed for that change to occur.

   A. $R_2 - R_1, R_3 \leftrightarrow R_2$ **B.** $R_2 \leftrightarrow R_1, R_3 \times \frac{1}{4}$ **C.** $R_2 \leftrightarrow R_3, R_3 \times 4$ **D.** $R_2 \leftrightarrow R_3, R_3 \div 4$

4. **WE7** Solve the following system of linear equations using Gaussian elimination and back substitution.

   a. $\begin{aligned} 6x - 4y &= 10 \\ -2x + 7y &= 8 \end{aligned}$ **b.** $\begin{aligned} x + 3y &= -2 \\ 3x - 9y &= -24 \end{aligned}$ **c.** $\begin{aligned} 9x + 8y &= 34 \\ -3x + 2y &= -2 \end{aligned}$ **d.** $\begin{aligned} -5x - 2y &= 19 \\ 4x + y &= -8 \end{aligned}$

5. **WE8** Solve the following system of linear equations using Gaussian elimination and back substitution.

   a. $\begin{aligned} 6x + 6y + z &= -12 \\ x + 4y - z &= -8 \\ 3x + 3y - 5z &= -6 \end{aligned}$ **b.** $\begin{aligned} -2x + 6y &= -20 \\ -x + 6y - 3 &= 26 \\ -3x - 4z &= 6 \end{aligned}$ **c.** $\begin{aligned} 6x - 6y + 2z &= -8 \\ -5x + 7y - 2z &= 16 \\ -6x + 7y - 5z &= 7 \end{aligned}$

   d. $\begin{aligned} -4x + 2y + 2z &= 38 \\ 4x - y + 2z &= -17 \\ -2x + y + 5z &= 35 \end{aligned}$ **e.** $\begin{aligned} 5x + 3y - z &= -23 \\ -x + 4y + 6z &= -15 \\ -2x + 4y + 6z &= -12 \end{aligned}$

6. Solve the following set of simultaneous equations, containing four variables, using Gaussian elimination and back substitution.

$$\begin{aligned} 2w + 8x - 4y + 2z &= -46 \\ -10w + x - 8y - 6z &= -25 \\ 4w - 5x + 2y - 5z &= -56 \\ -8w - 16x + 4y + 3z &= 33 \end{aligned}$$

7. The row echelon matrix of a system of linear equations is $\left[\begin{array}{ccc|c} 1 & 4 & -2 & 30 \\ 0 & 1 & 3 & 110 \\ 0 & 0 & 1 & 30 \end{array}\right]$.

   Assuming the independent variables are x, y and z, use back substitution to calculate the unique solution for the system.

Technology active

8. One group of students purchase 4 regular coffees and 3 large coffees for \$33. A second group purchase 2 regular and 5 large coffees for \$34. Use Gaussian elimination to calculate the cost of both coffee sizes.

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9. The cost of a movie ticket varies according to the classification of student, adult or pensioner. The table below summarises the cost for three different families to attend the cinema.

No. of students	No. of adults	No. of pensioners	Total cost($)
6	2	1	82
4	1	2	58
2	2	2	70

Use Gaussian elimination to determine the total cost for a family containing two students, one adult and two pensioners.

10. Convert the augmented matrix, $\left[\begin{array}{ccc|c} 5 & 2 & 4 & 32 \\ -4 & 7 & 8 & -15 \\ -3 & 11 & 1 & 9 \end{array}\right]$, into a system of linear equations.

11. A quadratic in the form, $y(x) = a_0 + a_1x + a_2x^2$, contains the points $A(2, 1)$, $B(3, 7)$ and $C(4, 17)$. If n points from a curve are known it is possible to develop a polynomial to the $n - 1$ degree through a process known as interpolating a polynomial. Points A, B and C can be translated into an augmented matrix given by,

$$\left[\begin{array}{ccc|c} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{array}\right] \quad \text{where} \quad \begin{array}{r} A(2, 1) = (x_1, y_1) \\ B(3, 7) = (x_2, y_2) \\ C(4, 17) = (x_3, y_3) \end{array}$$

The values for a_0, a_1 and a_2 are determined when the augmented matrix has reached reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 1 & a_2 \end{array}\right]$$

Develop an augmented matrix and perform elementary row operations to determine the values for a_0, a_1 and a_2.

12. In the previous questions we used augmented matrices in the form $[A\,|B]$ to find solutions. This method can be used to determine the inverse of matrix A. To achieve this, matrix A is augmented with the identity matrix I, $[A\,|I]$. The inverse is found by completing elementary row reductions until you produce $[I\,|A^{-1}]$. Follow this method to determine the inverse of matrix A where:

$$A = \left[\begin{array}{cc|c} 3 & 12 & -6 \\ 2 & 4 & -6 \\ 1 & 3 & -4 \end{array}\right].$$

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13. The initial stages in finding a solution to a system of linear equations was attempted below. Identify the location of the five errors in the solution below.

$$\begin{aligned} x - 5y + z &= -6 \\ 3x + y - 2z &= 4 \\ -5x + 4z - y &= -4 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & 4 & -1 & -4 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & 22 \\ 0 & -21 & 4 & -34 \end{array}\right]_{R_2 - 3R_1 \to R_2,\ R_3 + 5R_1 \to R_3}$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 0 & 1 & -\frac{5}{16} & \frac{11}{8} \\ 0 & 0 & -1 & 44 \end{array}\right]_{5R_3 - 21R_1 \to R_3,\ R_2 \div 16 \to R_2}$$

3.4 The three cases for solutions of systems of linear equations

The study of linear equations until this point has been focussed on the numerous methods that can be used to solve a system. You may have observed that all systems were common in that each had exactly one solution. That is, is each system had a unique solution. However, the solution set is not limited to only one solution.

Systems of linear equations have either:

- A unique (one) solution
- No solution
- Infinitely many solutions

A system containing at least one solution is defined as **consistent**. An **inconsistent** system has no solution.

3.4.1 Geometric interpretation of solutions of lines in two-dimensions

Recall the general form of a system in two dimensions. These systems are given by

$$\begin{aligned} a_1x + b_1y &= c_1 \quad [1] \\ a_2x + b_2y &= c_2 \quad [2]. \end{aligned}$$

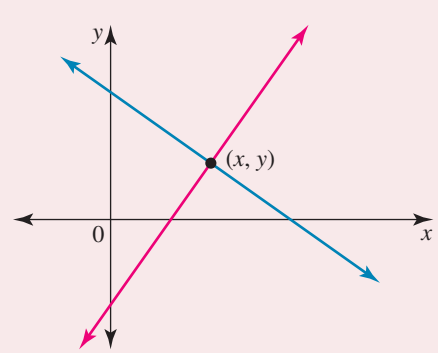
Systems in this form were previously expressed in the form of a matrix equation, $AX = B$. The unknown matrix X was solved using various techniques, with the most functional method being the inverse matrix approach. This was possible if A was non-singular ($\Delta \neq 0$). The coefficient matrix, A, will be point of reference when investigating the three possible solution types below.

A UNIQUE SOLUTION (NON-PARALLEL, DISTINCT LINES)

- **A is *non-singular* (det $A \neq 0$)**
- **The system is *consistent*.**
- **The two lines are *not parallel*.**
- **The two lines intersect at a one point that represents a *unique solution*.**

Properties of the coefficients

- $\dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$



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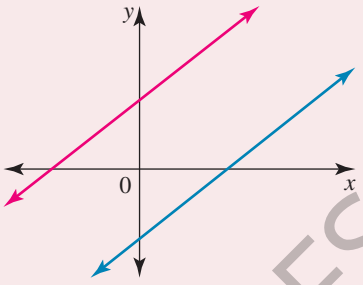
NO SOLUTION (PARALLEL BUT DISTINCT LINES)

- **A is *singular* (det $A = 0$)**
- **The system is *inconsistent*.**
- **The two lines are *parallel* and *distinct* (the lines will have different y-intercepts).**
- **The two lines do not intersect and there is *no solution*.**

Properties of the coefficients

- $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} \neq \dfrac{c_2}{c_1}$

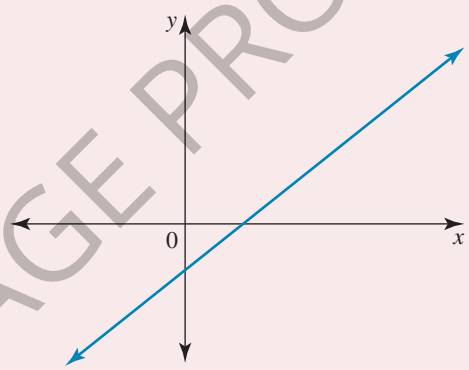


INFINITELY MANY SOLUTIONS (PARALLEL AND COINCIDENT LINES)

- **A is singular (det $A = 0$)**
- **The system is *consistent*.**
- **The two lines are *parallel* and *not distinct* (the lines will have the same y-intercept).**
- **The two lines are *coincident* meaning they overlap and there is an *infinite number of solutions*.**

Properties of the coefficients

- $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \dfrac{c_2}{c_1}$



WORKED EXAMPLE 9

Solve the following system of linear equations by using matrices and interpret the solution geometrically.

$$2x + 7y = 16$$
$$-4x - 14y = 20$$

THINK	WRITE
1. Define the matrices A, X and B.	$A = \begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$
2. Translate into a matrix equation.	$\begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$
3. Write as a general equation and solve for X.	$AX = B$ $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$
4. Recall the determinant rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and calculate for matrix A to determine if A is singular.	$\det(A) = \begin{vmatrix} 2 & 7 \\ -4 & -14 \end{vmatrix} = 2 \times -14 - 7 \times -4 = 0$

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5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.

$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.

6. Rearrange both equations into the form $y = mx + c$,

$$2x + 7y = 16$$
$$7y = 16 - 2x$$
$$\rightarrow y_1 = -\frac{2}{7}x + \frac{16}{7}$$

$$-4x - 14y = 20$$
$$14y = -4x - 20$$
$$\rightarrow y_2 = -\frac{2}{7}x - \frac{10}{7}$$

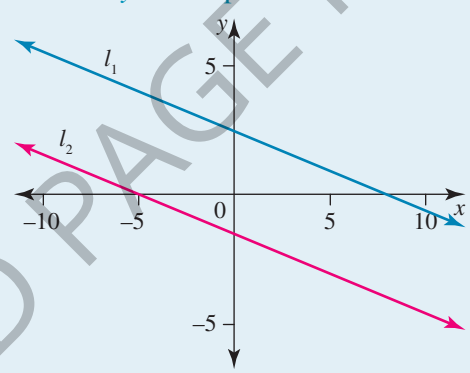
7. The lines are parallel because the gradients are equal.
The two lines are distinct because the y-intercepts are different.

Line [1]: $m_1 = \frac{-2}{7}$ y_1-intercept $= \frac{16}{7}$

Line [2]: $m_2 = -\frac{2}{7}$ y_2-intercept $= -\frac{10}{7}$

The lines are parallel, yet distinct as they have different y-intercepts.

8. Sketch the graphs. Note that the two lines are parallel with distinct y-intercepts, which means they will never intersect and hence have no solution.



9. State the answer.

There is no solution as the lines are parallel and distinct.

WORKED EXAMPLE 10

a. Solve the following system of linear equations by using matrices and interpret the solution geometrically.

b. Describe the solution set in parametric form.

$$\mathbf{3x - 4y = -2}$$
$$\mathbf{9x - 12y = -6}$$

THINK

WRITE

a. 1. Define the matrices A, X and B.

a. $A = \begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

2. Translate the linear equations into a matrix equation.

$$\begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

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3. Write as a general equation and solve for X.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

4. Recall the determinant rule

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and calculate for matrix A to determine if A is singular.

$$\det(A) = \begin{vmatrix} 3 & -4 \\ 9 & -12 \end{vmatrix} = 3 \times -12 - -4 \times 9 = 0$$

5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.

$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.

6. Rearrange both equations into the form $y = mx + c$.

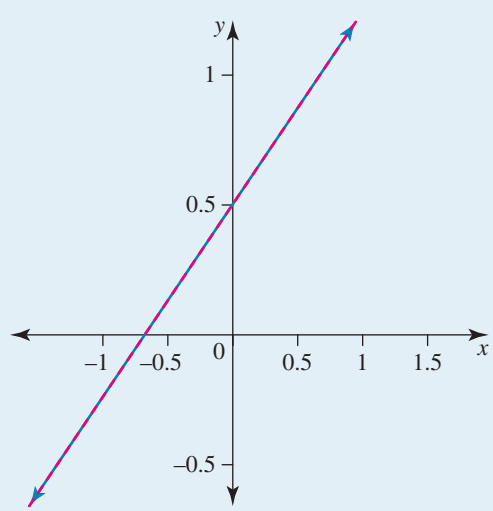
$$3x - 4y = -2$$
$$4y = 3x + 2$$
$$\rightarrow y_1 = \frac{3}{4}x + \frac{1}{2}$$

$$9x - 12y = -6$$
$$12y = 9x + 6$$
$$\rightarrow y_2 = \frac{3}{4}x + \frac{1}{2}$$

7. The lines are parallel because the gradients are equal.
The two lines are coincident because the y-intercepts are the same.

Line [1]: $m_1 = \frac{3}{4}$, y_1-int $= \frac{1}{2}$
Line [2]: $m_2 = \frac{3}{4}$, y_2-int $= \frac{1}{2}$
The two lines are coincident because they are identical.

8. Sketch the graphs. Note that the two lines overlap, meaning there are an infinite number of solutions.



b. 1. The two lines are identical. Rewrite either equation in parametric form. Begin by making x the subject of the equation.

b. $3x - 4y = -2$
$$3x = 4y - 2$$
$$x = \frac{4y - 2}{3}$$

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2. Express y in terms of t.	Let $y = t$ so that $x = \dfrac{4t-2}{3}$ As a coordinate: $\left(\dfrac{4t-2}{3}, t\right)$
3. State the answer.	There are an infinite number of solutions of the form $(x, y) = \left(\dfrac{4t-2}{3}, t\right)$.

3.4.2 Geometric interpretation of solutions of lines in three-dimensions

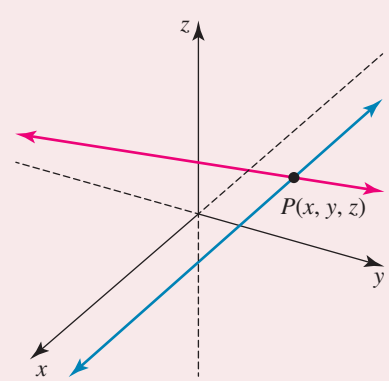
The vector equation of a line in three-dimensions was studied in Chapter 2. Recall that a line can be expressed in three different forms.

Vector equation form	Parametric form	Cartesian form
$\underset{\sim}{r} = \left(-5\hat{\imath} + 2\hat{\jmath} + 7\hat{k}\right) + k\left(3\hat{\imath} - 4\hat{\jmath} - 9\hat{k}\right)$ $\underset{\sim}{a} = -5\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$ $\underset{\sim}{d} = 3\hat{\imath} - 4\hat{\jmath} - 9\hat{k}$	$x = -5 + 3k =$ $y = 2 - 4k = k \in R$ $z = 7 - 9k =$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$	$\dfrac{x+5}{3} = \dfrac{y-2}{-4} = \dfrac{z-7}{-9}$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$

When examining two lines (l_1 and l_2) in three dimensions, three solution types are possible.

CONSISTENT SYSTEM – ONE UNIQUE SOLUTION

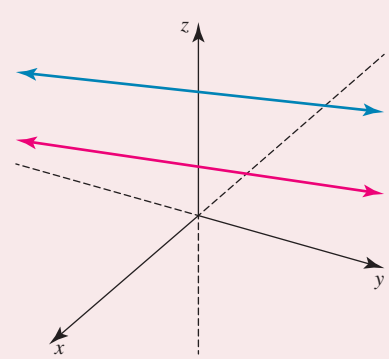
- **The system is *consistent*.**
- **The two lines are *not parallel*.**
- **The two lines intersect at a one point that represents a *unique solution*.**



INCONSISTENT SYSTEM – TWO CASES

Case 1: Parallel but distinct lines

- **The system is *inconsistent*.**
- **The two lines are *parallel* and *distinct*.**
- **The two lines do not intersect and there is *no solution*.**



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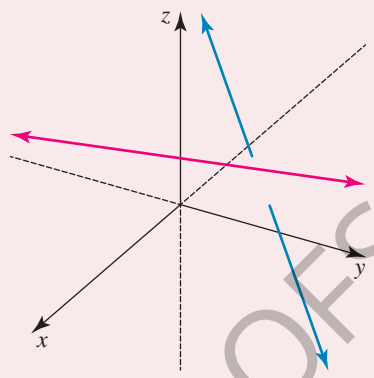
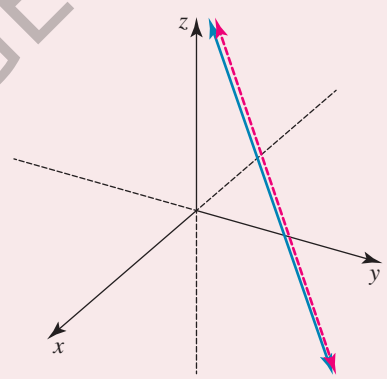
Case 2: Not parallel but distinct lines

- **The system is *inconsistent*.**
- **The two lines are referred to as skew lines and do not intersect, there is *no solution*.**



CONSISTENT SYSTEM – INFINITELY MANY SOLUTIONS (PARALLEL AND COINCIDENT LINES)

- **The system is *consistent*.**
- **The two lines are *parallel* and *not distinct*.**
- **The two lines are *coincident* meaning they overlap and there is an *infinite number of solutions*.**



WORKED EXAMPLE 11

a. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2.
b. Describe the geometric nature of the solution.

$$l_1: \left(5\hat{\imath} - 3\hat{\jmath} - 7\hat{k}\right) + k\left(-4\hat{\imath} + \hat{\jmath} + 5\hat{k}\right)$$
$$l_2: \left(-5\hat{\imath} + 2\hat{\jmath} - 9\hat{k}\right) + k\left(6\hat{\imath} - 4\hat{\jmath} + 7\hat{k}\right)$$

THINK

a. 1. Analyse the direction vector to determine if the lines are parallel.
They are not equal, nor is one a scalar multiple of the other.

