TOPIC 17 Polynomials

17.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

17.1.1 Why learn this?

Just as number is learned in stages, so too are graphs. You have been building your knowledge of graphs and functions over time. First you encountered linear functions, then quadratic and hyperbolic functions. Polynomials are higher-order functions represented by smooth and continuous curves. They can be used to model situations in many elds, such as business, science, architecture, design and engineering.

17.1.2 What do you know?

assesson

1. **THINK** List what you know about polynomials. Use a thinking tool such as a concept map to show your list.
2. **PAIR** Share what you know with a partner and then with a small group.
3. **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of polynomials.

LEARNING SEQUENCE

17.1 Overview
17.2 Polynomials
17.3 Adding, subtracting and multiplying polynomials
17.4 Long division of polynomials
17.5 Polynomial values
17.6 The remainder and factor theorems
17.7 Factorising polynomials
17.8 Solving polynomial equations
17.9 Review

learnon RESOURCES — ONLINE ONLY

Watch this eLesson: The story of mathematics: Mary Somerville
Searchlight ID: eles-2020

17.2 Polynomials

- A **polynomial** *in* x, sometimes denoted $P(x)$, is an expression containing only non-negative integer powers of x.
- The **degree** of a polynomial in x is the highest power of x in the expression. For example:

$3x + 1$	is a polynomial of degree 1, or linear polynomial.
$x^2 + 4x - 7$	is a polynomial of degree 2, or quadratic polynomial.
$-5x^3 + \frac{x}{2}$	is a polynomial of degree 3, or cubic polynomial.
10	is a polynomial of degree 0 (think of 10 as $10x^0$).

- Expressions containing a term similar to any of the following terms are not polynomials:

$$\frac{1}{x}, \quad x^{-2}, \quad \sqrt{x}, \quad 2^x, \quad \sin x, \quad \text{etc.}$$

For example, the following are not polynomials.

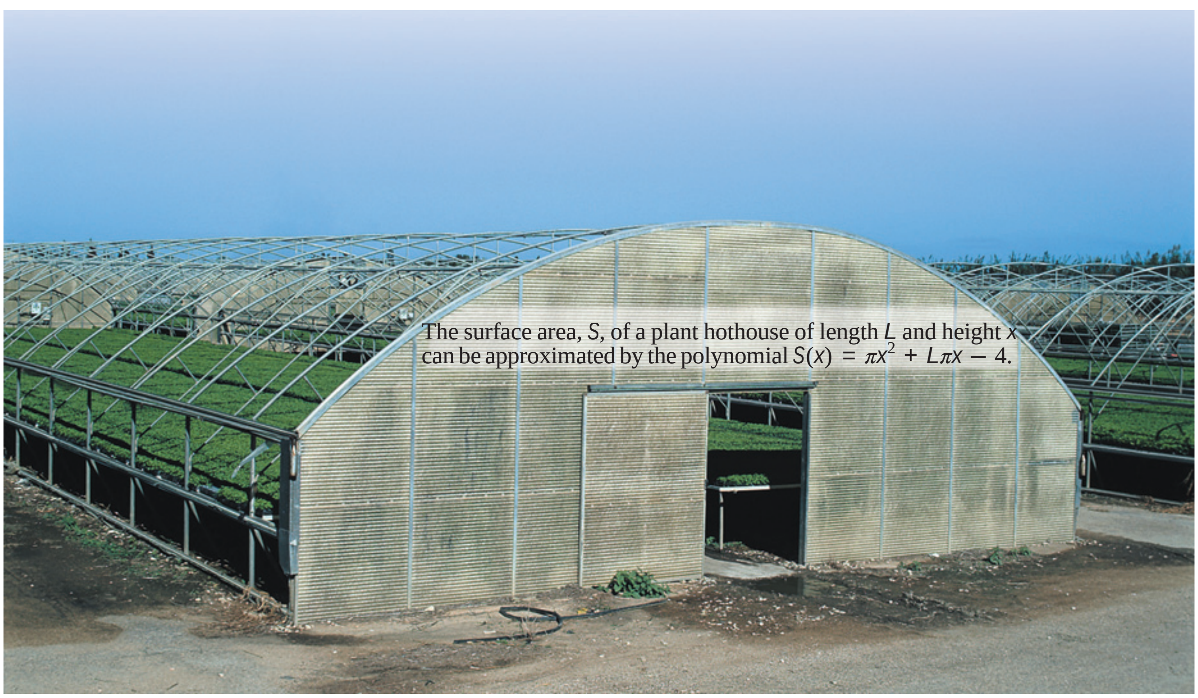
$$3x^2 - 4x + \frac{2}{x} \qquad -5x^4 + x^3 - 2\sqrt{x} \qquad x^2 + \sin x + 1$$

- In the expression $6x^3 + 13x^2 - x + 1$

x is the *variable*.
6 is the *coefficient* of x^3.
13 is the *coefficient* of x^2.
-1 is the *coefficient* of x.
$6x^3$, $13x^2$, $-x$ and $+1$ are all *terms*.
The *constant* term is $+1$.
The *degree* of the polynomial is 3.

- The leading term is $6x^3$ because it is the term that contains the highest power of x.
- The leading coefficient is 6.
- Any polynomial with a leading coefficient of 1 is called *monic*.