WRITE

a. $\underset{\sim}{d}_1 = -4\hat{\imath} + \hat{\jmath} + 5\hat{k}$
$\underset{\sim}{d}_2 = 6\hat{\imath} - 4\hat{\jmath} + 7\hat{k}$
$\underset{\sim}{d}_1 \neq m \times \underset{\sim}{d}_2$, where $m \in R$
The two lines are not parallel.

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2. Rewrite the equations of both lines in parametric form.

l_1: $x = 5 - 4k$
$y = -3 + k$
$z = -7 + 5k$
l_2: $x = -5 + 6k$
$y = 2 - 4k$
$z = -9 + 7k$

3. Both lines are in tems of the same parameter, k, where $k \in R$. Equate the x component of l_1 and l_2 to determine a possible value for k.

Solve for k by equating the x component of l_1 and l_2:
$5 - 4k = -5 + 6k$
$10 = 10k$
$k = 1$

4. Determine the position on both lines when the parameter is set to $k = 1$ by substituting $k = 1$ into the parametric equations.

Let $k = 1$, for lines, l_1 and l_2:
l_1: $x = 5 - 4 \times 1$
$y = -3 + 1$
$z = -7 + 5 \times 1$
$(x_1, y_1, z_1) = (1, -2, -2)$
l_2: $x = -5 + 6 \times 1$
$y = 2 - 4 \times 1$
$z = -9 + 7 \times 1$
$(x_2, y_2, z_2) = (1, -2, -2)$
$(x_1, y_1, z_1) = (x_2, y_2, z_2)$.

5. State the solution.

A unique solution exists at $(x, y, z) = (1, -2, -2)$.

b. Describe the geometric nature of the solution.

b. The two distinct lines intersect at one unique point, $(1, -2, -2)$.

In the above worked example, both vector equations are in terms of the parameter, k, where $k \in R$. When determining a point of intersection, it was simply found by equating the components, (x, y, z). In some of the more complex systems you may find that two lines are defined by different parameters, k_1 and k_2.

WORKED EXAMPLE 12

A system of linear equations in three variables contains the lines l_1 and l_2.

a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.

b. Describe the geometric nature of the system by referring to any possible solution/s.

$$l_1: \frac{x+6}{2} = \frac{y+2}{5} = \frac{z+4}{-3} = k_1, \quad l_2: \frac{x+3}{4} = \frac{y+6}{10} = \frac{z}{-6} = k_2$$

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THINK	WRITE
a. 1. Analyse the direction vector to determine if the lines are parallel. Both lines are in Cartesian form, $\frac{x-a_1}{d_1} = \frac{x-a_2}{d_2} = \frac{x-a_3}{d_3} = k$. The direction of the line can be expressed as a vector given by $\underset{\sim}{d}_1 = d_1\hat{\boldsymbol{\imath}} + d_2\hat{\boldsymbol{\jmath}} + d_3\hat{\boldsymbol{k}}$.	**a.** $\underset{\sim}{d}_1 = 2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$ $\underset{\sim}{d}_2 = 4\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}$ $\underset{\sim}{d}_1 = \frac{1}{2} \times \underset{\sim}{d}_2 \rightarrow \therefore \underset{\sim}{d}_1 \parallel \underset{\sim}{d}_2$ The two lines are parallel.
2. As the lines are parallel, they may overlap (coincident) or be distinct. Rewrite the equations of both lines parametric form.	l_1: $x = -6 + 2k_1$ $y = -2 + 5k_1$ $z = -4 - 3k_1$ l_2: $x = -3 + 4k_2$ $y = -6 + 10k_2$ $z = -6k_2$
3. Determine a point on the first line by setting the parameter to any given value. Let $k_1 = 0$.	For l_1, let $k_1 = 0$ l_1: $x = -6 + 2 \times 0$ $y = -2 + 5 \times 0$ $z = -4 - 3 \times 0$ The following point exists on l_1: $(x_1, y_1, z_1) = (-6, -2, -4)$
4. If the lines coincident, then the same point found on l_1 will satisfy the second line, l_2. Verify whether $(-6, -2, -4)$ lies on the second line.	Substitute $(-6, -2, -4)$ into the parametric form of l_2. $x = -3 + 4k_2$ $-6 = -3 + 4k_2$ $k_2 = \frac{-3}{4}$ $y = -6 + 10k_2$ $-2 = -6 + 10k_2$ $k_2 = \frac{2}{5}$ $z = -6k_2$ $-4 = -6k_2$ $k_2 = \frac{2}{3}$ The values for k_2 are not equal which indicates the point $(-6, -2, -4)$ does not lie on l_2.
5. State the solution.	A solution does not exist for the simultaneous system.
b. Describe the geometric nature of the solution.	**b.** The two lines are parallel and distinct which means they will never intersect.

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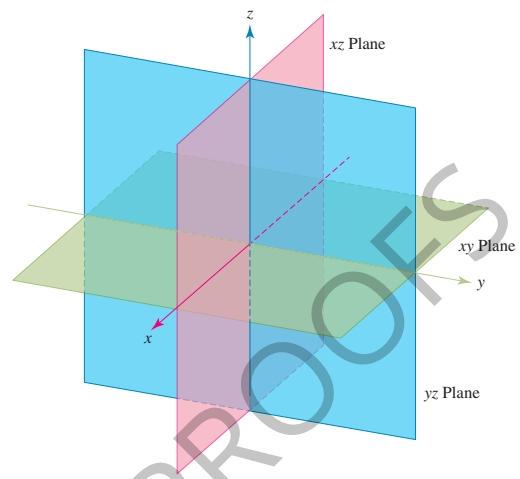
3.4.3 Geometric interpretation of solutions of planes in three-dimensions

Coordinate geometry in three dimensions was introduced in Chapter 23. These systems are defined by three variables to specify the location of a point $P(x, y, z)$. A right-handed system, consisting of three orthogonal planes is shown. Each of these planes could be written in Cartesian form, $ax + by + cx + d = 0$. When considering the equations of these planes simultaneously you are investigating the solution for a 3×3 linear system. In the system shown in the figure, the unique solution can be clearly seen as the intersection of the three planes at the origin, $P(0, 0, 0)$.

Systems comprising of three planes can have either one, zero or infinitely many solutions.



Review of properties of a plane

To begin this study, it important to refresh your memory on the properties of a plane and the associated terminology.

The equation of the plane passing through (a, b, c) and perpendicular to $\underset{\sim}{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ is given by:

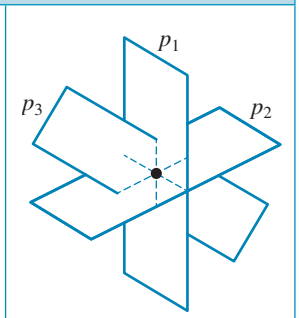
$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

or

$$(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{n} = 0$$

- To determine if two or more planes are parallel you must compare their normal vector $\underset{\sim}{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$. They are parallel if they are equal or a scalar multiple of the other.
- Coplanar objects are those lying in the same plane. Coplanar vectors occur when $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ (scalar triple product).
- The scalar triple product can be found, with or without technology, by calculating the determinant.

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Consistent system – one unique solution		
Geometric interpretation	Planes intersect at one unique point, $P(x_1, y_1, z_1)$	
$AX = B$	A is invertible.	
Coplanar	$\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$	

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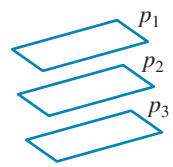
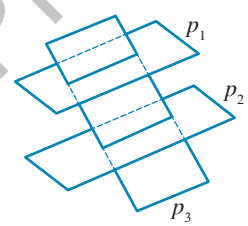
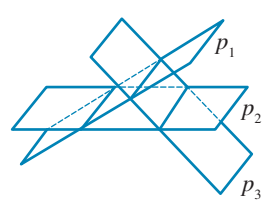
Inconsistent system – zero solutions		
Geometric interpretation	The normal vector of three planes are parallel yet distinct.	
$\boldsymbol{AX} = \boldsymbol{B}$	A is not invertible.	
Coplanar	$\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc\|c} 1 & a_{13} & a_{23} & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right]$	

Inconsistent system – zero solutions		
Geometric interpretation	Two planes are distinct with parallel normals. The third plane is not parallel.	
$\boldsymbol{AX} = \boldsymbol{B}$	A is not invertible.	
Coplanar	$\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$ The final row is a false statement, so this system has no solution (they intersect as lines in parallel pairs).	

Inconsistent system – zero solutions		
Geometric interpretation	The normals are not parallel.	
$\boldsymbol{AX} = \boldsymbol{B}$	A is not invertible.	
Coplanar	$\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$ The final row has a false statement, so this system has no solution. The planes intersect as pairs in a line.	

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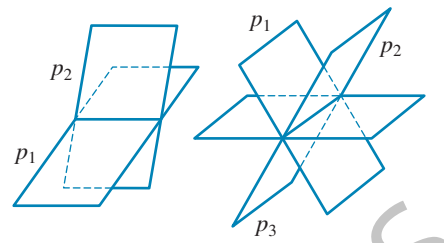
Consistent system – infinitely many solutions		
Geometric interpretation $AX = B$ **Coplanar** **Augmented matrix in triangular form**	The normals are coplanar and the planes intersect as a line. Infinitely many solutions can be defined parametrically. A is not invertible. $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & 0 \end{array}\right]$ The third equation is eliminated, and the free variable is expressed in terms of t, $z = t$.	
Geometric interpretation $AX = B$ **Coplanar** **Augmented matrix in triangular form**	The three planes are coincident. There are infinitely many solutions. A is not invertible. $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a}{a_{11}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	

WORKED EXAMPLE 13

The following descriptions are the geometric and/or algebraic properties of a system of linear equations in three variables. Make an inference on the type of solution set each system would produce based on the given information.

a. **The three planes are not parallel, $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ and the final row of the triangular matrix is a false statement.**

b. **The three normals are not parallel. The coefficient matrix in the equation $AX = B$ is non-singular.**

c. **When solving the system using Gaussian elimination the augmented matrix reaches the triangular form $\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$. The planes intersect as pairs in the form of parallel lines.**

THINK

a. The three planes are not parallel, hence the system could be consistent or inconsistent. The normals are coplanar therefore it can't be unique. The final row of the triangular matrix is false, so the planes may intersect in pairs of lines.

WRITE

a. It is an inconsistent system with zero solutions.

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b. As the coefficient matrix is invertible, and the planes are not parallel the only solution type is unique.

c. The final row of the augmented triangular matrix is false so the system is inconsistent. As the planes intersect as pairs in the form of parallel lines there is no unique solution.

b. It is a consistent system containing one unique solution.

c. It is an inconsistent system containing no solution.

WORKED EXAMPLE 14

Consider the following system of linear equations.

$$\begin{aligned} 3x + 2y + 5z &= 27 \\ x + 5y - 4z &= -2 \\ -4x + 3y - 6z &= -10 \end{aligned}$$

a. Determine the number of possible solutions by examining the normal vectors algebraically.
b. Convert the equations to an augmented matrix and then use technology to solve the system.
c. Describe how the solution can be interpreted geometrically.

THINK

a. 1. Determine if the normal vectors are equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.

2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.
The determinant may be calculated using technology. (See Chapter 2, Worked example 29).

$$\det\left(\begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{bmatrix}\right)$$

b. 1. Convert the linear system of equations into augmented matrix form where:
A = Coefficient matrix.
B = Value matrix.

WRITE

a. $\underset{\sim}{n}_1 = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$
$\underset{\sim}{n}_2 = \hat{\imath} + 5\hat{\jmath} - 4\hat{k}$
$\underset{\sim}{n}_3 = -4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$
The normals are not parallel, hence the planes are not parallel.

$$\begin{aligned} &\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 \times \underset{\sim}{n}_3 \\ &= (3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}) \cdot (\hat{\imath} + 5\hat{\jmath} - 4\hat{k}) \times (-4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}) \\ &= \begin{vmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{vmatrix} \\ &= 3\begin{vmatrix} 5 & -4 \\ 3 & -6 \end{vmatrix} - 2\begin{vmatrix} 1 & -4 \\ -4 & -6 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ -4 & -6 \end{vmatrix} \\ &= 105 \end{aligned}$$

As $\Delta \neq 0$, the planes are not coplanar and intersect at a unique point.

b. $[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array}\right]$

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2. The augmented matrix can be reduced into triangular form using technology (see Worked example 8).

$$\text{rref}\left(\left[\begin{array}{rrrr} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array}\right]\right)$$

Note: The same could be achieved using Gaussian elimination.

$$[A|B] = \left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The three planes intersect at $(x, y, z) = (-2, 4, 5)$.

c. State the type of solution and describe it geometrically.

c. The system is consistent and has one unique solution. Geometrically, the three planes intersect at a unique point.

WORKED EXAMPLE 15

Consider the following system of linear equations to answer the following.

$$\begin{aligned} -x - 5y - 8z &= -22 \\ 6x + 9y + 6z &= 69 \\ -2x - y + 2z &= -17 \end{aligned}$$

a. **Determine the number of possible solutions by examining the normal vectors algebraically.**
b. **Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.**
c. **Solve the system of linear equations, expressing the solution in parametric form.**
d. **Describe how the solution can be interpreted geometrically.**

THINK

a. 1. Examine the normal vectors to determine if they equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.

2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.

WRITE

a. $\underset{\sim}{n}_1 = -\hat{\imath} - 5\hat{\jmath} - 8\hat{k}$
$\underset{\sim}{n}_2 = 6\hat{\imath} + 9\hat{\jmath} + 6\hat{k}$
$\underset{\sim}{n}_3 = -2\hat{\imath} - \hat{\jmath} + 2\hat{k}$
None of the normal are parallel as they are not scalar multiples of each other.

$$\begin{aligned} \hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) &= (-\hat{\imath} - 5\hat{\jmath} - 8\hat{k}) \cdot (6\hat{\imath} + 9\hat{\jmath} + 6\hat{k}) \\ &\quad \times (-2\hat{\imath} - \hat{\jmath} + 2\hat{k}) \\ &= \begin{vmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{vmatrix} \\ &= -1\begin{vmatrix} 9 & 6 \\ -1 & 2 \end{vmatrix} - -5\begin{vmatrix} 6 & 6 \\ -2 & 2 \end{vmatrix} \\ &\quad + -8\begin{vmatrix} 6 & 9 \\ -2 & -1 \end{vmatrix} \\ &= 0 \end{aligned}$$

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The determinant may be calculated using technology (see Chapter 2, Worked example 29).

$$\det\left(\begin{bmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{bmatrix}\right)$$

As $\Delta = 0$ the three planes are coplanar and not parallel.
Geometrically, this could be the planes intersecting in a line or not at all.

b. 1. Convert the linear system of equations into augmented matrix form where,
A = Coefficient matrix.
B = Constant matrix.

$$[A|B] = \left[\begin{array}{ccc|c} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{array}\right]$$

2. The augmented matrix can be reduced into triangular form using technology (see Worked example 8).

$$\text{ref}\left(\begin{bmatrix} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{bmatrix}\right)$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 5 & 8 & 22 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c. 1. The third row of the triangular matrix is equivalent to, $0x + 0y + 0z = 0$.
This informs us that there are infinitely many solutions.

2. Express the first two rows of the triangular matrix as two equations in terms of x, y and z. Let the free variable $z = t$.

3. Rewrite equation [2] by substituting t for z and make y the subject.

4. Rewrite equation [1] substituting $z = t$ and $y = 3 - 2t$.
Simplify the LHS to then rearrange and make x the subject.

5. State the solution in terms of the parameter, t.

c. The system has infinitely many solutions because the planes intersect in a line.

$x + 5y + 8z = 22 \quad [1]$
$y + 2z = 3 \quad [2]$
Let $z = t$ as it is the free variable.

$y + 2t = 3$
$y = 3 - 2t \quad [3]$
Sub $y = 3 - 2t$ into [1]
$x + 5(3 - 2t) + 8t = 22$
$x + 15 - 10t + 8t = 22$
$x = 22 - 15 + 2t$
$x = 7 + 2t$
The intersection line is given by the parametric equations:
$x = 7 + 2t$
$y = 3 - 2t$
$z = t$

d. State the type of solution and describe it geometrically.

d. This is a consistent system with the solution representing the intersection of three planes in the form of a line.

Note: The solution could easily be converted to two other forms.

- Vector equation: $\underset{\sim}{r} = \left(7\hat{\imath} + 3\hat{\jmath} + 0\hat{k}\right) + t\left(2\hat{\imath} - 2\hat{\jmath} + \hat{k}\right)$
- Cartesian form: $\dfrac{x-7}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1} = t.$

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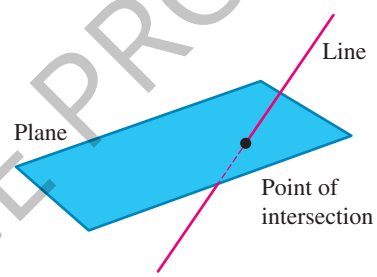
3.4.4 The intersection of lines and planes

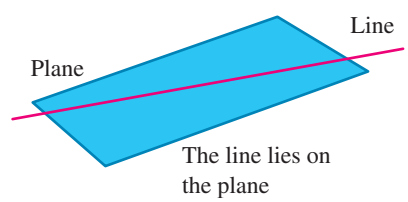
When considering the intersection of a plane and line in three-dimensions, there are three possible solution types.

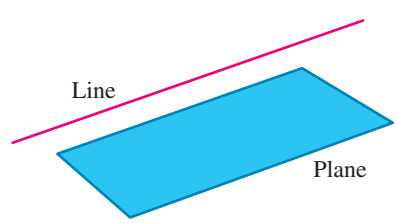
- **One (unique) solution** where the plane and line intersect at a single point $P(x, y, z)$.
- **Infinitely many solutions** in which the line lies on the plane.
- **No solution** as the line and plane are parallel yet distinct.

To examine these solutions geometrically, a vector approach will applied in the first instance. The plane and line are defined as follows.