An example of where polynomials are useful is shown below.



The surface area, S, of a plant hothouse of length L and height x can be approximated by the polynomial $S(x) = \pi x^2 + L\pi x - 4$.

Exercise 17.2 Polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a, b, f, 2–5, 7, 8, 11	Questions: 1c–e, g, i, 2–4, 6, 8, 10, 11	Questions: 1a, c, f–i, 2–4, 6–12

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. State the degree of each of the following polynomials.

a. $x^3 - 9x^2 + 19x + 7$ **b.** $65 + 2x^7$ **c.** $3x^2 - 8 + 2x$

d. $x^6 - 3x^5 + 2x^4 + 6x + 1$ **e.** $y^8 + 7y^3 - 5$ **f.** $\frac{1}{2}u^5 - \frac{u^4}{3} + 2u - 6$

g. $18 - \frac{e^5}{6}$ **h.** $2g - 3$ **i.** $1.5f^6 - 800f$

2. State the variable for each polynomial in question **1**.

3. Which polynomials in question **1** are:

a. linear
b. quadratic
c. cubic
d. monic?

4. State whether each of the following is a polynomial (P) or not (N).

a. $7x + 6x^2 + \frac{5}{x}$ **b.** $33 - 4p$ **c.** $\frac{x^2}{9} + x$

d. $3x^4 - 2x^3 - 3\sqrt{x} - 4$ **e.** $k^{-2} + k - 3k^3 + 7$ **f.** $5r - r^9 + \frac{1}{3}$

g. $\frac{4c^6 - 3c^3 + 1}{2}$ **h.** $2^x - 8x + 1$ **i.** $\sin x + x^2$

5. Consider the polynomial $P(x) = -2x^3 + 4x^2 + 3x + 5$.

a. What is the degree of the polynomial?
b. What is the variable?
c. What is the coef cient of x^2?
d. What is the value of the constant term?
e. Which term has a coef cient of 3?
f. Which is the leading term?

6. Consider the polynomial $P(w) = 6w^7 + 7w^6 - 9$.

a. What is the degree of the polynomial?
b. What is the variable?
c. What is the coef cient of w^6?
d. What is the coef cient of w?
e. What is the value of the constant term?
f. Which term has a coef cient of 6?

7. Consider the polynomial $f(x) = 4 - x^2 + x^4$.

a. What is the degree of the polynomial?
b. What is the coef cient of x^4?
c. What is the leading term?
d. What is the leading coef cient?

Understanding

8. A sports scientist determines the following equation for the velocity of a breaststroke swimmer during one complete stroke:

$$v(t) = 63.876t^6 - 247.65t^5 + 360.39t^4 - 219.41t^3 + 53.816t^2 + 0.4746t.$$

a. What is the degree of the polynomial?
b. What is the variable?
c. How many terms are there?
d. Use a graphics calculator or graphing software to draw the graph of this polynomial.
e. Match what happens during one complete stroke with points on the graph.

Reasoning

9. The distance travelled by a body after t seconds is given by $d(t) = t^3 + 2t^2 - 4t + 5$. Using a graphing calculator or suitable computer software, draw a graph of the above motion for $0 \leq t \leq 3$. Use the graph to help you answer the following:
a. What information does the constant term give?
b. What is the position of the body after 1 second?
c. Describe in words the motion in the first 2 seconds.

10. Write the following polynomials as simply as possible, arranging terms in descending powers of x.
a. $7x + 2x^2 - 8x + 15 + 4x^3 - 9x + 3$
b. $x^2 - 8x^3 + 3x^4 - 2x^2 + 7x + 5x^3 - 7$
c. $x^3 - 5x^2 - 11x - 1 + 4x^3 - 2x + x^2 - 5$

Problem solving

11. If $x^2 + 2x - 1 \equiv (x - 1)^2 + a(x + 1) + b$, find the values of a and b.
12. If $x^3 + 9x^2 + 12x + 7 \equiv x^3 + (ax + b)^2 + 3$, find the values of a and b.

Reflection

How can you tell what the degree of a polynomial is?

17.3 Adding, subtracting and multiplying polynomials

- To add or subtract polynomials, we simply add or subtract any like terms in the expressions.

WORKED EXAMPLE 1 TI | CASIO

Simplify each of the following.
a $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$
b $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$

THINK	WRITE
a 1 Write the expression.	**a** $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 + x^4 + 5x^2 - 4$
3 Identify any like terms and change the order.	$= x^4 + 5x^3 + 3x^2 + 5x^2 - 2x - 1 - 4$
4 Simplify by collecting like terms.	$= x^4 + 5x^3 + 8x^2 - 2x - 5$
b 1 Write the expression.	**b** $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 - x^4 - 5x^2 + 4$
3 Identify any like terms and change the order.	$= -x^4 + 5x^3 + 3x^2 - 5x^2 - 2x - 1 + 4$
4 Simplify by collecting like terms.	$= -x^4 + 5x^3 - 2x^2 - 2x + 3$

- If we expand linear factors, for example $(x + 1)(x + 2)(x - 7)$, we may also get a polynomial as the following worked example shows.

WORKED EXAMPLE 2 TI | CASIO

Expand and simplify:

a $x(x + 2)(x - 3)$

b $(x - 1)(x + 5)(x + 2)$.