- **Line** (l_1): $\underset{\sim}{r} = \underset{\sim}{a} + k\underset{\sim}{d}$ where $\underset{\sim}{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ is the position vector of a known point on the line and $\underset{\sim}{d} = d_1\hat{\boldsymbol{\imath}} + d_2\hat{\boldsymbol{\jmath}} + d_3\hat{\boldsymbol{k}}$ is the direction vector *parallel* to the line.
- **Plane** (p_1): $ax + by + cz + d = 0$, where $\underset{\sim}{n} = a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}$ is *perpendicular* to the plane.

Consistent system – one unique solution		
Examine the direction and normal vectors.	$\underset{\sim}{d} \cdot \underset{\sim}{n} \neq 0$. Therefore, the line and plane are *not parallel*.	
Geometric interpretation	The line and plane intersect at a unique point, $P(x, y, z)$ so *one solution* exists.	

Consistent system – infinitely many solutions		
Examine the direction and normal vectors.	$\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$ Therefore, the line and plane are *parallel*.	
Infinitely many or no solutions?	Find a point $P(x, y, z)$ that lies on the line l_1. This point *will satisfy* the equation of the plane p_1.	
Geometric interpretation	The line lies on the plane and *infinitely many solutions* exist.	

Inconsistent system – zero solutions		
Examine the direction and normal vectors.	$\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$ Therefore, the line and plane are *parallel*.	
Infinitely many or no solutions?	Find a point $P(x, y, z)$ that lies on the line l_1. This point *will not satisfy* the equation of the plane p_1.	
Geometric interpretation	The line is parallel yet distinct to the plane. *No solution* exists.	

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WORKED EXAMPLE 16

Consider a three-dimensional system containing the line l_1: $\underset{\sim}{r} = \left(2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}\right) + k\left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ and plane given by p_1: $2x + 7y - 4z = 61$.

a. Examine the line and plane to identify if they intersect.

b. If they do intersect, determine the solution, otherwise state the system has zero solutions.

THINK	WRITE
a. 1. Examine the normal $\underset{\sim}{n}$ and direction $\underset{\sim}{d}$ vectors. Determine if they are perpendicular by calculating the dot product.	a. $\underset{\sim}{n} = 2\hat{\imath} + 7\hat{\jmath} - 4\hat{k}, \underset{\sim}{d} = -\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ $\underset{\sim}{n} \cdot \underset{\sim}{d} = \left(2\hat{\imath} + 7\hat{\jmath} - 4\hat{k}\right) \cdot \left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ $= 2 \times -1 + 7 \times 2 + -4 \times -3$ $= 24$ $\underset{\sim}{n} \cdot \underset{\sim}{d} \neq 0$
2. State if the line and plane intersect.	As $\underset{\sim}{n} \cdot \underset{\sim}{d} \neq 0$ the normal and direction vectors are not perpendicular. The plane and line are not parallel hence a single point of intersection must exist.
b. 1. Convert the vector equation of the line into parametric equations.	b. l_1: $\underset{\sim}{r} = \left(2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}\right) + k\left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ $l_1: \begin{cases} x = 2 - k \\ y = 3 + 2k \\ z = -3 - 3k \end{cases}$
2 Substitute the parametric equations into the equation of the plane.	$2(2 - k) + 7(3 + 2k) - 4(-3 - 3k) = 61$
3 Solve for k.	$2(2 - k) + 7(3 + 2k) - 4(-3 - 3k) = 61$ $4 - 2k + 21 + 14k + 12 + 12k = 61$ $24k + 37 = 61$ $k = \dfrac{61 - 37}{24}$ $k = 1$
4 Determine the point of intersection by substituting $k = 1$ for the parametric equations.	$l_1, k = 1$: $\begin{cases} x = 2 - 1 = 1 \\ y = 3 + 2 \times 1 = 5 \\ z = -3 - 3 \times 1 = -6 \end{cases}$
5 State the solution.	The line and plane intersect at a single (unique) point $(1, 5, -6)$

When determining the intersection of a line and plane the vector approach is not the only method. In Worked example 16, the equation of the line was converted to parametric equations and subsequently substituted into the equation of the plane. A single value for k was found, indicating that a unique solution existed. The following worked example will examine systems containing infinitely many or no solutions.

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WORKED EXAMPLE 17

For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.

a. $l_1: \begin{cases} x = 2 + 4k \\ y = -3 + 2k, p_1: 3x + y + 7z = 24 \\ z = 3 - 2k \end{cases}$

b. $l_2: \begin{cases} x = 3 + 5k \\ y = -2 + 3k, p_2: -2x + 4y + 2z = 1 \\ z = 3 - k \end{cases}$

THINK	WRITE
a. 1. Substitute the parametric equations into the equation of the plane.	a. $3(2 + 4k) + 1(-3 + 2k) + 7(3 - 2k) = 18$
2. Solve for k.	$6 + 12k - 3 + 2k + 21 - 14k = 24$ $24 + 0k = 24$ $0k = 0$
3. Interpret the solution.	For any value of k the equation is true. The line lies on the plane meaning there are infinitely many solutions.
b. 1. Substitute the parametric equations into the equation of the plane.	b. $-2(3 + 5k) + 4(-2 + 3k) + 2(3 - k) = 1$
2. Solve for k.	$-6 - 10k - 8 + 12k + 6 - 2k = 1$ $0k - 8 = 1$ $0k = 9$
3. Interpret the solution.	There are no possible values for k that make this equation true. The line is parallel yet distinct from the plane meaning there are zero solutions.

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Units 3 & 4 > Area 2 > Sequence 2 > Concept 3

The three cases for solutions of systems of linear equations Summary screen and practice questions

Exercise 3.4 The three cases for solutions of systems of linear equations

Technology free

1. MC A unique solution exists in a square system of two dimensions if the lines exhibit which of the following geometric features?

   A. Parallel and distinct
   B. Coincidental
   C. Not parallel
   D. Homogenous

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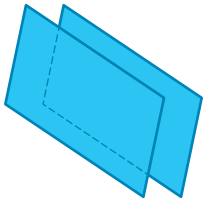
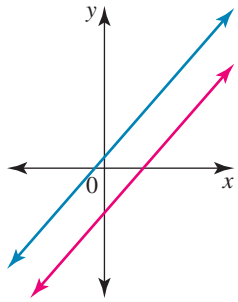
2. Refer to the following graphs and state how many solutions exist.

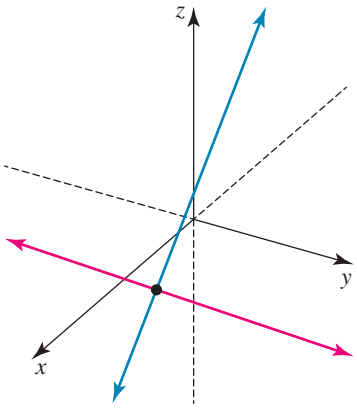
a.



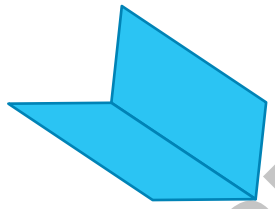
b.



c.



d.



3. WE9 Solve the following system of linear equations by using matrices and interpret the solution geometrically.

a. $8x + 2y = 12$
$4x - 7y = -10$

b. $-2x + y = -2$
$6x - 3y = 7$

c. $-3x + y = -14$
$-2x - 5y = 2$

d. $4x - 8y = 4$
$6x - 12y = -3$

4. WE10

a. Solve the following systems of linear equations by using matrices and interpret the solution geometrically.

b. Describe the solution set in parametric form.

i. $10x - 6y = 12$
$5x - 3y = 6$

ii. $x - 2y = -4$
$-4x + 8y = 16$

5. The following systems contain two lines in the form, $l_1: a_1x + b_1y = c_1$ and $l_2: a_2x + b_2y = c_2$. Determine the nature of the solution set by interpreting the values for a_n, b_n and c_n algebraically.

a. $x + 2y = 15$
$3x - y = 10$

b. $3x + 7y = 9$
$6x + 14y = 18$

c. $-16x + 6y = -14$
$-4x + \frac{3}{2}y = -4$

d. $-x + 4y = -3$
$-5x + 20y = -12$

6. In each case below, the number of solutions has been defined for the system. Determine the restrictions on the values for a, b and c that would make the statement on the solution type true.

a. $\left.\begin{array}{r} 5x - 2y = a \\ 10x - 4y = 24 \end{array}\right\}$ No solution

b. $\left.\begin{array}{r} 8x - 2y = -12 \\ -4x + by = c \end{array}\right\}$ Infinitely many solutions

7. WE11

a. Determine the unique solution for the following systems of linear equations defined by the lines l_1 and l_2.

b. Describe the geometric nature of the solution.

i. l_1: $(\hat{\imath} - \hat{\jmath} - 4\hat{k}) + k(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$
l_2: $(-15\hat{\imath} + 7\hat{\jmath} - 40\hat{k}) + k(-\hat{\imath} - 6\hat{\jmath} + 4\hat{k})$

ii. l_1: $(-1\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) + k(-\hat{\imath} - 4\hat{\jmath} - 4\hat{k})$
l_2: $(-9\hat{\imath} - 5\hat{\jmath} - 9\hat{k}) + k(7\hat{\imath} + 3\hat{\jmath} + 7\hat{k})$

iii. l_1: $(4\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) + k(6\hat{\imath} + \hat{\jmath} + 5\hat{k})$
l_2: $(60\hat{\imath} - 14\hat{\jmath} + 3\hat{k}) + k(-2\hat{\imath} + 4\hat{\jmath} + 5\hat{k})$

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8. a. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2.
   b. Describe the geometric nature of the solution.
   i. $l_1: \begin{cases} x = 3 - 3k \\ y = 3 - 4k \\ z = 4 + 4k \end{cases}$ $l_2: \begin{cases} x = -33 + k \\ y = -15 - 2k \\ z = -14 + 6k \end{cases}$
   ii. $l_1: k = \dfrac{x+1}{4} = \dfrac{y-5}{-3} = \dfrac{z+8}{4}$ $l_2: \begin{cases} x = 53 - 5k \\ y = -25 + 2k \\ z = -14 + 5k \end{cases}$

9. WE12 A system of linear equations in three variables contains the lines l_1 and l_2.
   a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.
   b. Describe the geometric nature of the system by referring to any possible solution/s.
   i. $l_1: \dfrac{x+4}{-6} = \dfrac{y-5}{1} = \dfrac{z+4}{8} = k_1$, $l_2: \dfrac{x+1}{-12} = \dfrac{y-2}{2} = \dfrac{z+7}{16} = k_2$
   ii. $l_1: \dfrac{x-4}{-5} = \dfrac{y+4}{-5} = \dfrac{z-1}{6} = k_1$, $l_2: \dfrac{x+18}{-2} = \dfrac{y+17}{7} = \dfrac{z-27}{2} = k_2$
   iii. $l_1: \dfrac{x-6}{-8} = \dfrac{y-3}{5} = \dfrac{z-1}{-6} = k_1$, $l_2: \dfrac{x+34}{4} = \dfrac{y-62}{6} = \dfrac{z+13}{7} = k_2$

10. A system of linear equations in three variables contains the lines l_1 and l_2.
   a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.
   b. Describe the geometric nature of the system by referring to any possible solution/s.
   i. $l_1: \dfrac{x+6}{-6} = \dfrac{y+4}{-4} = \dfrac{z+7}{-2} = k_1$
   $l_2: x = -1 - 30k_2, y = -7 - 20k_2, z = -10 - 10k_2$
   ii. $l_1: \begin{cases} x = 2 + k_1 \\ y = 6 - 3k_1 \\ z = 2 + 5k_1 \end{cases}$ $l_2: \begin{cases} x = 1 - 4k_2 \\ y = -9 + 3k_2 \\ z = 39 + k_2 \end{cases}$

11. WE13 The following descriptions relate to the geometric and/or algebraic properties of a system of linear equations in three variables. Make a justified assertion on the type of solution set each system would produce based on given information.
   a. The three planes are coincident.
   b. The upper triangular augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$.
   c. The three planes are not parallel, $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ and the final row of the triangular matrix is a false statement. The three planes meet along a line defined by the vector equation, $\underset{\sim}{r} = (4\hat{\imath} - 6\hat{\jmath} - 11\hat{k}) + k(-3\hat{\imath} + 13\hat{\jmath} + 2\hat{k})$.

Technology active

12. WE14 Consider the following system of linear equations to answer the following.

a. $5x + 8y + z = -5$
$-7x - 4y - 5z = -11$
$-3x - y - 6z = 4$

b. $-7x + 7y + 42z = 9$
$5x - 2y + 8z = -11$
$7x - 4y - 4z = 2$

c. $3x + 6y - 2z = -25$
$8x - 4y + 6z = -58$
$-5x - 5y - 3z = 43$

d. $-6x - 6y - z = -64$
$x - y + 2z = 4$
$x + 7y - 7z = 24$

e. $6x + 3y - 24z = 8$
$-3x - y + 6z = -2$
$4x + 8z = 10$

f. $-4x - 5y - 4z = 5$
$4x + 5y + 4z = -7$
$7x - 6y + 7z = 4$

   i. Determine the number of possible solutions by examining the normal vectors algebraically.
   ii. Convert the equations to an augmented matrix and then use technology to solve the system.
   iii. Describe how the solution can be interpreted geometrically.

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13. In each of the following cases, determine the equation of the line formed by the intersection between the two planes. Give all possible answers in parametric form.

a. $p_1: 8x - 2y + 3z = 1$
$p_2: -8x + 6y + z = -5$

b. $p_1: -3x - 2y + z = 1$
$p_2: 7x + y + 5z = 6$

c. $p_1: \frac{2}{3}x - \frac{7}{6}y + \frac{1}{2}z = 2$
$p_2: -4x + 7y + -3z = -11$

14. Investigate the number of possible solutions by examining the normal vectors algebraically. Describe how the systems could be interpreted geometrically.

a. $2x + 3y - 4z = 8$
$2x + 3y - 4z = -5$
$2x + 3y - 4z = -4$

b. $-6x + 8y + 5z = 11$
$-12x + 16y + 10z = -5$
$24x - 32y - 20z = 0$

c. $4x + 7y - 4z = 5$
$16x + 28y - 16z = -3$
$7x - z = -1$

d. $-7x + 4y - z = 5$
$-42x + 24y - 6z = -1$
$-3x - 2y + 5z = 7$

15. The equations below are of three coincidental planes. Determine the real values for d_2 and a_3.

$$p_1: 8x + 12y - 4z = 16$$
$$p_2: 4x + 6y - 2z = d_2$$
$$p_3: a_3x + 18y - 6z = 24$$

16. a. Determine the line of intersection between the following planes. Express the solution as a vector equation.

$$p_1: -4x - 2y + 5z = -18 \quad p_2: 5x + 3y - \frac{11}{2}z = 26$$

b. Calculate $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$. How does this relate to the solution obtained in **a**.?

17. **WE15** Consider the following systems of linear equations to answer the following.

a. $3x + 2y - 16z = 22$
$-4x - 4y + 20z = -28$
$x + 3y - 3z = 5$

b. $6x + 5y - 4z = 42$
$-3x + 4y + 28z = 135$
$x - 4y - 20z = -109$

c. $-3x + y + \frac{11}{2}z = 40$
$5x + 4y + 5z = 58$
$3x + 5y + \frac{19}{2}z = 92$

d. $4x + 2y - 18z = 58$
$8x + 8y - 32z = 120$
$x + y - 4z = 15$

i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
iii. Solve the system of linear equations, expressing the solution in parametric form.
iv. Describe how the solution can be interpreted geometrically.

18. Determine if the solution set is valid for the following systems.

a. $-5x + 7y + 44z = 79$
$9x + 8y - 38z = 2$
$4x - 2y - 28z = -38$
Solution: $\begin{cases} x = -6 + 6t \\ y = 7 - 2t \\ z = t \end{cases}$

b. $-7x - 3y + 22z = 136$
$2x + 2y - 4z = -24$
$2x - y - 10z = -63$
Solution: $\begin{cases} x = -25 + 4t \\ y = 13 - 2t \\ z = t \end{cases}$

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19. **WE16**

a. Consider a three-dimensional system containing the line $l_1: \underset{\sim}{r} = \left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right) + k\left(4\hat{\imath} + 6\hat{\jmath} - 2\hat{k}\right)$ and plane given by $p_1: 5x - 2y + 4z = 8$.

i. Examine the line and plane to identify if they intersect.

ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

b. Consider a three-dimensional system containing the line $l_1: \underset{\sim}{r} = \left(2\hat{\imath} + 7\hat{\jmath} - 9\hat{k}\right) + k\left(3\hat{\imath} + \hat{\jmath} - \hat{k}\right)$ and plane given by $p_1: 2x - 3y + 3z = 4$.

i. Examine the line and plane to identify if they intersect.

ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

c. Consider a three-dimensional system containing the line $l_1: \underset{\sim}{r} = \left(\hat{\imath} - 2\hat{\jmath} + 5\hat{k}\right) + k\left(2\hat{\imath} + 4\hat{\jmath} - 6\hat{k}\right)$ and plane given by $p_1: -3x - 2y + z + 34 = 0$.

i. Examine the line and plane to identify if they intersect.

ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

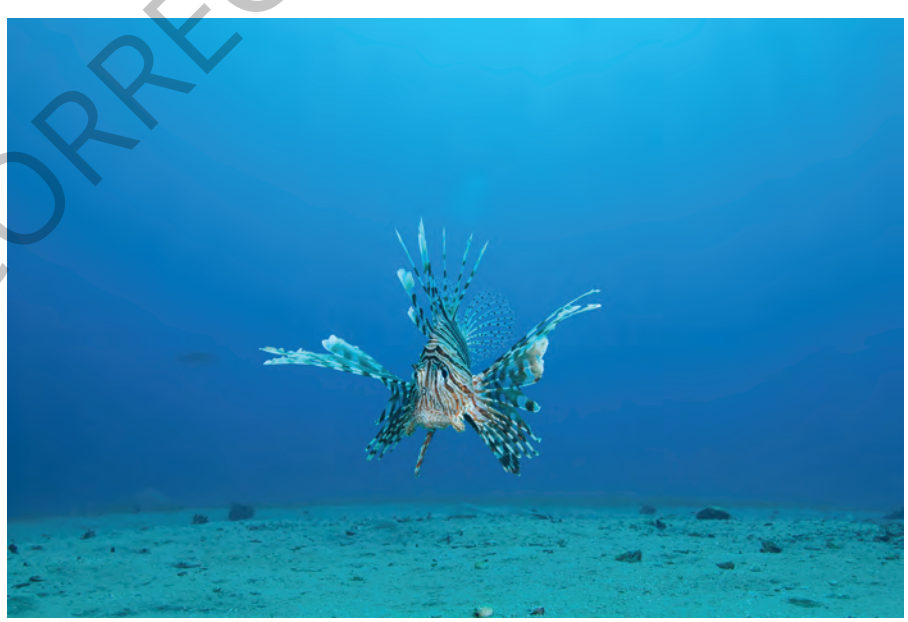
20. **WE17** For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.

a. $l_1: \begin{cases} x = 2 + 4k \\ y = -1 - 2k \\ z = 7 - 3k \end{cases}, p_1: 5x - 3y + 2z - 47 = 0$

b. $l_2: \begin{cases} x = 5 - 2k \\ y = 4 - 9k \\ z = 1 - 3k \end{cases}, p_2: 3x - 2y + 4z = 37$

c. $l_1: \begin{cases} x = 1 + 3k \\ y = -3 + 2k \\ z = -2 - k \end{cases}, p_1: 5x - 7y + z - 24 = 0$

21. An object is travelling a straight-line path according to the vector equation, $l_2: \left(-2\hat{\imath} - 3\hat{\jmath} - 5\hat{k}\right) + k\left(-6\hat{\imath} - 4\hat{\jmath} - \hat{k}\right)$. The sea floor is a flat plane modelled by the equation, $24x - 16y - 31z = -16$. Determine if the object will ever intersect the sea floor and calculate a solution if possible.



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3.5 Using technology for matrix calculations

3.5.1 Matrix operations using a graphic calculator

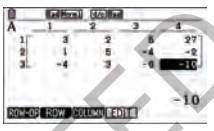
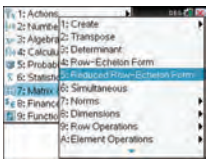
Until this point, systems of linear equations, and associated operations, have been completed algebraically. Augmented matrices can be reduced to row-echelon form by either:

- Completing elementary row operations algebraically.
- Using technology (for example: graphics calculator, MS Excel, Geogebra, Wolfram, etc.)

In Worked example 14, the unknown matrix X, was calculated through technology, where,

$$[A|B] = \left[\begin{array}{rrr|r} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array}\right]$$

The solution can be calculated using either a TI or Casio graphics calculator as shown below.

	Casio	**TI**
Defining a matrix.		
If the matrix is invertible, a reduced row-echelon matrix is produced using the *rref* command.	$\left[\begin{array}{rrrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$	
The matrix in reduced row-echelon form.	$\left[\begin{array}{rrrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$	
Unique solution	$(x, y, z) = (-2, 4, 5)$	$(x, y, z) = (-2, 4, 5)$

Most of the basic matrix operations can be completed using technology. It is important that you are competent in this area when answering technology active questions. The aim of the worked examples that follow is to build your proficiencies in the use of technology. Begin each matrix investigation by first defining the associated matrices.

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WORKED EXAMPLE 18

Refer to the matrices below and use technology to calculate the following matrix operations.

$$A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$$

a. $2A - B$ **b.** $\det A$

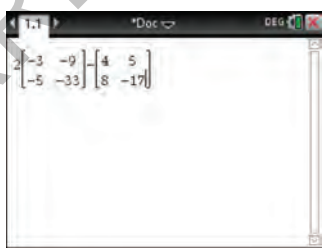
TI | THINK **WRITE**

a. 1. On a Calculator page, select
2

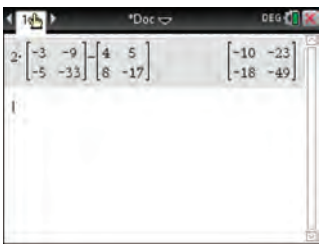
2. Complete the entry line as
−3
−9
−5
−33

3. Complete the entry line as
−
Then select
2

4. Press the button, followed by the button.
Complete the entry line as
4
5
8
−17

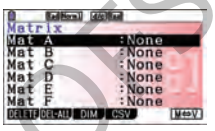


5. Press the ENTER button.
The answer appears on the screen.



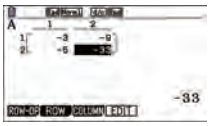
CASIO | THINK **WRITE**

a. 1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.

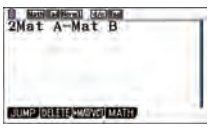


2. Press the EXE button.
Complete the entry line as
2
2
Press the EXE button.

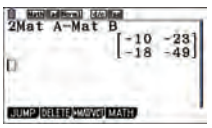
3. Complete the entry line as
−3
−9
−5
−33
Press the EXIT button when finished.
Repeat steps 1-3 to define Matrix B.



4. On a Run-Matrix screen, select
2
SHIFT+2 (Mat)
ALPHA
$X.\theta$, T
−
SHIFT+2 (Mat)
ALPHA
Log
Press the EXE button.



5. The answer appears on the screen.



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b. 1. On a Calculator page, select
MENU
7: Matrix & Vector
3: Determinant
2

2. Complete the entry line as
−3
−9
−5
−33
Press the ENTER button.
The answer appears on the screen.

b. 1. On a Run-Matrix screen, select
OPTN
MAT/VCT
DET
SHIFT+2 (Mat)
ALPHA
$X.\theta, T$

2. Press the EXE button.
The answer appears on the screen.

3.5.2 Powers of matrices

A logical extension of matrix multiplication is to calculate the power of a matrix, where

$$A^1 = A$$
$$A^2 = A \times A$$
$$A^3 = A^2 \times A$$

In general form, $A^n = A^{n-1}A$, where n is a positive integer. But what dimensions can matrix A have?

Matrix powers – an investigation

We can investigate powers of matrices by completing the following steps. Remember to use pronumerals for the elements of A, not constant values.

1. a. Let matrix A be any 3×2 matrix.
   b. Find A^2.
   c. What do you notice?
2. a. Let matrix A be any 2×2 matrix.
   b. Find A^2.
3. a. Let matrix A be any 2×3 matrix.
   b. Find A^2.
   c. What do you notice?
4. a. What general conclusion can you make concerning the order of a matrix that is to be raised to a power?
   b. Justify your conclusion by referring to the dimensions of matrices involved in a product.

From the above investigations we can conclude that $A \times A$ must be conformable; that is, the number of columns of the first factor in the product should be the same as the number of rows in the second factor.

That is, A must be a *square* matrix where $n \times n$ is multiplied by $n \times n$ to get A^2.

Hence powers of matrices are only defined for square matrices.

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WORKED EXAMPLE 19

If $A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$ use technology to solve:

a. A^4 **b.** $(A \times B)^2$

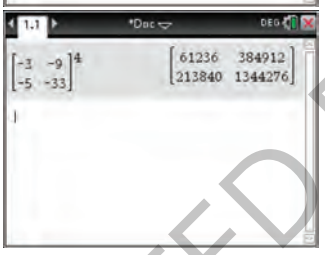
TI | THINK **WRITE**

a. 1. On a Calculator page, select

2. Complete the entry line as

$\begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}^4$

3. Press the ENTER button.
The answer appears on the screen.



b. 1. On a Calculator page, select

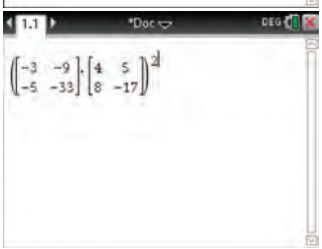
(

)
^2

2. Complete the entry line as

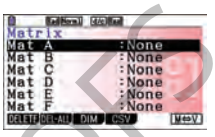
$\begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix} -$
$\begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$

Press the EXIT button twice when finished.

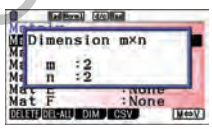


CASIO | THINK **WRITE**

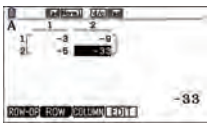
a. 1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.



2. Press the EXE button.
Complete the entry line as
2
2
Press the EXE button.

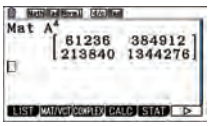


3. Complete the entry line as
$-3 \quad -9$
$-5 \quad -33$
Press the EXIT button twice when finished.

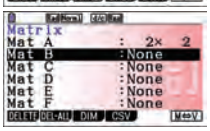


4. On a Run-Matrix screen, select
SHIFT
2
ALPHA
$X.\theta$, T
^
4

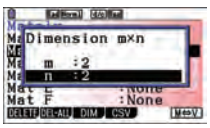
5. Press the EXE button.
The answer appears on the screen.



b. 1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.



2. Highlight Mat B and press the EXE button.
Complete the entry line as
2
2
Press the EXE button.



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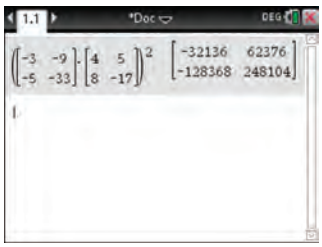
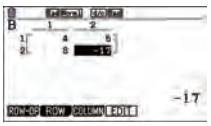
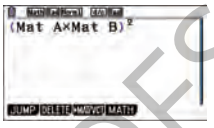
3. Press the EXE button. The answer appears on the screen.



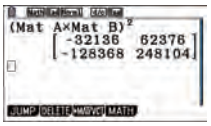
3. Complete the entry line as
5 5
8 −17
Press the EXIT button twice when finished.



4. On a Run-Matrix screen, select
(
SHIFT
2
ALPHA
$X.\theta$, T
×
SHIFT
2
ALPHA
log
)
^
2



5. Press the EXE button. The answer appears on the screen.



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Units 3 & 4 > Area 2 > Sequence 2 > Concept 4

Using technology for matrix calculations Summary screen and practice questions

Exercise 3.5 Using technology for matrix calculations

Technology active

1. **WE18** Refer to the matrices below to calculate the following matrix operations.

$$A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}, C = \begin{bmatrix} 9 \\ -9 \end{bmatrix}, D = \begin{bmatrix} 2 & -6 & 4 \\ 1 & -4 & 10 \\ -2 & 6 & -6 \end{bmatrix}, E = \begin{bmatrix} -42 \\ -49 \\ 48 \end{bmatrix}$$

a. $11A - 8B$
b. $\frac{1}{5}(AB)$
c. $\det(B)$
d. $|D|$
e. A^{-1}
f. D^{-1}
g. $X = D^{-1}E$
h. $X = B^{-1}C$

2. Consider $A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 5 & -4 \\ 3 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 & 4 \\ 6 & -1 & 1 \\ 0 & -3 & 5 \end{bmatrix}$, when examining if the following statements are true or false.

a. $A^3 \times B^2 = B^2 \times A^3$
b. $A^3B^3 = (AB)^3$

3. **WE19** If $A = \begin{bmatrix} 6 & -2 \\ 1 & 7 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} -10 \\ -2 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 3 \\ 5 & -1 & 2 \end{bmatrix}, E = \begin{bmatrix} 10 \\ 9 \\ 24 \end{bmatrix}$,

find:

a. A^3
b. $A^2 \times \left(B^{-1}\right)^3$

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c. $2D^3$

d. Determine X if,

$$\begin{bmatrix} 122 & 121 & 120 \\ 119 & 118 & 117 \\ 116 & 115 & 114 \end{bmatrix} = D^2 - 5X$$

4. Determine if the following systems are either consistent or inconsistent by completing the 'ref' or 'rref' operation on a graphics calculator.

a. $A|B = \left[\begin{array}{ccc|c} 10 & -8 & -2 & 3 \\ 14 & -19 & -5 & -12 \\ 15 & -12 & -3 & 7 \end{array}\right]$

b. $A|B = \left[\begin{array}{cc|c} -8 & -7 & 12 \\ 16 & 14 & -20 \end{array}\right]$

c. $A|B = \left[\begin{array}{cccc|c} 3 & 4 & 7 & -8 & -96 \\ 2 & 6 & -1 & 0 & 26 \\ -4 & 7 & 2 & 6 & 72 \\ 4 & 3 & -2 & -1 & 8 \end{array}\right]$

d. $A|B = \left[\begin{array}{ccc|c} 5 & -3 & 1 & 19 \\ 6 & -9 & 8 & 23 \\ 12 & -3 & -4 & 41 \end{array}\right]$

5. One week's production at a café, in matrix form is given by,

$$A = \begin{array}{c} \begin{array}{ccc} \text{Coffee} & \text{Tea} & \text{Juice} \end{array} \\ \begin{bmatrix} 78 & 34 & 14 \\ 17 & 11 & 10 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Units of material} \\ \text{Units of labour} \end{array}$$

Assume that in this week's production, the cost for each of material is \$0.35 and the cost of each unit of labour is \$0.58. Calculate the total cost to manufacture coffee, tea and juice.

6. Consider the two matrices, $A = \begin{bmatrix} -8 & 2 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -6 \\ 4 & -9 \end{bmatrix}$, to then calculate the following.

a. $(A - B)(A + B)$ **b.** $(A + B)^2$

7. A nilpotent matrix exists if $A^2 = 0$. Nilpotent matrices are singular. Determine if the following matrices are nilpotent.

a. $A = \begin{bmatrix} 8 & -32 \\ 2 & -8 \end{bmatrix}$ **b.** $\begin{bmatrix} 9 & -81 \\ 1 & -9 \end{bmatrix}$

8. An idempotent matrix exists if $A^2 = A$. Idempotent matrices are singular. Determine if the following matrices are idempotent.

a. $A = \begin{bmatrix} 6 & -3 \\ 10 & -5 \end{bmatrix}$ **b.** $B = \begin{bmatrix} -6 & 6 \\ -7 & 7 \end{bmatrix}$

9. If $A = \begin{bmatrix} -2 & -4 \\ 7 & 0 \end{bmatrix}$, then validate the mathematical statement $A^2 + 2A = -28I$

10. Demonstrate, using numerical methods, that $A^2B = A(AB)$ given, $A = \begin{bmatrix} 5 & -3 & -2 \\ 4 & 0 & -6 \\ 2 & 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 & 1 \\ 9 & 8 & 1 \\ 2 & 9 & 2 \end{bmatrix}$.

11. The sum of three numbers is 25. When double the first number and twice the second number are added to the third number the sum is 43. The last number is one less than the first. Develop a suitable matrix equation to calculate the values for each number.

12. **a.** Develop a suitable matrix equation to determine suitable values for c given,

$$\begin{bmatrix} 5 & 5 & c \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ c \end{bmatrix} = -100$$

b. Validate the solution from **a**.

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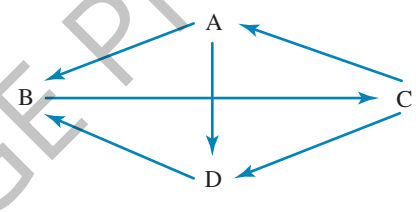
3.6 Dominance and Leslie matrices

3.6.1 Dominance matrices

Have you ever wondered how tennis players are seeded or ranked? It obviously has something to do with their performance against past opponents. In a knock-out competition, one loss and you are out of the competition. Only the winners continue to play. Dominance matrices are often used to determine player rankings.

The following example will explain how matrices are used to establish the seedings or rankings of players in round-robin situations where each player plays every other player, thereby creating a more just system of ranking.

Consider 4 players Anna, Belle, Carla and Diane (A, B, C, D), who on past performances have shown that A defeats D and B, D defeats B, C defeats A and D, and B defeats C.



This situation can be represented on a **digraph** — a network diagram that has arrows on the edges, where A → B indicates that A defeats B.

The information from the digraph can be converted into matrix form (a *dominance matrix*) as below:

$$M = \begin{array}{c|cccc} \text{defeats} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 0 & 0 & 1 & 0 \\ C & 1 & 0 & 0 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

where 1s are used to indicate 'defeats' and 0s to indicate otherwise. Obviously A can't defeat A so a ' 0' is used along the leading diagonal.

Notice also that:

1. there are as many 1s as there are paths
2. corresponding elements occur on either side of the leading diagonal. That is, if A defeats B (1), then a 0 will be stored in the B defeats A element on the opposite side of the leading diagonal.

When the elements of each of the rows are added they yield a **dominance vector**, showing how many players each has defeated.

$$V = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

This result can be readily checked from the original digraph by counting the number of arrows *out of* each node.

Note, from now on the row/column labels will be omitted.

It can be seen from this information that A and C are ranked equally, and B and D are ranked equally; this can be written as

$$\left\{ \begin{matrix} A \\ C \end{matrix} \right\} \left\{ \begin{matrix} B \\ D \end{matrix} \right\}$$

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So we still need to distinguish between A and C, and B and D to establish the ranking.

We assume in most ranking situations that if A defeats B and B defeats C then A will defeat C. This relationship is described as being *transitive*, where if variable $a < b$ and $b < c$, then $a < c$.

In our example, A defeats B who defeats C, and A defeats D who defeats B. This means that A has second-order influence over C and B but not D. (A doesn't defeat anyone who defeats D). The matrix M^2 can be used to investigate second-order influence.

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Notice that the leading diagonal is still 0. It is impossible for a player to have second-order influence over themselves. Row 1 represents the second-order influence of player A over the other players. The element 2 in row 3 occurs because C defeats 2 players (A and D) who defeat B.

We can find the second-order dominance vector, $V_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, but how much importance should it be given?

If we assign equal importance, we calculate the sum of the first-order influence and the second-order influence.

$$M + M^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

This gives a dominance vector $\begin{bmatrix} 4 \\ 3 \\ 5 \\ 2 \end{bmatrix} = V_1 + V_2$ and allows us to rank the competitors in the order C, A, B, D.

If we wanted to investigate third-order influence, we could calculate M^3 $(= M^2M)$.

$$M^2M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ giving } V_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

At this stage, notice that the leading diagonal is no longer 0. If there were more players in the tournament, we could continue finding powers of M, but with 4 players, we stop at M^3. In general, if there are m players, we stop at M^{m-1}.

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In most scenarios, it is probably unfair to assign equal importance to first-, second- and third-order influence. We can allocate arbitrary constants to weight the influence; that is, $M + xM^2 + yM^3$. The resulting dominance vector can be found by calculating $V_1 + xV_2 + yV_3$.