THINK	WRITE
a 1 Write the expression.	**a** $x(x + 2)(x - 3)$
2 Expand the last two linear factors.	$= x(x^2 - 3x + 2x - 6)$ $= x(x^2 - x - 6)$
3 Multiply the expression in the grouping symbols by x.	$= x^3 - x^2 - 6x$
b 1 Write the expression.	**b** $(x - 1)(x + 5)(x + 2)$
2 Expand the last two linear factors.	$= (x - 1)(x^2 + 2x + 5x + 10)$
3 Multiply the expression in the second pair of grouping symbols by x and then by -1.	$= (x - 1)(x^2 + 7x + 10)$ $= x^3 + 7x^2 + 10x - x^2 - 7x - 10$
4 Collect like terms.	$= x^3 + 6x^2 + 3x - 10$

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Expanding the product of two linear factors
Searchlight ID: doc-5366

Exercise 17.3 Adding, subtracting and multiplying polynomials

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–c, 3a–c, 4, 5a–c, 6, 7, 9, 12	Questions: 1b–d, 2b–d, 3b–d, 4, 5b, d, f, h, i, 6, 8, 10, 12	Questions: 1c–e, 2c–e, 3–5, 6b, d, f, 7–13

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE1a Simplify each of the following.

a. $(x^4 + x^3 - x^2 + 4) + (x^3 - 14)$

b. $(x^6 + x^4 - 3x^3 + 6x^2) + (x^4 + 3x^2 + 5)$

c. $(x^3 + x^2 + 2x - 4) + (4x^3 - 6x^2 + 5x - 9)$

d. $(2x^4 - 3x^3 + 7x^2 + 9) + (6x^3 + 5x^2 - 4x + 5)$

e. $(15x^4 - 3x^2 + 4x - 7) + (x^5 - 2x^4 + 3x^2 - 4x - 3)$

2. WE1b Simplify each of the following.

a. $(x^4 + x^3 + 4x^2 + 5x + 5) - (x^3 + 2x^2 + 3x + 1)$

b. $(x^6 + x^3 + 1) - (x^5 - x^2 - 1)$

c. $(5x^7 + 6x^5 - 4x^3 + 8x^2 + 5x - 3) - (6x^5 + 8x^2 - 3)$

d. $(10x^4 - 5x^2 + 16x + 11) - (2x^2 - 4x + 6)$

e. $(6x^3 + 5x^2 - 7x + 12) - (4x^3 - x^2 + 3x - 3)$

3. WE2a Expand and simplify each of the following.

a. $x(x + 6)(x + 1)$
b. $x(x - 9)(x + 2)$
c. $x(x - 3)(x + 11)$
d. $2x(x + 2)(x + 3)$
e. $-3x(x - 4)(x + 4)$
f. $5x(x + 8)(x + 2)$
g. $x^2(x + 4)$
h. $-2x^2(7 - x)$
i. $(5x)(-6x)(x + 9)$
j. $-7x(x + 4)^2$

4. WE2b Expand and simplify each of the following.

a. $(x + 7)(x + 2)(x + 3)$
b. $(x - 2)(x + 4)(x - 5)$
c. $(x - 1)(x - 4)(x + 8)$
d. $(x - 1)(x - 2)(x - 3)$
e. $(x + 6)(x - 1)(x + 1)$
f. $(x - 7)(x + 7)(x + 5)$
g. $(x + 11)(x + 5)(x - 12)$
h. $(x + 5)(x - 1)^2$
i. $(x + 2)(x - 7)^2$
j. $(x + 1)(x - 1)(x + 1)$

5. Expand and simplify each of the following.

a. $(x - 2)(x + 7)(x + 8)$
b. $(x + 5)(3x - 1)(x + 4)$
c. $(4x - 1)(x + 3)(x - 3)(x + 1)$
d. $(5x + 3)(2x - 3)(x - 4)$
e. $(1 - 6x)(x + 7)(x + 5)$
f. $3x(7x - 4)(x - 4)(x + 2)$
g. $-9x(1 - 2x)(3x + 8)$
h. $(6x + 5)(2x - 7)^2$
i. $(3 - 4x)(2 - x)(5x + 9)(x - 1)$
j. $2(7 + 2x)(x + 3)(x + 4)$

6. Expand and simplify each of the following.

a. $(x + 2)^3$
b. $(x + 5)^3$
c. $(x - 1)^3$
d. $(x - 3)^4$
e. $(2x - 6)^3$
f. $(3x + 4)^4$

Understanding

7. Simplify $2(ax + b) - 5(c - bx)$.

8. Expand and simplify $(x + a)(x - b)(x^2 - 3bx + 2a)$.

Reasoning

9. If $(x - 3)^4 = ax^4 + bx^3 + cx^2 + dx + e$, nd a, b, c, d and e.

10. Simplify $(2x - 3)^3 - (4 - 3x)^2$.

11. Find the difference in volume between a cube of side $\frac{3(x-1)}{2}$ and a cuboid whose sides are x, $(x + 1)$ and $(2x + 1)$.

Problem solving

12. Find the constants a, b and c if

$$\frac{5x - 7}{(x-1)(x+1)(x-2)} \equiv \frac{a}{(x-1)} + \frac{b}{(x+1)} + \frac{c}{(x-2)}.$$

13. Write $\frac{3x - 5}{(x^2+1)(x-1)}$ in the form $\frac{ax + b}{(x^2+1)} + \frac{c}{(x-1)}$ and hence nd the values of a, b and c.

Re ection

How do you add or subtract polynomials?

17.4 Long division of polynomials

- The reverse of expanding is factorising (expressing a polynomial as a product of its linear factors).
- Before learning how to factorise, you must be familiar with long division of polynomials. You will remember in earlier levels doing long division questions.
- The same process can be used to divide polynomials by polynomial factors.

Consider $(x^3 + 2x^2 - 13x + 10) \div (x - 3)$ or

$$x-3\,\overline{)\,x^3+2x^2-13x+10}$$

x into x^3 goes x^2 times (consider only the leading terms). Write x^2 at the top.

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \end{array}$$

$x^2 \times (x - 3) = x^3 - 3x^2$
Write the $x^3 - 3x^2$.

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;x^3-3x^2 \end{array}$$

Subtract.
$(x^3 - x^3 = 0, 2x^2 - -3x^2 = 5x^2)$

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2 \end{array}$$

Note: Subtracting a negative is the same as changing the sign and adding.

Bring down the $-13x$.

$$\begin{array}{rl} & x^2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \end{array}$$

x into $5x^2$ goes $5x$. Write $+5x$ at the top.

$$\begin{array}{rl} & x^2+5x \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \end{array}$$

$5x \times (x - 3) = 5x^2 - 15x$

Write the $5x^2 - 15x$.

$$\begin{array}{rl} & x^2+5x \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \\ & \;\;\;\;\;\;\;\;\;\;5x^2-15x \end{array}$$

Subtract.

Note: $5x^2 - 5x^2 = 0,\ -13x - -15x = +2x$

$$\begin{array}{rl} & x^2+5x \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \\ & \;\;\;\;\;\;\;\;\;\;\underline{5x^2-15x} \\ & \;2x \end{array}$$

Bring down the 10.

$$\begin{array}{rl} & x^2+\;\;5x \\ x-3 & \overline{)\,x^3+2x^3-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \\ & \;\;\;\;\;\;\;\;\;\;\underline{5x^2-15x} \\ & \;2x+10 \end{array}$$

x into $2x$ goes 2. Write $+2$ at the top.

$$\begin{array}{rl} & x^2+\;\;5x+2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \\ & \;\;\;\;\;\;\;\;\;\;\underline{5x^2-15x} \\ & \;2x+10 \end{array}$$

Write the $2x - 6$.

$$\begin{array}{rl} & x^2+\;\;5x+2 \\ x-3 & \overline{)\,x^3+2x^2-13x+10} \\ & \;\;\underline{x^3-3x^2} \\ & \;\;\;\;\;\;\;\;\;\;5x^2-13x \\ & \;\;\;\;\;\;\;\;\;\;\underline{5x^2-15x} \\ & \;2x+10 \\ & \;2x-6 \end{array}$$

Subtract to get 16.

$$\begin{array}{r}
x^2 + 5x + 2 \leftarrow \text{Quotient} \\
x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\
\underline{x^3 - 3x^2} \qquad\qquad \\
5x^2 - 13x \quad\; \\
\underline{5x^2 - 15x} \quad\; \\
2x + 10 \\
\underline{2x - 6} \\
16 \leftarrow \text{Remainder}
\end{array}$$

Answer: $x^2 + 5x + 2$ remainder 16

WORKED EXAMPLE 3

Perform the following long divisions and state the quotient and remainder.

a $(x^3 + 3x^2 + x + 9) \div (x + 2)$

b $(x^3 - 4x^2 - 7x - 5) \div (x - 1)$

c $(2x^3 + 6x^2 - 3x + 2) \div (x - 6)$

THINK

a 1 Write the question in long division format.

2 Perform the long division process.

3 Write the quotient and remainder.

b 1 Write the question in long division format.

2 Perform the long division process.

3 Write the quotient and remainder.

c 1 Write the question in long division format.

2 Perform the long division process.

3 Write the quotient and remainder.

WRITE

a

$$\begin{array}{r}
x^2 + x - 1 \leftarrow Q \\
x + 2\overline{)x^3 + 3x^2 + x + 9} \\
\underline{x^3 + 2x^2} \qquad\quad \\
x^2 + x \quad\; \\
\underline{x^2 + 2x} \quad\; \\
-x + 9 \\
\underline{-x - 2} \\
11 \leftarrow R
\end{array}$$

Quotient is $x^2 + x - 1$; remainder is 11.

b

$$\begin{array}{r}
x^2 - 3x - 10 \leftarrow Q \\
x - 1\overline{)x^3 - 4x^2 - 7x - 5} \\
\underline{x^3 - x^2} \qquad\qquad \\
-3x^2 - 7x \quad\; \\
\underline{-3x^2 + 3x} \quad\; \\
-10x - 5 \\
\underline{-10x + 10} \\
-15 \leftarrow R
\end{array}$$

Quotient is $x^2 - 3x - 10$; remainder is -15.

c

$$\begin{array}{r}
2x^2 + 18x + 105 \leftarrow Q \\
x - 6\overline{)2x^3 + 6x^2 - 3x + 2} \\
\underline{2x^3 - 12x^2} \qquad\qquad \\
18x^2 - 3x \quad\; \\
\underline{18x^2 - 108x} \quad\; \\
105x + 2 \\
\underline{105x - 630} \\
632 \leftarrow R
\end{array}$$

Quotient is $2x^2 + 18x + 105$; remainder is 632.