If we choose $x = 0.5$ and $y = 0.25$, the dominance vector would be

$$V_1 + 0.5V_2 + 0.25V_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 0.5\begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} + 0.25\begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.75 \\ 2.75 \\ 4.25 \\ 2 \end{bmatrix}$$

This would rank the four players as C, A, B, D.

When you compare the final seeding with the initial information, we can see that A and C both won 2 games, but the wins by A were against the lower ranked B and D. Players B and D both won 1 game but B managed to defeat the higher placed C. This justifies the seeding as produced.

WORKED EXAMPLE 20

We want to seed 4 badminton players, Oliver, Benjamin, Brooke and Millicent. In past matches, Millicent defeated Oliver and Brooke, both Oliver and Brooke defeated Benjamin, Benjamin defeated Millicent, and Oliver defeated Brooke.

a. Construct a digraph to represent this information.

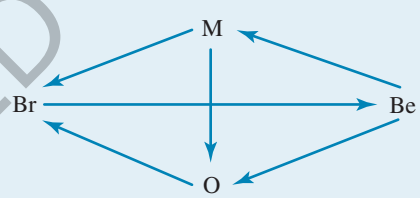
b. By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.

THINK

a. A diagraph represents the matches played in the past. M → O indicates that M defeats O.

WRITE

a.



b. 1. The information from the digraph is translated to a dominance matrix, M. The 1s indicate 'defeats'.

b.

$$M = \begin{array}{c} \begin{array}{cccc} \text{O} & \text{Be} & \text{Br} & \text{M} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Oliver} \\ \text{Benjamin} \\ \text{Brooke} \\ \text{Millicent} \end{array}$$

2. M^2 helps us investigate the second order influence.

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

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3. Dominance vectors are calculated by adding the elements of each row from the matrices. The resulting dominance vector is the sum of V_1 and V_2.

$$V_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 5 \end{bmatrix}$$

4. State the ranked order of players according to the resulting dominance vector, $V_1 + V_2$.

The players would be ranked Millicent, Oliver, Benjamin, Brooke.

3.6.2 Leslie matrices

There are many factors that affect the size of populations of animal species. The first of these is the survival rate; that is, the proportion of animals that survive until the next year (or whatever time period is being used). Clearly, a more accurate picture of the survival rate can be obtained if we use survival rates specific to the different age groups. For example, a saltwater crocodile may produce many offspring in a year but not many of the babies survive their first year. However, once they reach 1 year of age it is much more likely that they will get to 2 years of age. A second factor influencing the population of a species is its fecundity or how many babies are produced per year. Again, fecundity rates that are specific to particular age groups will yield a more accurate analysis.

To see how matrices can be used to model animal populations, consider the following example.

A wildlife officer is interested in the number of a certain species of rat in a habitat and has compiled the following information on this species.

We have a population breakdown by age:

Age (x) (at the end of the year)	Number (N) (of females)
1	55
2	32
3	25

It is assumed that the rats do not live longer than three years and that there are the same number of males and females.

Fecundity rates, F_x, (the number of female offspring per female per year) have been calculated:

Age (x)	Fecundity (F_x)
1	0.6
2	2.4
3	0.5

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The survival rates, S_x, of each age group have also been estimated:

Age (x)	Survival rate (S_x)
1	0.5
2	0.6

Given this information and an initial starting population, it is possible to predict the future population.

Note that the 1-year-old rats that survive the year become 2-year-old rats in the next year.

As each age group of females produce babies, the number of 1-year-old rats in the next year can be calculated as:

$$\begin{aligned}\text{Number of } 1-\text{year}-\text{old rats} = {} & \text{number of } 1-\text{year}-\text{olds} \times F_1 \\ & \text{number of } 2-\text{year}-\text{olds} \times F_2 \\ & \text{number of } 3-\text{year}-\text{olds} \times F_3\end{aligned}$$

The number of 2-year-old rats in the next year $= 0.5 \times$ number of 1-year-olds.

The number of 3-year-old rats in the next year $= 0.6 \times$ number of 2-year-olds.

It is useful to formulate the relationships within a population of rats in any year, and between the populations of rats over several years, using matrices. The population of rats at the end of year 1 can be written:

$$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

A matrix called the **Leslie matrix** can be used to estimate the next year's population given the current population. The Leslie matrix is given by:

$$L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 5 & 0.6 & 0 \end{bmatrix}$$

The estimated rat population at the end of the second year is N_2, where $N_2 = L \times N_1$.

A Leslie matrix has fecundity rates across the top row, survival rates along the sub-diagonal and 0s elsewhere.

$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$ where F_i is the fecundity rate of the i th age group and S_i is the survival rate of the i th age group.

We can also calculate the number of females after any year using the Leslie matrix and the number of females after year 1.

Consider the number of females after years $2, 3, 4, \ldots, r$.

After year 2: $N_2 = L \times N_1$

After year 3: $N_3 = L \times N_2 = L \times L \times N_1 = L^2 \times N_1$

After year 4: $N_4 = L \times N_3 = L \times L^2 \times N_1 = L^3 \times N_1$

After year r: $N_r = L^{r-1} \times N_1$

In general terms, $N_r = L \times N_{r-1}$ and $N_r = L^{r-1} \times N_1$.

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WORKED EXAMPLE 21

Using the Leslie matrix, $L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$, and the female rat population in year 1, $N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$.

a. **State the meaning of the number 2.4 in the matrix.**
b. **Calculate the female rat population at the start of year 2.**
c. **Calculate the total rat population after 6 years, assuming equal numbers of male and female rats.**

THINK	WRITE
a. The first row of the Leslie matrix gives fecundity rates. The second column refers to 2-year-olds.	a. The number 2.4 refers to the number of female rats born on average to a 2-year-old rat.
b. 1. The population in the next year is found by multiplying the Leslie matrix by the population this year.	b. $N_2 = L \times N_1$
2. Perform the matrix multiplications.	$N_2 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix} = \begin{bmatrix} 122.3 \\ 27.5 \\ 19.2 \end{bmatrix}$
3. Write the answer.	The total female population is $122.3 + 27.5 + 19.2 = 169$.
c. 1. Calculate the population after 6 years using the formula $N_r = L^{r-1} \times N_1$.	c. $N_6 = L^5 \times N_1$
2. Perform the matrix multiplications.	$N_6 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}^5 \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix} = \begin{bmatrix} 521.9 \\ 171.9 \\ 73.3 \end{bmatrix}$
3. Add the female populations.	The total female population is $521.9 + 171.9 + 73.3 = 767.1$.
4. Calculate the total rat population assuming that there are equal numbers of males and females.	Number of males and females $= 2 \times 767.1$ $= 1534.2$
5. Answer the question.	The total rat population is 1534 rats.

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studyon
Units 3 & 4 > Area 2 > Sequence 2 > Concept 5
Dominance and Leslie matrices Summary screen and practice questions

Exercise 3.6 Dominance and Leslie matrices

Technology free

1. WE20 The top four chess players in a tournament are Breanna, Kayley, Teagan and Cameron. In past matches, Cameron defeated Breanna and Teagan, both Breanna and Teagan defeated Kayley, Kayley defeated Cameron, and Breanna defeated Teagan.
   a. Construct a digraph to represent this information.
   b. By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.

2. Three friends have noticed that when they played chess, Mair defeated Ann and Janine, and Ann defeated Janine. Use dominance matrices to rank these players.
3. A round-robin netball match was arranged for house competitions where Barnes lost to all but Cunningham, Cunningham lost to Leslie but defeated Hamilton. No teams went undefeated.
   a. If it is decided to give equal importance to first- and second-order influence, use dominance matrices to rank the students' houses.
   b. If house points are allocated as 20, 15, 10, 5 for the overall ranking, how many points did each house receive?
4. Five schools are debating in a round-robin tournament — the following table shows the results.

	Clifton	Warwick	Goondiwindi	Stanthorpe	Ipswich
Clifton	—	L	W	W	L
Warwick		—	W	W	L
Goondiwindi			—	L	W
Stanthorpe				—	W

   Clifton lost to Warwick and Ipswich and so on.
   a. Construct a dominance matrix of this information.
   b. A total of 15 points is divided in the ratio 5:4:3:2:1 and awarded according to the ranking of the schools at the end of the tournament. If this division of points is allocated according to the figures produced by the dominance matrix sum $M + 0.8M^2 + 0.5M^3$, list the number of points each school wins.
5. Suppose that only 15% of baby saltwater crocodiles live to 1 year of age and that 60% of 1-year-olds survive to 2 years of age. In one season, 40 crocodile eggs hatch. How many of these survive to:
   a. 1 year of age?
   b. 2 years of age?

Technology active

6. WE21 The change in the rat population in a certain habitat is described by the following Leslie matrix:

$$L = \begin{bmatrix} 0.7 & 1.4 & 0.7 \\ 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

   The initial female rat population is $N_1 = \begin{bmatrix} 33 \\ 18 \\ 8 \end{bmatrix}$.
   a. State the meaning of the number 1.4 in the matrix.
   b. State the meaning of the number 0.5 in the matrix.

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c. Calculate the female rat population for year 3.
d. Calculate the total rat population after 9 years, assuming equal numbers of male and female rats.

7. The numbers of a particular species of bird have been monitored over three years and the following Leslie matrix has been developed to model the population. The birds are aged 1, 2 or 3 years.

$$L = \begin{bmatrix} 0 & 3.1 & 2.2 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

a. Determine how many new birds do 1-year-old birds produce on average.
b. Determine how many new birds do 2-year-old birds produce on average.
c. Calculate what proportion of 1-year-old birds survive the first year.
d. If the female population in year 1 is $N_1 = \begin{bmatrix} 245 \\ 101 \\ 32 \end{bmatrix}$, calculate the population of female birds in year 3.

8. Consider the following data associated with sheep flocks.

Age (years)	Birth rate	Survival rate	Number (of females) initially
0	0	0.90	450
1	0.50	0.85	2200
2	0.45	0.50	860
3	0.40	0	55

a. Construct a Leslie matrix for the sheep flock.
b. Use the Leslie matrix to predict the sheep population in 5 years time.

9. On a farm, the following data are recorded for the sheep flock in a particular year.

Age	Number of females at start of year	Number of females at the end of the year	Number of females born to this group
0	330	280	0
1	880	870	350
2	150	0	12

Assume that there are no female sheep aged over 2 years.

a. Calculate the survival rate of:
   i. the 0-year-old group
   ii. the 1-year-old group
   iii. the 2-year-old group.
b. Calculate is the birth rate for:
   i. the 0-year-old group
   ii. the 1-year-old group
   iii. the 2-year-old group.
c. Construct a Leslie matrix for the sheep flock.
d. Use the matrix to predict the sheep population in 3 years time.

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10. The Leslie matrix (L) below shows the fecundity rate of a female mouse population while N_1 shows the population in the first year.

$$L = \begin{bmatrix} 0.8 & 2.2 & 0.6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} N_1 = \begin{bmatrix} 65 \\ 28 \\ 31 \end{bmatrix}$$

Calculate the total number of rats in the sixth year. It is assumed that there is equal number of male and female rats.

3.7 Applications of matrices

3.7.1 Markov chains and eigenvectors

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates the probability associated with the state of a system as it makes various transitions. It answers questions such as, 'What is the probability that James will be late to work today given that he was late yesterday?' or 'What is the likelihood that the next note in a piece of music will be an A, given that the previous note was a G'? or, 'What can be said about the long-term behaviour of the piece of music?'.

In Palmdale only two brands of soft drink are sold — Blue or Red. Suppose, at the start of the first month, 50% of the population drank Blue and 50% of the population drank Red. The people of Palmdale do not always stick to the same drink. At the end of the month some of them change to the other drink. Suppose at the end of each month 40% of people who drink Blue change to Red and 70% of those who drink Red change to Blue.

As time goes by the proportions are as follows.

	Start	End of month 1	End of month 2
Blue	50%	$50\% \times 60\% + 70\% \times 50\% = 65\%$	$65\% \times 60\% + 70\% \times 35\% = 63.5\%$
Red	50%	$50\% \times 30\% + 40\% \times 50\% = 35\%$	$35\% \times 30\% + 40\% \times 65\% = 36.5\%$

What is the long-term behaviour of the way in which the market share is distributed? Even the calculation of the behaviour for three months is complicated. The proportion of market share each month is called the **state** of the system. The sequence of states from month to month forms a **Markov chain**.

A **transition matrix** can be used to simplify the calculations. As its name suggests, the matrix assists in the calculation of the transition in drinking habits from one month to the next.

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} \textit{Blue} & \textit{Red} \end{array} \\ \text{To} \begin{array}{c} \textit{Blue} \\ \textit{Red} \end{array} & \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \end{array} \quad \textit{From Red to Blue}$$

$$T = \begin{bmatrix} 9.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$$

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Note that the columns of a transition matrix always sum to 1. At any time, the state is represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where x is the proportion of Blue and y is the proportion of Red. Thus, if the initial state, S_0, is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, then the state after:

1 month, S_1, is: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$

2 months, S_2, is: $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.635 \\ 0.365 \end{bmatrix}$.

Often we are interested in the long-term behaviour of the market or the **steady state**, as it is often called. The steady state will be a value of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}.$$

This means the market share remains the same from one month to the next.

In this case $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives $x = 0.6x + 0.7y$.

Recalling that x and y are proportions that add to 1.

$$x + y = 1$$

Solving the simultaneous equations gives:

$$x = 0.6364$$
$$y = 0.3636.$$

That is, the steady state or long-term behaviour of the market is that *Blue* has 63.64% of the market and *Red* has 36.36%.

MARKOV CHAINS AND LONG TERM BEHAVIOUR

1. **A transition matrix, M, is used to calculate the likelihood of certain states given the states at a previous stage.**
2. **The sequence of states, S_i, is called a Markov chain.**
3. **The values of M are arranged as follows:**

$$\begin{array}{cc} & \textbf{From} \\ & \begin{array}{cc} A & B \end{array} \\ \textbf{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

4. **If a system can be in one of r states, the $r \times 1$ column matrix (or vector) p gives the likelihood of each of the states.**
5. **If a system is initially in a state p after n stages the system is in a state given by $M^n p$.**
6. **The sum of columns in a transition matrix is 1.**
7. **The sum of probabilities in the state matrix is 1.**
8. **The long-term behaviour of the system can be found by considering $M^{50}p$, first checking that M^{51} is the same as M^{50}.**

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WORKED EXAMPLE 22

The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & x \end{bmatrix}$.

a. Calculate the value of x.

b. If the state of the system is $\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$**, calculate the state of the system 2 stages later, using technology.**

THINK

a. Form an equation by adding the elements of the third column. The sum of each column must be 1.

b. 1. The state of the system will be $p = M^2 p_0$ where p_0 is the initial state of the system. (Use a graphics calculator to make the calculations, see instructions.)

2. Write the answer.

WRITE

a. $0.5 + 0.4 + x = 1$

$x = 0.1$

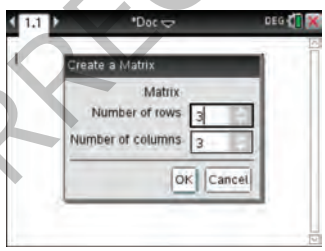
b. $p = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2 \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$

The steady state will be $p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}$.

TI | THINK **WRITE**

1. On a Calculator page, select

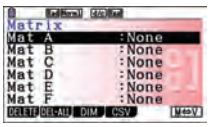
2. Complete the entry line as
3
3
Press the OK button.



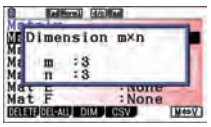
3. Complete the entry line as
$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2$

CASIO | THINK **WRITE**

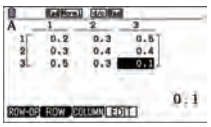
1. On a Run-Matrix screen, select
MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.



2. Press the EXE button.
Complete the entry line as
3
3
Press the EXE button.



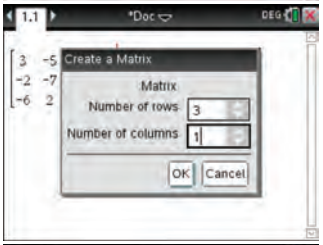
3. Complete the entry line as
0.2 0.3 0.5
0.3 0.4 0.4
0.5 0.3 0.1
Press the EXIT button when finished.



UNCORRECTED PAGE PROOFS

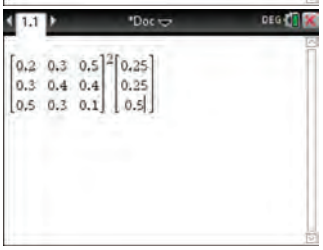
4. Press the button, followed by the button.

5. Complete the entry line as
3
1
Press the OK button.

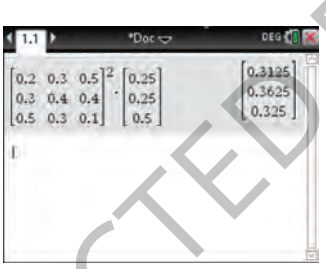


6. Complete the entry line as

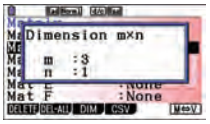
$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$



7. Press the ENTER button. The answer appears on the screen.



4. Highlight the Mat B row and press the EXE button. Complete the entry line as
3
1
Press the EXE button.

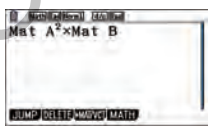


5. Complete the entry line as

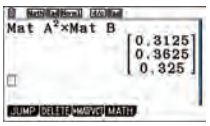
$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$

Press the EXIT button twice when finished.

6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
ALPHA
$X.\theta$, T
^
2
×
Mat
ALPHA
Log



7. Press the EXE button. The answer appears on the screen. example.



WORKED EXAMPLE 23

The Nee Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 60%. If it is not raining on a particular day, the chance that it will rain on the following day is 50%.

a. Construct a transition matrix to represent this situation.

b. Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.

c. In the long term, determine what proportion of days it rains in the Nee Islands.

THINK

a. 1. Label the columns and rows with the possible states and enter the appropriate values for the transition matrix.

WRITE

a.