WORKED EXAMPLE 4

TI | CASIO

State the quotient and remainder for $(x^3 - 7x + 1) \div (x + 5)$.

THINK

1 Write the question in long division format. Note that there is no x^2 term in this equation. Include $0x^2$ as a 'placeholder'.

2 Perform the long division process.

3 Write the quotient and remainder.

WRITE

$$
\begin{array}{r l}
 & x^2 - 5x + 18 \leftarrow Q \\
x + 5 \overline{)} & x^3 + 0x^2 - 7x + 1 \\
 & \underline{x^3 + 5x^2} \\
 & -5x^2 - 7x \\
 & \underline{-5x^2 - 25x} \\
 & 18x + 1 \\
 & \underline{18x + 90} \\
 & -89 \leftarrow R
\end{array}
$$

Quotient is $x^2 - 5x + 18$; remainder is -89.

WORKED EXAMPLE 5

Find the quotient and the remainder when $x^4 - 3x^3 + 2x^2 - 8$ is divided by the linear expression $x + 2$.

THINK

1 Set out the long division with each polynomial in descending powers of x. If one of the powers of x is missing, include it with 0 as the coef cient.

WRITE

$x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8}$

2 Divide x into x^4 and write the result above.

$$
\begin{array}{r l}
 & x^3 \\
x + 2 \overline{)} & x^4 - 3x^3 + 2x^2 + 0x - 8
\end{array}
$$

3 Multiply the result x^3 by $x + 2$ and write the result underneath.

$$
\begin{array}{r l}
 & x^3 \\
x + 2 \overline{)} & x^4 - 3x^3 + 2x^2 + 0x - 8 \\
 & x^4 + 2x^3
\end{array}
$$

4 Subtract and bring down the remaining terms to complete the expression.

$$
\begin{array}{r l}
 & x^3 - 5x^2 \\
x + 2 \overline{)} & x^4 - 3x^3 + 2x^2 + 0x - 8 \\
 & \underline{-(x^4 + 2x^3)} \\
 & -5x^3 + 2x^2 + 0x - 8
\end{array}
$$

5 Divide x into $-5x^3$ and write the result above.

6 Continue this process to complete the long division.

$$
\begin{array}{r l}
 & x^3 - 5x^2 \\
x + 2 \overline{)} & x^4 - 3x^3 + 2x^2 + 0x - 8 \\
 & \underline{-(x^4 + 2x^3)} \\
 & -5x^3 + 2x^2 + 0x - 8 \\
 & \underline{-(-5x^3 - 10x^2)} \\
 & 12x^2 + 0x - 8 \\
 & \underline{-(12x^2 + 24x)} \\
 & -24x - 8 \\
 & \underline{-(-24x - 48)} \\
 & 40
\end{array}
$$

7 The polynomial $x^3 - 5x^2 + 12x - 24$, at the top, is the quotient.

The quotient is $x^3 - 5x^2 + 12x - 24$.

8 The result of the nal subtraction, 40, is the remainder.

The remainder is 40.

learnON RESOURCES — ONLINE ONLY

Try out this interactivity: Long division of polynomials
Searchlight ID: int-2793

Exercise 17.4 Long division of polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3, 4a, b, 5, 6a–c, 7a–c, 8, 10	Questions: 1e–h, 2e–h, 3a, c, e, 4c, d, 5, 6d–f, 7d–f, 8, 10	Questions: 1g–j, 2e–h, 3b, d, f, 4e, f, 5, 6e–h, 7d–f, 8–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE3a Perform the following long divisions and state the quotient and remainder.

a. $(x^3 + 4x^2 + 4x + 9) \div (x + 2)$
b. $(x^3 + 2x^2 + 4x + 1) \div (x + 1)$
c. $(x^3 + 6x^2 + 3x + 1) \div (x + 3)$
d. $(x^3 + 3x^2 + x + 3) \div (x + 4)$
e. $(x^3 + 4x^2 + 3x + 4) \div (x + 2)$
f. $(x^3 + 6x^2 + 2x + 2) \div (x + 2)$
g. $(x^3 + x^2 + x + 3) \div (x + 1)$
h. $(x^3 + 8x^2 + 5x + 4) \div (x + 8)$
i. $(x^3 + x^2 + 4x + 1) \div (x + 2)$
j. $(x^3 + 9x^2 + 3x + 2) \div (x + 5)$

2. WE3b State the quotient and remainder for each of the following.

a. $(x^3 + 2x^2 - 5x - 9) \div (x - 2)$
b. $(x^3 + x^2 + x + 9) \div (x - 3)$
c. $(x^3 + x^2 - 9x - 5) \div (x - 2)$
d. $(x^3 - 4x^2 + 10x - 2) \div (x - 1)$
e. $(x^3 - 5x^2 + 3x - 8) \div (x - 3)$
f. $(x^3 - 7x^2 + 9x - 7) \div (x - 1)$
g. $(x^3 + 9x^2 + 2x - 1) \div (x - 5)$
h. $(x^3 + 4x^2 - 5x - 4) \div (x - 4)$

3. WE3c Divide the rst polynomial by the second and state the quotient and remainder.

a. $3x^3 - x^2 + 6x + 5, x + 2$
b. $4x^3 - 4x^2 + 10x - 4, x + 1$
c. $2x^3 - 7x^2 + 9x + 1, x - 2$
d. $2x^3 + 8x^2 - 9x - 1, x + 4$
e. $4x^3 - 10x^2 - 9x + 8, x - 3$
f. $3x^3 + 16x^2 + 4x - 7, x + 5$

4. Divide the rst polynomial by the second and state the quotient and remainder.

a. $6x^3 - 7x^2 + 4x + 4, 2x - 1$
b. $6x^3 + 23x^2 + 2x - 31, 3x + 4$
c. $8x^3 + 6x^2 - 39x - 13, 2x + 5$
d. $2x^3 - 15x^2 + 34x - 13, 2x - 7$
e. $3x^3 + 5x^2 - 16x - 23, 3x + 2$
f. $9x^3 - 6x^2 - 5x + 9, 3x - 4$

Understanding

5. State the quotient and remainder for each of the following.

a. $\dfrac{-x^3 - 6x^2 - 7x - 16}{x + 1}$
b. $\dfrac{-3x^3 + 7x^2 + 10x - 15}{x - 3}$
c. $\dfrac{-2x^3 + 9x^2 + 17x + 15}{2x + 1}$
d. $\dfrac{4x^3 - 20x^2 + 23x - 2}{-2x + 3}$

6. WE4 State the quotient and remainder for each of the following.

a. $(x^3 - 3x + 1) \div (x + 1)$
b. $(x^3 + 2x^2 - 7) \div (x + 2)$
c. $(x^3 - 5x^2 + 2x) \div (x - 4)$
d. $(-x^3 - 7x + 8) \div (x - 1)$
e. $(5x^2 + 13x + 1) \div (x + 3)$
f. $(2x^3 + 8x^2 - 4) \div (x + 5)$
g. $(-2x^3 - x + 2) \div (x - 2)$
h. $(-4x^3 + 6x^2 + 2x) \div (2x + 1)$

7. WE5 Find the quotient and the remainder when each polynomial is divided by the linear expression given.

a. $x^4 + x^3 + 3x^2 - 7x, x - 1$
b. $x^4 - 13x^2 + 36, x - 2$
c. $x^5 - 3x^3 + 4x + 3, x + 3$
d. $2x^6 - x^4 + x^3 + 6x^2 - 5x, x + 2$
e. $6x^4 - x^3 + 2x^2 - 4x, x - 3$
f. $3x^4 - 6x^3 + 12x, 3x + 1$

Reasoning

8. Find the quotient and remainder when $ax^2 + bx + c$ is divided by $(x - d)$.

9. A birthday cake in the shape of a cube had side length $(x + p)$ cm. The cake was divided between $(x - p)$ guests. The left-over cake was used for lunch the next day. There were q^3 guests for lunch the next day and each received $c^3\text{cm}^3$ of cake, which was then all nished.

Find q in terms of p and c.

Problem solving

10. When $x^3 - 2x^2 + 4x + a$ is divided by $x - 1$ the remainder is zero. Use long division to determine the value of a.

11. When $2x^2 + ax + b$ is divided by $x - 1$ the remainder is zero but when $2x^2 + ax + b$ is divided by $x - 2$ the remainder is 9. Use long division to determine the value of a and b.

Re ection

Can you think of an alternative way to divide polynomials?

17.5 Polynomial values

- Consider the polynomial $P(x) = x^3 - 5x^2 + x + 1$.
- The value of the polynomial when $x = 3$ is denoted by $P(3)$ and is found by substituting $x = 3$ into the equation in place of x. That is:

$P(3) = (3)^3 - 5(3)^2 + (3) + 1$
$P(3) = 27 - 5(9) + 3 + 1$
$P(3) = 27 - 45 + 4$
$P(3) = -14$

WORKED EXAMPLE 6 TI | CASIO

If $P(x) = 2x^3 + x^2 - 3x - 4$, nd:

a $P(1)$ **b** $P(-2)$ **c** $P(a)$ **d** $P(2b)$ **e** $P(x + 1)$.

THINK	WRITE
a 1 Write the expression.	**a** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with 1.	$P(1) = 2(1)^3 + (1)^2 - 3(1) - 4$
3 Simplify.	$= 2 + 1 - 3 - 4$ $= -4$

b **1** Write the expression.	**b** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with -2.	$P(-2) = 2(-2)^3 + (-2)^2 - 3(-2) - 4$
3 Simplify.	$= 2(-8) + (4) + 6 - 4$ $= -16 + 4 + 6 - 4$ $= -10$
c **1** Write the expression.	**c** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with a.	$P(a) = 2a^3 + a^2 - 3a - 4$
3 No further simpli cation is possible, so stop here.	
d **1** Write the expression.	**d** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with $2b$.	$P(2b) = 2(2b)^3 + (2b)^2 - 3(2b) - 4$
3 Simplify.	$= 2(8b^3) + 4b^2 - 6b + 4$ $= 16b^3 + 4b^2 - 6b + 4$
e **1** Write the expression.	**e** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with $(x + 1)$.	$P(x+1) = 2(x+1)^3 + (x+1)^2 - 3(x+1) - 4$
3 Expand the right-hand side and collect like terms.	$= 2(x+1)(x+1)(x+1) + (x+1)(x+1) - 3(x+1) - 4$ $= 2(x+1)(x^2 + 2x + 1) + x^2 + 2x + 1 - 3x - 3 - 4$ $= 2(x^3 + 2x^2 + x + x^2 + 2x + 1) + x^2 - x - 6$ $= 2(x^3 + 3x^2 + 3x + 1) + x^2 - x - 6$ $= 2x^3 + 6x^2 + 6x + 2 + x^2 - x - 6$ $= 2x^3 + 7x^2 + 5x - 4$

learnON RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Substitution into quadratic equations
Searchlight ID: doc-5367