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} \text{Rain} & \text{Dry} \end{array} \\ \text{To} \begin{array}{c} \text{Rain} \\ \text{Dry} \end{array} & \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} \end{array}$$

$$\text{So } M = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}.$$

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b. 1. Define the initial state; that is, define the matrix for the state on Tuesday.

2. Find the state on Friday by calculating $M^3\begin{bmatrix}1\\0\end{bmatrix}$, where M is the transition matrix. The matrix M is raised to the power of 3 as Friday is three days after Tuesday.

3. Answer the question.

c. 1. Find the long-term term behaviour by considering a large (say $n = 50$) power of the transition matrix, M.

2. Check that M^{51} is the same. They are the same (to 4 decimal places), which indicates that a steady-state solution is found.

3. Choose a starting value, say $\begin{bmatrix}1\\0\end{bmatrix}$, and calculate the long-term proportions. Note: Any starting value could have been used.

4. Answer the question.

b. The state on Tuesday (rain) is $\begin{bmatrix}1\\0\end{bmatrix}$.

The state on Friday, $\begin{bmatrix}x\\y\end{bmatrix}$, is given by:

$$\begin{aligned}\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}\\ &= \begin{bmatrix}0.556 & 0.555\\0.444 & 0.445\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &= \begin{bmatrix}0.556\\0.444\end{bmatrix}.\end{aligned}$$

The likelihood of rain on Friday is 0.556.

c. Consider $M^{50} = \begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}^{50}$

$$= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}$$

and $M^{51} = \begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}^{51}$

$$= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}$$

If the starting value is $\begin{bmatrix}1\\0\end{bmatrix}$, the long-term behaviour is given by $M^{50}\begin{bmatrix}1\\0\end{bmatrix}$

$$\begin{aligned}&= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &= \begin{bmatrix}0.5556\\0.4444\end{bmatrix}.\end{aligned}$$

So, the long-term probability of rain is 0.5556 and dry is 0.4444.

3.7.2 Eigenvalues

For any $n \times n$ matrix A, if there is a $n \times 1$ column matrix, X, such that $AX = \lambda X$, then X is called an **eigenvector** of A and λ is called an **eigenvalue** of the matrix, A. That is, for matrix A, the eigenvalue λ acts as a scalar on the eigenvector X. It is important to remember that eigenvalues are numbers whereas eigenvectors are vectors.

In the vector equation $AX = \lambda X$, both λ and X are unknowns. To solve this, we need to rewrite the equation as a linear system by expressing λX as λIX.

$$AX = \lambda X \qquad [1]$$

$$AX - \lambda X = 0$$

$AX - \lambda IX = 0$ where I is the identity matrix.

$$(A - \lambda I)X = 0 \qquad [2]$$

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In order for the linear system to have a non-trivial (nonzero) solution, $A - \lambda I$ must be singular. That is, $\det(A - \lambda I) = 0$. It follows from equation [2] that:

$$\det(A - \lambda I) = 0. \qquad [3]$$

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$.

If A is a 2×2 matrix, equation [3] is a quadratic equation in λ.

$$\begin{aligned}\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= ad - a\lambda - bc - d\lambda + \lambda^2 \\ &= \lambda^2 - \lambda(a+d) + ad - bc\end{aligned}$$

If A is a 3×3 matrix then equation [3] is an equation in λ of degree 3.

$$\begin{aligned}\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) &= \det\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix} \\ &= -\lambda^3 + \lambda^2(a+e+i) + \lambda(fh - ei + cg + bd - ea - ai) \\ &\quad + cdh - ecg + bfg - bdi + eai - afh\end{aligned}$$

In general, any $n \times n$ matrix will have n eigenvalues. Additionally, the sum of the diagonal entries of the square matrix is equal to the sum of the eigenvalues (check this in Worked example 24).

Eigenvectors of a matrix, A, are important because they represent objects that are in a sense unchanged by a transformation represented by the matrix, A. In physics, economics and biology, eigenvectors are important features in mathematical modelling. In the previous section on Markov chains, the values for a steady state were to be found by solving the equation $\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$. If we substitute $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and use eigenvector notation, we obtain $X = TX$.

Thus, the steady state, X, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

EIGENVECTORS AND EIGENVALUES

For a matrix, A, if there is a column matrix (vector), X, such that:

$$AX = \lambda X$$

then X is called an eigenvector of A and λ is called an eigenvalue of the matrix, A.
The steady state, $X = TX$, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

WORKED EXAMPLE 24

If $A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$, determine:

a. **eigenvalues associated with the matrix**
b. **eigenvectors associated with the matrix.**

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THINK	WRITE
a. 1. To find the eigenvalues, λ, substitute matrix values into the expression $A - \lambda I$.	**a.** $A - \lambda I = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $= \begin{bmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{bmatrix}$
2. Substitute for $A - \lambda I$ in $\det(A - \lambda I) = 0$.	So $\begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} = 0$
3. Find the determinant and solve for λ.	$(1-\lambda)(2-\lambda) - 12 = 0$ $\lambda^2 - 3\lambda - 10 = 0$ $(\lambda - 5)(\lambda + 2) = 0$ Using the Null Factor Law $\lambda = 5$ and $\lambda = -2$
4. Write the eigenvalues.	Thus, the eigenvalues are 5 and -2.
b. 1. Recall the relationship linking the eigenvectors with the eigenvalues.	**b.** $AX = \lambda X$
2. Substitute $\lambda = 5$ and $\lambda = -2$.	$AX = 5X$ or $AX = -2X$
3. Use $\lambda = 5$ first. Substitute values for matrix A and assign unknowns to matrix X.	Let $X = \begin{bmatrix} a \\ b \end{bmatrix}$ then with $\lambda = 5$, $AX = 5X$.
4. Substitute for A and X. Consider the first row of the product. (You will obtain the same result if the second row is used.)	$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 5\begin{bmatrix} a \\ b \end{bmatrix}$ $a + 6b = 5a$ $6b = 4a$ $b = \frac{4a}{6}$ $= \frac{2a}{3}$
5. As you cannot determine independent values for a and b, state the relationship between them.	X is any vector of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$.
6. Repeat this procedure for $\lambda = -2$. Again, substitute for A and X. Consider the first row of the product only.	$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -2\begin{bmatrix} a \\ b \end{bmatrix}$ $a + 6b = -2a$ $6b = -3a$ $b = \frac{-3a}{6}$ $= -\frac{a}{2}$
7. State the eigenvectors for $\lambda = -2$.	X is any vector of the form $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.
8. Answer the question.	The eigenvectors for the given matrix are any vectors of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$ or $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.

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3.7.3 Leontief Matrices

Wassily Leontief was born in St Petersburg in 1906. Although he trained in Russia and Germany his main contribution to economic theory took place while he was a professor of economics at Harvard University in the United States. He received the Nobel Prize for economics in 1973 and died in 1999. Leontief developed input–output analysis, a technique greatly valued by economists involved in planning and development.

Consider an industry or economy that produces energy (E), raw materials (R) and manufactured goods (M). There is an inter-relationship between these three activities. To produce energy you require raw materials and manufactured goods. Likewise, to produce raw materials requires energy and manufactured goods.

The Leontief model looks at an economy such as this and relates input and output. It answers questions such as: 'what inputs (how much of each of E, R and M) are needed to produce a certain output?'.

Matrices are used to record the relationship between the industries.

Suppose the relationship can be quantified and represented by a consumption matrix, C.

$$C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$$

This matrix should be interpreted as follows. The figure of 0.4 in the second row, first column means that every unit of E produced requires 0.4 units of R.

Output and input can be represented using 3×1 matrices, p and q. For example, $p = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ represents an output of 3 units of E, 4 units of R and 7 units of M.

The input, q, required to give an output, p, can be obtained by using the equation:

$$q = Cp.$$

WORKED EXAMPLE 25

Consider the consumption matrix

$$\boldsymbol{C} = \begin{array}{c} \begin{array}{ccc} \boldsymbol{E} & \boldsymbol{R} & \boldsymbol{M} \end{array} \\ \begin{bmatrix} \mathbf{0.2} & \mathbf{0.3} & \mathbf{0.3} \\ \mathbf{0.4} & \mathbf{0.1} & \mathbf{0.3} \\ \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.1} \end{bmatrix} \end{array} \begin{array}{c} \\ \boldsymbol{E} \\ \boldsymbol{R} \\ \boldsymbol{M} \end{array}$$

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a. State the meaning of the value 0.4 in the third row of the matrix.
b. Calculate the inputs needed to produce 1 unit of E, 0 units of M and 0 units of R.
c. Calculate the inputs needed to produce 2 units of E, 2 units of R and 3 units of M.

THINK

a. Consider the row and column for element **0.4**. The row is linked to M and the column is linked to R.

b. 1. Write the output as a 3×1 matrix.

2. Recall the equation relating input to output, $q = Cp$.

3. Multiply the consumption matrix given, C, by the output matrix p.

4. State the answer as a sentence.

c. 1. Write the output as a 3×1 matrix.

2. Recall the equation relating input to output, $q = Cp$.

3. Multiply the consumption matrix given, C, by the output matrix p.

4. State the answer as a sentence.

WRITE

a. This entry means it takes 0.4 units of M to make 1 unit of R.

b. Let $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the amount of $E = 1$ with the quantities of R and M being 0.

$q = Cp$

$$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$$

The input q required is 0.2 units of E, 0.4 units of R and 0.3 units of M.

c. $q = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$q = Cp$

$$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$$

The input needed is 1.9 units of E, 1.9 units of R and 1.7 units of M.

In the previous worked example, you will notice that the products of the process $p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ exceeded the inputs $q = \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$. This surplus, $p - q = \begin{bmatrix} 0.1 \\ 0.1 \\ 1\ 3 \end{bmatrix}$, can be used to meet external demand, d.

In a well-balanced economy, it is possible to choose a production, p, such that it matches a given demand, d, and $p - q = d$. Note that, as $q = Cp$, this equation is equivalent to $p - Cp = d$. That is, production less consumption inputs required is equal to demand.

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This gives:

$$\begin{aligned} p - Cp &= d \\ (I - C)p &= d \\ p &= (I - C)^{-1} d \end{aligned}$$

where I is the identity matrix.

You may think that, provided $(I - C)^{-1}$ exists, the equation $p = (I - C)^{-1} d$ ensures that we can always find production levels, p, such that the demand will always be met. This is not always the case, for although the values can be computed in theory, in practice some of the elements of p may turn out to be negative. This would not make sense and thus the process would fail for that value of d.

WORKED EXAMPLE 26

Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$

a. Determine the production, p, that will match the demand $d = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.

b. Demonstrate that it is always possible to find values for production that will exactly match a given demand, d.

THINK	WRITE
a. 1. Recall the production equation $p = (I - C)^{-1} d$ to determine production, given demand.	a. $p = (I - C)^{-1} d$.
2. Substitute values for C and d.	$p = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}\right]^{-1} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$
3. Determine p using a graphics calculator.	$p = \begin{bmatrix} 19.687 \\ 17.187 \\ 18.645 \end{bmatrix}$
4 Write the answer.	When the demand is: $E = 5, R = 2$ and $M = 4$ the production that exactly matches it is: $E = 19.69, R = 17.19$ and $M = 18.65$.
b. 1. Use the equation, $p = (I - C)^{-1} d$ to find p.	b. $(I - C)^{-1} = \begin{bmatrix} 2.40 & 1.35 & 1.25 \\ 1.56 & 2.19 & 1.25 \\ 1.49 & 1.42 & 2.08 \end{bmatrix}$
2. Recognise that the values of production need to be positive.	As long as $(I - C)^{-1}$ exists and has positive entries, it will be always possible to find values of p to match the demand, d.

The next worked example uses a two-industry economy and illustrates how we can form a consumption matrix from information about the economy.

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WORKED EXAMPLE 27

Consider the following two industry input–output models. To produce one tonne of coal requires 0.2 tonnes of coal and 0.3 tonnes of steel. To produce one tonne of steel requires 0.4 tonnes of coal and 0.2 tonnes of steel.

a. Construct the consumption matrix for this economy.

b. Determine the production required to meet the external demand of 80 tonnes of coal and 20 tonnes of steel.

THINK

a. Construct the consumption matrix using C and S across the top and down the side, where C is coal and S is steel.

b. 1. Recall the formula relating production, p, to demand, d and substitute the consumption matrix calculated in a. and the external demand given, d.

2. Write the answer in words.

WRITE

a.
$$C = \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} C \\ S \end{array}$$

b. $p = (I - C)^{-1} d$

$$= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 138.5 \\ 76.9 \end{bmatrix}$$

The production that meets the demand is 138.5 tonnes of coal and 76.9 tonnes of steel.

3.7.4 Cryptology

Cryptology is the science of rendering a communication meaningless to anyone intercepting the message. It is an area of mathematics that has grown rapidly, and is used on a daily basis through secure messaging apps and internet banking.

Enciphering is the coding of a message (plain text) into a form that is meaningless to a third party. **Deciphering** is the reverse process; taking the code and translating it back into the original plain text message.

Consider the simplest of ciphers — a mono-alphabetic substitution; that is, the substitution of one letter for another. A cipher such as this is easy to crack if one has enough text to work with. A disadvantage of substitution ciphers is that they preserve the frequencies of individual letters, making it relatively easy to break the code by statistical methods. The percentage frequency of each letter in English is:

A	B	C	D	E	F	G	H	I	J	K	L	M
8.2	1.5	2.8	4.3	12.7	2.2	2.0	6.1	7.0	0.2	0.8	4.0	2.4
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
6.7	7.5	1.9	0.1	6.0	6.3	9.1	2.8	1.0	2.3	0.1	2.0	0.1

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Naturally, coded texts vary in content, and the frequency of letters in a code will not follow these general proportions exactly. However, they do provide a starting point for someone attempting to decipher a text. The appearance of apostrophes and frequent letter associations in common words such as 'the' may also provide clues.

Enciphering

In the discussion to follow, we assume that each plain-text and cipher-text letter, except Z, is assigned the numerical value that specifies its position in the standard alphabet.

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
1	2	3	4	5	6	7	8	9	10	11	12	13
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
14	15	16	17	18	19	20	21	22	23	24	25	0

In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by a procedure that involves modulo arithmetic. Modulo arithmetic plays an important role in cryptology.

Let us consider this first.

Recall that we say:

$$45 \equiv 3 \text{ modulo } 7, \text{ because the remainder when } 45 \text{ is divided by } 7 \text{ is } 3.$$

This statement is usually abbreviated to:

$$45 \equiv 3 \pmod 7 .$$

When operating in mod 7 there are only 7 distinct numbers 0, 1, 2, 3, 4, 5 and 6. Every other number can be linked to one of these remainders.

As there are 26 letters in the English alphabet, modulo 26 is frequently used in cryptology.

For example, $87 = 9 \pmod{26}$. This means that the remainder when 87 is divided by 26 is 9. Another way of looking at this is $87 = 26 \times 3 + 9$.

For negative numbers, we still look for a positive remainder. For example, $-38 = 14 \pmod{26}$. That is, $-38 = 26 \times -2 + 14$.

For modulo 26, $Z_{26} = \{0, 1, 2, \ldots, 25\}$.

If a is a number in Z_m, then a number, a^{-1}, in Z_m is called a *reciprocal* or *multiplicative inverse* of a modulo m if $aa^{-1} = a^{-1}a = 1 \pmod m$.

Not every element of Z_m will have a multiplicative inverse.

The easiest way to search for a numbers inverse is to use the multiplication table and search along its row for the number 1.

- For instance, if we choose the number 2 in mod 26, and then searched its row for the number 1, we wont find it.
- The reciprocal of 3 in mod 26 is 9 because the product of the two numbers is 1.

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The results for multiplication modulo 26 are listed in the table below.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

From the table we can see that 9 is the multiplicative inverse of 3 (mod 26) as $9 \times 3 = 1 \pmod{26}$. That is, $3^{-1} \bmod 26 = 9$.

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WORKED EXAMPLE 28

Simplify each of the following.

a. **200 mod 26** **b.** **5^{-1} mod 26** **c.** **-9 mod 26**

THINK	WRITE
a. Since mod 26 refers to the remainder when dividing by 26, divide 200 by 26 and find the remainder.	**a.** $200 \div 26 = 7$ remainder 18. $200 \bmod 26 = 18$.
b. Use the table to find the multiplicative inverse of 5. 5^{-1} means $5^{-1} \times 5 = 1$.	**b.** From the table, $5 \times 21 = 1 \pmod{26}$. Therefore $5^{-1} \bmod 26 = 21$.
c. Divide -9 by 26 and find the positive remainder. Use $-9 = -1 \times 26 + 17$ rather than $-9 = 0 \times 26 - 9$.	**c.** $-9 \div 26 = -1$ remainder 17. Therefore $-9 \bmod 26 = 17$.

Now we can use this to transform plain text to cipher text. In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by the following procedure

Step 1 Choose a 2×2 matrix with integer entries $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix will be used to perform the encoding.

Step 2 Group successive plain-text letters into pairs, adding an arbitrary 'dummy' letter to fill out the pair if the plain text has an odd number of letters, and replace each plain-text letter by its last numerical value.

Step 3 Successively convert each plain-text pair, p_1p_2, into a column matrix: $P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, and form the product AP. We will call P a *plain-text* matrix and AP the corresponding *cipher-text* matrix.

Step 4 Convert each cipher-text matrix into its alphabetic equivalent using modulo 26 if the numbers are greater than 26.

WORKED EXAMPLE 29

Use the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ to encipher the message: THE GAME IS UP.

THINK	WRITE
1. Group the letters in pairs, doubling the last if there is an odd number.	TH EG AM EI SU PP
2. Convert each letter to a number from 0 to 25. Use the table on page 171.	20 8 5 7 1 13 5 9 19 21 16 16
3. Premultiply the first pair of numbers by the given cipher matrix	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 40 \\ 36 \end{bmatrix}$
4. Convert each number in the resulting matrix to module 26; that is, find the remainder when each number is divided by 26. In this case, $48 = 22 \pmod{26}$ and $36 = 10 \pmod{26}$	$= \begin{bmatrix} 22 \\ 10 \end{bmatrix} \bmod 26$

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5. Repeat steps 3 and 4 for the remaining pairs.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix} \bmod 26$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \end{bmatrix} \bmod 26$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \end{bmatrix} \bmod 26$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 19 \\ 21 \end{bmatrix} = \begin{bmatrix} 59 \\ 61 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \bmod 26$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 16 \\ 16 \end{bmatrix} = \begin{bmatrix} 48 \\ 48 \end{bmatrix} = \begin{bmatrix} 22 \\ 22 \end{bmatrix} \bmod 26$$

So 20 8 5 7 1 13 5 9 19 21 16 16 becomes 22 10 17 19 15 1 19 23 7 9 22 22

6. Convert each number to a letter of the alphabet using the table under Enciphering.

The enciphered message is V J Q S O A S W G I V V.

Because the plain text was grouped in pairs and enciphered by a 2×2 matrix, the cipher in the previous worked example is referred to as a *2-cipher*. It is possible to group the plain text in triples and encipher by a 3×3 matrix with integer entries; this is called a *3-cipher*. In general, for an *n-cipher*, plain text is grouped into sets of n letters and enciphered by an $n \times n$ matrix with integer entries.

Deciphering

To decipher a message that has been encrypted using matrix A, we use the inverse of A, A^{-1}. That is, for the enciphering 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the deciphering 2×2 matrix, A^{-1} is given by

$$A^{-1} = (ad - bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \pmod{26}$$

where $(ad - bc)^{-1}$ is the multiplicative inverse of $(ad - bc)$ in modulo 26.

Note that not all numbers have an inverse in mod 26. The inverse of 3 is 9 because $9 \times 3 = 27 = 1 \pmod{26}$; but there is no inverse for 2 or 4 or 13. (See the table on page 172.)

Note also that the inverse of -3 is different from the inverse of 3. The inverse of 3 is 9 but $-3 \times 9 = -27 = 25 \pmod{26}$. The inverse of -3 is 17 as $-3 \times 17 = -51 = 1 \pmod{26}$.

WORKED EXAMPLE 30

Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

DPOOOHZZVOQOVYYVQQ

THINK	WRITE
1. To find the inverse matrix, A^{-1}, first calculate the determinant of matrix A.	$\det A = 2 \times 2 - 1 \times 1$ $= 3$

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2. Find the multiplicative inverse (mod 26) of $\det A$ using the relationship that $(\det A)^{-1} \times (\det A) = 1$ for modulo 26. (Use the table.)

From the table, the multiplicative inverse of 3 is 9.

3. Find A^{-1}.

$$9 \times 3 = 27 = 1 \pmod{26}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = 9\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -9 \\ -9 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$$

4. Convert the elements in the resulting matrix to modulo 26.

5. Convert the cipher text to a string of numbers. (Use the table.) Group the numbers in pairs.

DPOOOHZZVOQOVYYVQQ gives
4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17

6. Multiply the first pair of numbers by the deciphering matrix $\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$.

$$\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}\begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 344 \\ 356 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 18 \end{bmatrix} \bmod 26$$

7. Convert the elements in the resulting matrix to modulo 26.

8. Repeat steps 6 and 7 for all pairs of numbers. List the decoded numbers.

4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17
gives 6 18 5 5 16 9 0 0 1 20 15 13 15 18 18 15 23 23

9. Convert the decoded numbers to text.

This gives FREEPIZZATOMORROWW.

studyon

Units 3 & 4 > Area 2 > Sequence 2 > Concept 6 > **Applications of matrices** Summary screen and practice questions

Exercise 3.7 Applications of matrices

Technology free

1. WE22 The transition matrix for a system is $\begin{bmatrix} 0.3 & 0.2 & 0.50 \\ 0.3 & 0.2 & 0.25 \\ 0.4 & x & 0.25 \end{bmatrix}$.

a. Calculate the value of x.

b. If the current state of the system is $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

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c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

d. If the current state of the system is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

2. The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.2 & c \\ 0.3 & b & 0.25 \\ a & 0.1 & 0.20 \end{bmatrix}$.

a. Calculate the values of a, b and c.

b. If the current state of the system is $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$, calculate the state of the system after each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

3. WE23 The Robinson Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 70%. If it is not raining on a particular day, the chance that it will rain on the following day is 52%.

a. Construct a transition matrix to represent this situation.

b. Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.

c. Using technology, determine the long term proportion of days it rains in the Nee Islands.

4. WE24 If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, find the:

a. eigenvalues associated with the matrix.

b. eigenvectors associated with the matrix.

5. If $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$, find the:

a. eigenvalues associated with the matrix.

b. eigenvectors associated with the matrix.

6. If matrix M is a 2×2 transition matrix, then the columns of M will each add to give 1. Show that $\lambda = 1$ is always an eigenvalue for such a transition matrix.

7. a. Show that the eigenvalues of a matrix, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ are found by solving the following equation for λ:

$$\lambda^2 - 5\lambda - 2 = 0$$

b. Show that the matrix A is a solution to the following equation.

$$A^2 - 5A - 2I = 0$$

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Technology active

8. **WE25** Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} G \\ H \\ T \end{array}$,
   a. state the meaning of the value 0.6 in the third row of the matrix
   b. calculate the inputs needed to produce: 1 unit of H
   c. calculate the inputs needed to produce 12 units of G, 15 units of H and 30 units of T.

9. Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} G \\ H \\ T \end{array}$, calculate the inputs needed to produce:
   a. 3 units of G.
   b. 8 units of G, 25 units of H and 10 units of T.

10. **WE26** Given the consumption matrix at right: $C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} G \\ H \\ T \end{array}$
   a. calculate the production needed to match the demand, $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$
   b. show that a suitable production can be found to match any demand.

11. Given the consumption matrix
$$C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} G \\ H \\ T \end{array}:$$
   a. calculate the production needed to match the demand $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$
   b. show that a suitable production can be found to match any demand.

12. Consider the following three-industry economy:
The external demands for A, B and C are 20, 30 and 10 respectively. The exchange matrix is:
$$M = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.5 & 0.6 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}.$$
   a. What is the meaning of the entry 0.5 in the matrix?
   b. Calculate the required output for each of the three industries.

13. **WE27** A two-industry economy consists of coal and steel. To produce one tonne of coal requires 0.3 tonnes of coal and 0.4 tonnes of steel. To produce one tonne of steel requires 0.45 tonnes of coal and 0.2 tonnes of steel.
   a. Construct the consumption matrix.
   b. What production is required to meet the demand of 45 tonnes of coal and 25 tonnes of steel?

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14. Consider the following hypothetical three-industry economy.

To manufacture 1 unit of *A* requires	0.2 units of *A*	0.3 units of *B*	0.1 units of *C*
To manufacture 1 unit of *B* requires	0.1 units of *A*	0.2 units of *B*	0.2 units of *C*
To manufacture 1 unit of *C* requires	0.2 units of *A*	0.1 units of *B*	0.3 units of *C*

a. Construct the consumption matrix for this economy.
b. What inputs are required to produce 10 units of *A*, 20 units of *B* and 60 units of *C*?
c. How much of each should be produced to meet the external demand of 45 units of *A*, 30 units of *B* and 40 units of *C*?

15. WE28 Simplify each of the following.
a. $300 \bmod 26$ b. $7^{-1} \bmod 26$ c. $-14 \bmod 26$ d. $200 \bmod 26$ e. $8^{-1} \bmod 26$
f. $-5 \bmod 26$ g. $431 \bmod 26$ h. $15^{-1} \bmod 26$ i. $-15 \bmod 26$

16. WE29 Encode the message: LEAVE IMMEDIATELY using the enciphering matrix $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$.

17. Encode the message: THE IDES OF MARCH using the enciphering matrix $\begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$.

18. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has been used to encrypt the following message:
O L Y P X F R K R M E L.
a. What is $A^{-1} \bmod 26$?
b. Use this matrix to decipher the message.

19. WE30 Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$
I V W L A E D C O U E D V W K R Y R.

20. Encode the message: ALAN TURING IS A LEGEND using a 3×3 enciphering matrix $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

3.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple Familiar

1. Given the following systems of linear equations,
a. $7x + 2y - 4z = 24$
$x - 8y + 2z = 22$
b. $2x - y + 3z = 25$
$3x + 5y - 4z = -3$
$8x + 2y - 5z = 26$
i. state the number of m equations and n variables in the system
ii. justify whether the system is square or not
iii. determine if $x = 6, y = -1, z = 4$ is a valid solution set all equations in the system.

2. MC Identify which of the following matrices is row echelon form.
A. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -7 \end{bmatrix}$ B. $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 2 & 5 \end{bmatrix}$ C. $\begin{bmatrix} 4 & 3 & 3 \\ 0 & 2 & 8 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 9 & 4 \\ 0 & 1 & 7 \end{bmatrix}$

3. State if the following are *true* of *false*.
a. A consistent system could have an infinite number of solutions.
b. A linear system in three unknowns can have no solution.
c. A line is formed when three planes intersect. This system is inconsistent.

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d. Two parallel lines in three dimensions are distinct. This system is inconsistent.
e. A 3×3 linear system is consistent because $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$.

4. **MC** A system consists of the following equations.

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= d_1 \\ a_2x_2 + b_2y_2 + c_2z_2 &= d_2 \\ a_3x_3 + b_3y_3 + c_3z_3 &= d_3 \end{aligned}$$

The solution for this system is $x = -2 + 4t, y = 3 + 8t, z = 4 - t$ where $t \in R$. In geometric terms, the system and solution are best described by which of the following:

A. It is an inconsistent system.
B. It is a consistent system containing three planes. The planes intersecting with the solution forming a line. Infinitely many solutions exist.
C. It is a consistent system containing three planes. The planes intersecting at the unique position defined by $(x, y, z) = (-2, 3, 4)$
D. It is an inconsistent system containing three planes. Two of the planes are parallel and distinct.

5. Consider $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -4 \\ -3 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, when calculating if the following statements are true or false.

a. $A^2(A + B) = A^3 + AB$
b. $(A^3 - B^3)A^3 = A^3A^3 - A^3B^3$

6. In a round robin tennis competition, Tim recorded two losses to Nelly and Peter. Nelly's only loss was to Peter. Peter lost to Earnest. Use dominance matrices and the formula $M + 0.5M^2$ to rank the players.

7. Brooke Cove is very monsoonal area with prevalent heavy rainfall. If it is raining on a particular day, the chance that it will rain the next day is 89%. If it is not raining on a particular day, the chance that it will rain on the following day is 71%.

a. Construct a transition matrix to represent this situation.
b. Using the transition matrix, calculate the probability that, if it is raining on Wednesday, it will also be raining on Saturday of the same week.
c. In the long term, determine on what proportion of days does it rain in the Brooke Cove.

8. Solve the following system of linear equations by using an inverse matrix.

$$\begin{aligned} 7x + 2y + 5z &= 17 \\ 3x + 8y + z &= 9 \\ 2x - 7y + z &= -15 \end{aligned}$$

9. Solve the following system of linear equations using Gaussian elimination and back substitution.

a. $\begin{aligned} 6x - 12y &= -18 \\ -2x + 7y &= 12 \end{aligned}$
b. $\begin{aligned} -2x + 6y + z &= -14 \\ -x + 6y - 3z &= -4 \\ -30y - 4z &= -30 \end{aligned}$

10. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2. Describe the geometric nature of the solution.

$$\begin{aligned} l_1&: \left(-3\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 7\hat{\boldsymbol{k}}\right) + k\left(6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}\right) \\ l_2&: \left(6\hat{\boldsymbol{\imath}} + 9\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}\right) + k\left(-3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) \end{aligned}$$

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11. Consider the following system,

$$\begin{aligned} -6x + 2y - 5z &= -22 \\ 8x - 3y - 4z &= 49 \\ -3x - 7y - 4z &= -48 \end{aligned}$$

i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to solve the system.
iii. Describe how the solution can be interpreted geometrically.

12. Consider the following system of linear equations,

$$\begin{aligned} 2x + 4y + 12z &= 22 \\ 8x + 3y - 30z &= 23 \\ 8x - 4y - 72z &= -12 \end{aligned}$$

i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
iii. Solve the system of linear equations, expressing the solution in parametric form.
iv. Describe how the solution can be interpreted geometrically.

Complex familiar

13. Determine a general solution for the variable matrix X in the following matrix equations. (It is assumed that square matrices are non-singular).
a. $2X - 3B = C$ **b.** $BX - 2X + C = 0$ **c.** $A(X + I) + 2C = B$ **d.** $XA + B = XC - 2D$

14. The sum of three numbers is 10. When triple the first number is added to the sum of the second and third number the result is 26. The second number is 6 less than the third. Develop suitable matrix equation and apply Gaussian elimination techniques to determine a unique solution.

15. Determine if the following systems are either consistent or inconsistent by completing the 'ref' or 'rref' operation on a graphics calculator.

a. $A \mid B = \left[\begin{array}{ccc|c} 36 & 24 & 16 & 84 \\ -9 & -6 & -4 & -21 \\ 6 & 4 & \frac{8}{3} & 14 \end{array}\right]$ **b.** $A \mid B = \left[\begin{array}{cc|c} -3 & 6 & 12 \\ 6 & -12 & -40 \end{array}\right]$

16. The numbers of a particular species of lizard, all aged 1, 2 or 3 years, have been monitored over many years and the following Leslie matrix has been developed to model the population.

$$L = \begin{bmatrix} 0 & 5.30 & 3.50 \\ 0.20 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

a. On average, how many new lizards do 1-year-old lizards produce?
b. On average, how many new lizards do 2-year-old lizards produce?
c. What proportion of 1-year-old lizards survive the first year?
d. If the initial female population is $N_1 = \begin{bmatrix} 35 \\ 46 \\ 22 \end{bmatrix}$, calculate the female population in 3 years time.

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Complex Unfamiliar

17. Determine the solution/s to the following system where $x, y, z \in R$.

$$\begin{aligned} 7x^3 + 2y^2 + 3z &= -35 \\ 2x^3 + 5y^2 - 5z &= 24 \\ 3x^3 + y^2 - 4z &= -19 \end{aligned}$$

18. The linear equations below form a consistent system with infinitely many solutions.

$$\begin{aligned} p_1&: -4x - 2y - 4z = -12 \\ p_2&: -x + y + 5z = 9 \\ p_3&: 2x + b_3y + 4z = d_3 \end{aligned}$$

Calculate the unknowns values, b_3 and d_3.

19. A consistent linear system is given by:

$$\begin{aligned} 3x + 3y + 6z &= 3a \\ 10x + 10z &= 10b \\ 8x + 4y + 12z &= 4c \end{aligned}$$

Use Gaussian elimination to determine the relationship between a, b and c that must exist if the initial statement about the system type is true.

20. A study was carried out on a dingo population that inhabited a remote sand island. The population has the following characteristics.

- Half the dingoes survive the first year.
- Of the surviving dingoes, half survive the second year.
- In the first year of life, these dingoes produce no offspring.
- On average, 6 dingoes are produced in the second year and 8 in the third year.
- The current population is given by the initial state matrix,

$$V_1 = \begin{bmatrix} 60 \\ 40 \\ 36 \end{bmatrix} \quad \begin{array}{l} \ldots.0 < t \leq 1 \\ \ldots.1 < t \leq 2 \\ \ldots.2 < t \leq 3 \end{array}$$

a. Determine the dingo population in 5 and 10 years' time.

b. Compare the populations at both 5 and 10 years to show that a common ratio doesn't exist in terms of the number of dingoes per age group.

c. It is possible that a population, consisting of three age groups, will continuously abide by a fixed ratio. For example,

$$X = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \quad \begin{array}{l} \ldots.0 < t \leq 1 \\ \ldots.1 < t \leq 2 \\ \ldots.2 < t \leq 3 \end{array}$$

Determine an alternative for V_1 so that the ratio between age groups remains constant.