Complete this digital doc: WorkSHEET 17.1
Searchlight ID: doc-14618

Exercise 17.5 Polynomial values

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2–8, 11	Questions: 1a, e–h, 2–7, 9, 11	Questions: 1a, i–l, 2–12

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE6 If $P(x) = 2x^3 - 3x^2 + 2x + 10$, nd the following.
   a. $P(0)$ b. $P(1)$ c. $P(2)$ d. $P(3)$
   e. $P(-1)$ f. $P(-2)$ g. $P(-3)$ h. $P(a)$
   i. $P(2b)$ j. $P(x + 2)$ k. $P(x - 3)$ l. $P(-4y)$
2. Copy the following table.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a								
b								
c								
d								

Complete columns 2 to 5 of the table for each of the following polynomials.
a. $P(x) = x^3 + x^2 + x + 1$ b. $P(x) = x^3 + 2x^2 + 5x + 2$
c. $P(x) = x^3 - x^2 + 4x - 1$ d. $P(x) = x^3 - 4x^2 - 7x + 3$

Understanding

3. Find the remainder when each polynomial in question **2** is divided by $(x - 1)$ and complete column 6 of the table.
4. Find the remainder when each polynomial in question **2** is divided by $(x - 2)$ and complete column 7 of the table.
5. Find the remainder when each polynomial in question **2** is divided by $(x + 1)$ and complete column 8 of the table.
6. Find the remainder when each polynomial in question **2** is divided by $(x + 2)$ and complete column 9 of the table.
7. Copy and complete:
   a. A quick way of nding the remainder when $P(x)$ is divided by $(x + 8)$ is to calculate ____________.
   b. A quick way of nding the remainder when $P(x)$ is divided by $(x - 7)$ is to calculate ____________.
   c. A quick way of nding the remainder when $P(x)$ is divided by $(x - a)$ is to calculate ____________.

Reasoning

8. If $P(x) = 2(x - 3)^5 + 1$, nd:
   a. $P(2)$ b. $P(-2)$ c. $P(a)$ d. $P(-2a)$.
9. When $x^2 + bx + 2$ is divided by $(x - 1)$, the remainder is $b^2 - 4b + 7$. Find the possible values of b.
10. If $P(x) = -2x^3 - 3x^2 + x + 3$, nd:
    a. $P(a) + 1$ b. $P(a + 1)$.

Problem solving

11. If $P(x) = 3x^3 - 2x^2 - x + c$ and $P(2) = 8P(1)$, nd the value of c.
12. If $P(x) = 5x^2 + bx + c$ and $P(-1) = 12$ while $P(2) = 21$, nd the values of b and c.

Re□ection

Is there a quick way to □nd a remainder when dividing polynomials?

17.6 The remainder and factor theorems

17.6.1 The remainder theorem

- In the previous exercise, you may have noticed that:

  the remainder when $P(x)$ is divided by $(x - a)$ is equal to $P(a)$.

  That is, $R = P(a)$.

  This is called the **remainder theorem**.
- If $P(x) = x^3 + x^2 + x + 1$ is divided by $(x - 2)$, the quotient is $x^2 + 3x + 7$ and the remainder is $P(2)$, which equals 15. That is:

$$(x^3 + x^2 + x + 1) \div (x - 2) = x^2 + 3x + 7 + \frac{15}{x - 2} \quad \text{and}$$
$$(x^3 + x^2 + x + 1) = (x^2 + 3x + 7)(x - 2) + 15$$

- In general, if $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, we can write:

$$P(x) \div (x - a) = Q(x) + \frac{R}{(x - a)} \text{ where } R = P(a)$$
$$\square\ P(x) = (x - a)Q(x) + R$$

17.6.2 The factor theorem

- The remainder when 12 is divided by 4 is zero, since 4 is a factor of 12.
- Similarly, if the remainder (R) when $P(x)$ is divided by $(x - a)$ is zero, then $(x - a)$ is a factor of $P(x)$.
- Since $R = P(a)$, nd a value of a that makes $P(a) = 0$, then $(x - a)$ is a factor.

  If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

  This is called the **factor theorem**.
- Imagine $P(x)$ could be factorised as follows:

  $P(x) = (x - a)Q(x)$, where $Q(x)$ is 'the other' factor of $P(x)$.
- If $P(a) = 0$, $(x - a)$ is a factor.

WORKED EXAMPLE 7

Without actually dividing, nd the remainder when $x^3 - 7x^2 - 2x + 4$ is divided by:

a $x - 3$ **b** $x + 6$.