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Answers

3 Solving systems of linear equations and the application of matrices

Exercise 3.2 Solving linear equations using matrix algebra

1. a. and d. are non-linear.
2. a. i. $m = 2, n = 3$
   ii. Not a square system as $m \neq n$
   iii. Not a valid solution.
   b. i. $m = 3\ , n = 3$
   ii. It is a square system as $m = n$
   iii. The solution set satisfies all equations in the system.
3. a. $x = 6, y = -2$ b. $x = 2, y = 3$
   c. $x = 6, y = 4$ d. $x = -4, y = 8$
4. a. $x = -1, y = 0, z = -5$
   b. $x = 5, y = 7, z = 5$
   c. $x = 6, y = -3, z = 2$
   d. $x = -4, y = -5, z = -2$
5. 1st place is awarded 10 points
   2nd place is awarded 7 points
   3rd place is awarded 5 points
   4th place is awarded 3 points
6. The cost per individual item are:
   Lolly bag \$1
   Pizza \$8
   Meat pie \$5
   Hamburger \$6
7. C
8. B
9. $X = (E + I)^{-1}(G - F)$
10. a. $X = \left(A^{-1}C\right)B^{-1}$ b. $X = BA^{-1}$
    c. $X = A^{-1}IB^{-1}$ d. $X = (A + I)^{-1}C$
11. a. $X = (A + B)^{-1}(D - C)$
    b. $X = (I2 + 3A)^{-1}B$
    c. $X = (A + 3B)^{-1}C$
    d. $X = (7A)^{-1}(B + C - 7AI)$
12. a. $X = \begin{bmatrix} -\frac{7}{6} & -\frac{11}{6} \\ -\frac{1}{3} & -\frac{10}{9} \end{bmatrix}$ b. $X = \begin{bmatrix} -\frac{1}{18} & -\frac{7}{9} \\ -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$
    c. $X = \begin{bmatrix} -\frac{43}{84} & -\frac{53}{84} \\ -\frac{11}{28} & -\frac{9}{28} \end{bmatrix}$ d. $X = \begin{bmatrix} \frac{64}{203} & \frac{78}{203} \\ \frac{-2}{7} & \frac{-2}{7} \end{bmatrix}$
13. a. $X = \begin{bmatrix} -\frac{5}{18} & -\frac{19}{16} \\ \frac{7}{8} & \frac{1}{16} \end{bmatrix}$
    b. $X = \begin{bmatrix} -\frac{7}{20} & \frac{3}{10} \\ \frac{13}{20} & -\frac{11}{30} \end{bmatrix}$
    c. $X = \begin{bmatrix} \frac{30}{217} & \frac{11}{31} \\ -\frac{2}{217} & -\frac{9}{31} \end{bmatrix}$
    d. $X = \begin{bmatrix} -\frac{31}{45} & \frac{23}{45} \\ -\frac{16}{15} & -\frac{14}{15} \end{bmatrix}$
14. True, $LHS = RHS$
15. $\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}$

Exercise 3.3 Solving a system of linear equations using the Gaussian technique of elimination

1. B 2. C 3. B
4. a. $x = 3, y = 2$ b. $x = -5, y = 1$
   c. $x = 2, y = 2$ d. $x = 1, y = -12$
5. a. $x = 0, y = -2, z = 0$
   b. $x = -2, y = -4, z = 0$
   c. $x = 3, y = 5, z = 2$
   d. $x = -5, y = 5, z = 4$
   e. $x = -3, y = -3, z = -1$
6. $w = 2, x = -3, y = 4, z = -5$
7. $x = 10, y = 20, z = 30$
8. $L = 5,\ \ R = \frac{9}{2}$
9. $A = 22,\ \ C = 5,\ \ P = 8$
   Two children, 1 adult and 2 pensioners will cost
   Cost = \$48
10. $5x + 2y + 4z = 32$
    $-4x + 7y + 8z = -15$
    $-3x + 11y + z = 9$
11. $y(x) = 2x^2 - 4x + 1$
12. $A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{5}{3} & -\frac{8}{3} \\ \frac{1}{9} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{6} & \frac{-2}{3} \end{bmatrix}$
13. $\begin{bmatrix} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & \boxed{4} & \boxed{-1} & -4 \end{bmatrix}$
    $\begin{bmatrix} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & \boxed{22} \\ 0 & \boxed{-21} & 4 & -34 \end{bmatrix}_{R_2 - 3R_1, R_3 + 5R_1}$
    $\begin{bmatrix} 1 & -5 & 1 & -6 \\ 0 & 1 & \frac{-5}{16} & \frac{11}{8} \\ 0 & 0 & -1 & \boxed{44} \end{bmatrix}_{5R_3 - 21R_1, R_2 \div 16}$

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Exercise 3.4 The three cases for solutions of systems of linear equations

1. C
2. a. No solution
   b. No solution
   c. One unique solution if the lines intersect; no solution if they are skew
   d. Infinitely many solutions
3. a. Unique solution $(x, y) = (1, 2)$ is the intersection of two lines that are not parallel.
   b. Zero solutions.
   The lines are parallel, yet distinct as they have different y-intercepts.
   c. Unique solution $(x, y) = (4, -2)$ is the intersection of two lines that are not parallel.
   d. Zero solutions. The lines are parallel, yet distinct as they have different y-intercepts.
4. i. a. There are an infinite number of solutions.
   b. Parametric form $(x, y) = \left(\frac{3t+6}{5}, t\right)$.
   ii. a. There are an infinite number of solutions.
   b. Parametric of form $(x, y) = (2t - 4, t)$.
5. a. One unique solution. The lines are not parallel and intersect.
   b. There are infinitely many solutions. The lines are coincident, hence the same line.
   c. Zero solution exist. The two lines are parallel but distinct.
   d. Zero solutions exist. The two lines are parallel but distinct.
6. a. $c \neq 12$ b. $b = 1, c = 6$
7. i. a. A unique solution exists at $(-19, -17, -24)$.
   b. The two distinct lines intersect at one unique point.
   ii. a. A unique solution exists at $(-2, -2, -2)$.
   b. The two distinct lines intersect at one unique point.
   iii. a. A unique solution exists at $(46, 14, 38)$.
   b. The two distinct lines intersect at one unique point.
8. i. a. A unique solution exists at $(-24, -33, 40)$.
   b. The two distinct lines intersect at one unique point.
   ii. a. A unique solution exists at $(23, -13, 16)$.
   b. The two distinct lines intersect at one unique point.
9. i. a. Zero solutions exist.
   b. The two are parallel and distinct which means they will never intersect.
   ii. a. The system has a unique solution.
   b. The two lines intersect at $(-16, -24, 25)$.
   iii. a. The system has a unique solution.
   b. The two lines intersect at $(-50, 38, -41)$.
10. i. a. Zero solutions exist.
   b. The two are parallel and distinct which means they will never intersect.
   ii. a. The system has a unique solution.
   b. The two lines intersect at $(9, -15, 37)$.
11. a. Coincident planes are the same plane. There are infinitely many solutions.
   b. One unique solution exists.
   c. The system is consistent with infinitely many solutions existing along the line,
   $\underset{\sim}{r} = (4\hat{\imath} - 6\hat{\jmath} - 11\hat{k}) + k(-3\hat{\imath} + 13\hat{\jmath} + 2\hat{k})$.
12. a. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
   ii. The system is **consistent** and has one unique solution. The three planes intersect at $(6, -4, -3)$.
   iii. The planes intersect at one unique point.
   b. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
   ii. The final row is a false statement, so this system has no solution.
   iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.
   c. i. The planes are not coplanar. This implies that the planes intersect at a unique point.
   ii. The system is consistent and has one unique solution. The three planes intersect at $(7, -1, -1)$.
   iii. Geometrically the solution indicates the planes intersect at one unique point.
   d. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
   ii. The system is consistent and has one unique solution. The three planes intersect at $(3, 7, 4)$.
   iii. Geometrically the solution indicates the planes intersect at one unique point.
   e. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
   ii. The final row is a false statement, so this system has no solution.
   iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.
   f. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
   ii. The final row is a false statement, so this system has no solution.
   iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.
13. a. $x = \frac{-1-5t}{8}, \ y = -1 - t, z = t$
   b. $x = 1 - t, \ y = -1 + 2t, z = t$
   c. No solution.
14. a. The planes are parallel yet distinct. It is an inconsistent system with zero solutions.
   b. The planes are parallel yet distinct. It is an inconsistent system with zero solutions.
   c. Two planes are parallel yet distinct. It is an inconsistent system with zero solutions.
   d. Two planes are parallel yet distinct. It is an inconsistent system with zero solutions.
15. $d_2 = 8, a_3 = 12$
16. a. $x = 1 - 2t, y = 7 - \frac{3}{2}t, z = t$
   Vector equation of line:
   $$\underset{\sim}{r} = (\hat{\imath} + 7\hat{\jmath}) + t\left(-2\hat{\imath} - \frac{3}{2}\hat{\jmath} + \hat{k}\right)$$

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b. The vector product of the normals is parallel to the directional vector of the line of intersection formed by the two planes.

17. a. $x = 8 + 6t, y = -1 - t, z = t$
b. $x = -13 + 4t, y = 24 - 4t, z = t$
c. $x = -6 + t, y = 22 - 2.5t, z = t$
d. $x = 14 + 5t, y = 1 - t, z = t$

18. a. The parametric equation satisfies all equations in the system. It is a valid solution set.
b. The parametric equation satisfies all equations in the system. It is a valid solution set.

19. a. Infinitely many solutions
b. Zero solutions
c. Unique solution at $P(5, 6, -7)$

20. a. Unique solution at $P(6, -3, 4)$
b. No solution (line is parallel yet distinct from the plane).
c. Infinitely many solutions (the line lies on the plane).

21. $(x, y, z) = \left(\frac{-1124}{49}, \frac{-831}{49}, \frac{-416}{49}\right)$

Exercise 3.5 Using technology for matrix calculations

1. a. $\begin{bmatrix} -65 & -139 \\ -119 & -227 \end{bmatrix}$ b. $\begin{bmatrix} \frac{-84}{5} & \frac{138}{5} \\ \frac{-284}{5} & \frac{536}{5} \end{bmatrix}$

c. -108 d. 4

e. $\begin{bmatrix} \frac{-11}{18} & \frac{1}{6} \\ \frac{5}{54} & \frac{-1}{18} \end{bmatrix}$ f. $\begin{bmatrix} -9 & -3 & -11 \\ \frac{-7}{2} & -1 & -4 \\ \frac{-1}{2} & 0 & \frac{-1}{2} \end{bmatrix}$

g. $\begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$ h. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. a. $LHS \neq RHS$
$\therefore$ Statement is false.
b. $LHS \neq RHS$
$\therefore$ The statement is false.

3. a. $\begin{bmatrix} 178 & -250 \\ 125 & 303 \end{bmatrix}$
b. $\begin{bmatrix} \frac{1114}{343} & \frac{586}{343} \\ \frac{599}{343} & \frac{281}{343} \end{bmatrix}$
c. $\begin{bmatrix} 74 & -62 & 30 \\ 30 & -152 & 84 \\ 210 & -48 & 26 \end{bmatrix}$
d. $\begin{bmatrix} \frac{-108}{5} & -24 & \frac{-121}{5} \\ \frac{-104}{5} & 21 & \frac{-123}{5} \\ \frac{-91}{5} & \frac{-123}{5} & \frac{-108}{5} \end{bmatrix}$

4. a. It is an inconsistent system.
b. It is an inconsistent system.
c. It is a consistent system.
d. It is a consistent system.

5. Total cost for coffee is \$37.16.
Total cost for tea is \$18.28.
Total cost for juice is \$10.70.

6. a. $\begin{bmatrix} 71 & 4 \\ 63 & -28 \end{bmatrix}$ b. $\begin{bmatrix} 45 & 44 \\ -11 & 12 \end{bmatrix}$

7. a. A is nilpotent. b. A is nilpotent.

8. a. A is idempotent. b. A is idempotent.

9. Statement is true, LHS = RHS

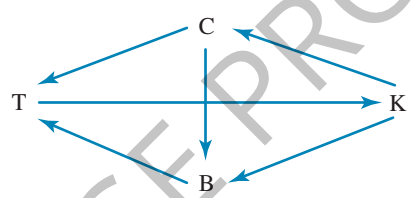
10. Statement is true, LHS = RHS

11. The unknown numbers are 8, 10 and 7.

12. a. $c = \frac{-20}{3}$
b. Statement is true, LHS=RHS

Exercise 3.6 Dominance and Leslie matrices

1. a.



b. The players would be ranked Cameron, Breanna, Kayley, Teagan.

2. Players can be ranked Mair, Ann, Janine

3. a. The houses are ranked Hamilton, Leslie, Cunningham, Barnes.
b. Hamilton received 20 points.
Leslie received 15 points.
Cunningham received 10 points.
Barnes received 5 points.

4. a. $M = \begin{array}{c} \begin{array}{ccccc} C & W & G & S & I \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} C \\ W \\ G \\ S \\ I \end{array}$
b. Warwick 5 points, Ipswich 4 points, Stanthorpe 3 points, Clifton 2 points, Goondiwindi 1 point

5. a. 6 b. 3.6

6. a. 1.4 females per year b. 50% live to age 3
c. 91 female rats d. 673 (336.6 females)

7. a. 0 new birds b. 3.1 new birds
c. 0.2 survive d. 338 females

8. a. $L = \begin{bmatrix} 0 & 0.50 & 0.45 & 0.4 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.85 & 0 & 0 \\ 0 & 0 & 0.50 & 0 \end{bmatrix}$
b. 7460 sheep (3730 females)

9. a. i. 0.85 ii. 0.99 iii. 0
b. i. 0 ii. 0.4 iii. 0.08
c. $\begin{bmatrix} 0 & 0.40 & 0.08 \\ 0.85 & 0 & 0 \\ 0 & 0.99 & 0 \end{bmatrix}$
d. 769 female sheep, 1538 total population.

10. $N_9 = \begin{bmatrix} 2705 \\ 870 \\ 223 \end{bmatrix}$
7596 (3798 female).

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Exercise 3.7 Applications of matrices

1. a. $x = 0.6$
   b. i. $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$
   iii. $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.392 \end{bmatrix}$
   c. i. $P = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.35 \\ 0.25 \\ 0.40 \end{bmatrix}$
   iii. $P = \begin{bmatrix} 0.355 \\ 0.255 \\ 0.390 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.393 \end{bmatrix}$
   d. i. $P = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$
   iii. $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$
2. a. $a = 0.5, b = 0.7, c = 0.55$
   b. i. $P = \begin{bmatrix} 0.455 \\ 0.345 \\ 0.210 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.274 \\ 0.428 \\ 0.299 \end{bmatrix}$
   iii. $P = \begin{bmatrix} 0.305 \\ 0.456 \\ 0.239 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.284 \\ 0.470 \\ 0.246 \end{bmatrix}$
   c. i. $P = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.375 \\ 0.395 \\ 0.230 \end{bmatrix}$
   iii. $P = \begin{bmatrix} 0.281 \\ 0.447 \\ 0.273 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.296 \\ 0.465 \\ 0.240 \end{bmatrix}$
3. a. $\begin{bmatrix} 0.7 & 0.52 \\ 0.3 & 0.48 \end{bmatrix}$
   b. 0.636
   c. The long-term probability of rain is 0.63415 and dry is 0.36585
4. a. $\lambda = 7, \lambda = -2$ b. $\begin{bmatrix} a \\ \frac{-4a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ a \end{bmatrix}$
5. a. $\lambda = -0.2, \lambda = 1$ b. $\begin{bmatrix} a \\ \frac{7a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ -a \end{bmatrix}$
6. $\lambda = 1$
7. a. Sample responses can be found in the worked solutions in the online resources.
   b. Sample responses can be found in the worked solutions in the online resources.
8. a. This entry means it takes 0.6 units of T to make 1 unit of G.
   b. The input q is 0.5 units of G, 0.4 units of H and 0.1 units of T.
   c. The input q is 17.1 units of G, 17.4 units of H and 11.7 units of T.
9. a. The input needed is 0.6 units of G, 0.6 units of H and 1.2 units of T.
   b. The input needed is 7.6 units of G, 14.6 units of H and 6.7 units of T.
10. a. The production that exactly matches the demand is 109 units of G, 171 units of H and 119 units of T.
    b. It will always be possible to find values of P to match the demand d.
11. a. The production that exactly matches the demand is 83 units of G, 143 units of H and 81 units of T.
    b. It will always be possible to find values of P to match the demand d.
12. a. To produce 1 unit of A requires 0.5 units of C.
    b. The required output is 37 units of A, 50 units of B and 59 units of C.
13. a. $c = \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.3 & 0.45 \\ 0.4 & 0.20 \end{bmatrix} \end{array} \begin{array}{c} C \\ S \end{array}$
    b. The production that meets the demand is 124 tonnes of coal and 93 tonnes of steel.
14. a. $C = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}$
    b. The input q is 16 units of A, 13 units of B and 23 units of C.
    c. The production that meets the demand is 90 units of A, 83 units of B and 94 units of C.
15. a. 14 b. 15 c. 12
    d. 18 e. Not possible f. 21
    g. 15 h. 7 i. 11
16. The enciphered message is M K P G K X M Z V Y U N C S U G
17. The enciphered message is BHCXGCIPYSNMTQ
18. a. $A^{-1}\text{mod}26 = \begin{bmatrix} 8 & 11 \\ 11 & 4 \end{bmatrix}$
    b. R E L A X B E H A P P Y or RELAX BE HAPPY
19. YOU SHOULD SEE ENIGMA.
20. C Z O A W J R X K K R D O D O N E Z B P P.

3.8 Review: exam practice

1. a. i. $m = 2, n = 3$
   ii. The system is not square because $m \neq n$
   iii. The solution is valid for all equations in the system.
   b. i. $m = 3, n = 3$
   ii. The system is square because $m = n$
   iii. The solution is valid for all equations in the system.
2. D
3. a. True b. True c. False
   d. True e. True
4. B
5. a. True: $LHS = RHS$ b. False: $LHS \neq RHS$
6. The players are ranked Peter, Nelly, Earnest and Tim.
7. a. $\begin{bmatrix} 0.89 & 0.71 \\ 0.11 & 0.29 \end{bmatrix}$
   b. 0.8666
   c. The long-term probability of rain is 0.86585 and dry is 0.13415.
8. The unique solution is the point $(x, y, z) = (-6, 2, 11)$.
9. a. The unique solution is the point $(x, y) = (1, 2)$.
   b. The unique solution is the point $(x, y, z) = (10, 1, 0)$.

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10. A unique solution exists at $(3, 6, -1)$.
The two distinct lines intersect at one unique point.

11. a. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
    b. The system is consistent and has one unique solution. The three planes intersect at $(7, 5, -2)$.
    c. The planes intersect at one unique point, $(7, 5, -2)$.

12. a. The three planes are coplanar, and not parallel. Geometrically, this could be the planes intersecting in a line or not at all.
    b. $[A\,|B] = \left[\begin{array}{ccc|c} 1 & 0 & -6 & 1 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$
    The system has infinitely many solutions because the planes intersect in a line.
    c. $x = 1 + 6t$
    $y = 5 - 6t$
    $z = t$
    d. This is a consistent system with the solution representing the intersection of three planes in the form of a line.

13. a. $X = \frac{1}{2}(C + 3B)$
    b. $X = (A - 2)^{-1} B$
    c. $X = \left[A^{-1}(B - 2C)\right] - I$
    d. $X = (-2D - B)(A - C)^{-1}$

14. $(x, y, z) = (8, -2, 4)$

15. a. The system is consistent (infinite number of solutions).
    b. It is a inconsistent system (parallel yet distinct lines).

16. a. 0
    b. 5.3
    c. 0.2
    d. $\begin{bmatrix} 77.4 \\ 64.2 \\ 1.8 \end{bmatrix}$; 143 female lizards

17. $(x, y, z) = (-2, \pm 3, 1)$

18. The line is defined by the parametric equations:

$$x = -1 + t, y = 8 - 4t, z = t$$

$$p_3\colon 2x + \frac{3}{2}y + 4z = 10$$

$$\therefore b_3 = \frac{3}{2}, d_3 = 10$$

19. The relationship is $0 = a + b - c$

20. a. $X_5 = \begin{bmatrix} 2076 \\ 852 \\ 85 \end{bmatrix}$

    $X_{10} = \begin{bmatrix} 83\,856 \\ 20\,246 \\ 5382 \end{bmatrix}$

    b. $\begin{bmatrix} 24t \\ 10t \\ t \end{bmatrix} \neq \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$
    The ratios are not equivalent at 5 and 10 years

    c. $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$
    This is in a ratio of 16: 4: 1. If the initial dingo population conforms to $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$, the population each year that follows will remain in the same ratio.

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