THINK	WRITE
a 1 Name the polynomial.	**a** Let $P(x) = x^3 - 7x^2 - 2x + 4$.
2 The remainder when $P(x)$ is divided by $(x - 3)$ is equal to $P(3)$.	$R = P(3)$ $= 3^3 - 7(3)^2 - 2(3) + 4$ $= 27 - 7(9) - 6 + 4$ $= 27 - 63 - 6 + 4$ $= -38$
b The remainder when $P(x)$ is divided by $(x + 6)$ is equal to $P(-6)$.	**b** $R = P(-6)$ $= (-6)^3 - 7(-6)^2 - 2(-6) + 4$ $= -216 - 7(36) + 12 + 4$ $= -216 - 252 + 12 + 4$ $= -452$

WORKED EXAMPLE 8 TI | CASIO

$(x - 2)$ is a factor of $x^3 + kx^2 + x - 2$. Find the value of k.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 + kx^2 + x - 2$.
2 The remainder when $P(x)$ is divided by $(x - 2)$ is equal to $P(2) = 0$.	$0 = P(2)$ $= 2^3 + k(2)^2 + 2 - 2$ $0 = 8 + 4k$
3 Solve for k.	$4k = -8$ $k = -2$

Exercise 17.6 The remainder and factor theorems

assesson

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–d, 3a–d, 4, 7a–d	Questions: 1, 2e–h, 3e–h, 4–6, 7d–g, 8	Questions: 1, 2g–j, 3f–h, 4–9

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE7 Without actually dividing, nd the remainder when $x^3 + 3x^2 - 10x - 24$ is divided by:

a. $x - 1$ **b.** $x + 2$ **c.** $x - 3$ **d.** $x + 5$
e. $x - 0$ **f.** $x - k$ **g.** $x + n$ **h.** $x + 3c$.

2. Find the remainder when the rst polynomial is divided by the second without performing long division.

a. $x^3 + 2x^2 + 3x + 4, x - 3$
b. $x^3 - 4x^2 + 2x - 1, x + 1$
c. $x^3 + 3x^2 - 3x + 1, x + 2$
d. $x^3 - x^2 - 4x - 5, x - 1$
e. $2x^3 + 3x^2 + 6x + 3, x + 5$
f. $-3x^3 - 2x^2 + x + 6, x + 1$
g. $x^3 + x^2 + 8, x - 5$
h. $x^3 - 3x^2 - 2, x - 2$
i. $-x^3 + 8, x + 3$
j. $x^3 + 2x^2, x - 7$

Understanding

3. WE8 **a.** The remainder when $x^3 + kx + 1$ is divided by $(x + 2)$ is -19. Find the value of k.

b. The remainder when $x^3 + 2x^2 + mx + 5$ is divided by $(x - 2)$ is 27. Find the value of m.

c. The remainder when $x^3 - 3x^2 + 2x + n$ is divided by $(x - 1)$ is 1. Find the value of n.

d. The remainder when $ax^3 + 4x^2 - 2x + 1$ is divided by $(x - 3)$ is -23. Find the value of a.

e. The remainder when $x^3 - bx^2 - 2x + 1$ is divided by $(x + 1)$ is 0. Find the value of b.

f. The remainder when $-4x^2 + 2x + 7$ is divided by $(x - c)$ is -5. Find a possible whole number value of c.

g. The remainder when $x^2 - 3x + 1$ is divided by $(x + d)$ is 11. Find the possible values of d.

h. The remainder when $x^3 + ax^2 + bx + 1$ is divided by $(x - 5)$ is -14. When the cubic polynomial is divided by $(x + 1)$, the remainder is -2. Find a and b.

4. MC *Note:* There may be more than one correct answer.

a. When $x^3 + 2x^2 - 5x - 5$ is divided by $(x + 2)$, the remainder is:

A. -5 **B.** -2 **C.** 2 **D.** 5

b. Which of the following is a factor of $2x^3 + 15x^2 + 22x - 15$?

A. $(x - 1)$ **B.** $(x - 2)$ **C.** $(x + 3)$ **D.** $(x + 5)$

c. When $x^3 - 13x^2 + 48x - 36$ is divided by $(x - 1)$, the remainder is:

A. -3 **B.** -2 **C.** -1 **D.** 0

d. Which of the following is a factor of $x^3 - 5x^2 - 22x + 56$?

A. $(x - 2)$ **B.** $(x + 2)$ **C.** $(x - 7)$ **D.** $(x + 4)$

5. Find one factor of each of the following cubic polynomials.

a. $x^3 - 3x^2 + 3x - 1$
b. $x^3 - 7x^2 + 16x - 12$
c. $x^3 + x^2 - 8x - 12$
d. $x^3 + 3x^2 - 34x - 120$

Reasoning

6. Prove that each of the following is a linear factor of $x^3 + 4x^2 - 11x - 30$ by substituting values into the cubic function: $(x + 2), (x - 3), (x + 5)$.

7. Without actually dividing, show that the first polynomial is exactly divisible by the second (that is, the second polynomial is a factor of the first).

a. $x^3 + 5x^2 + 2x - 8, x - 1$
b. $x^3 - 7x^2 - x + 7, x - 7$
c. $x^3 - 7x^2 + 4x + 12, x - 2$
d. $x^3 + 2x^2 - 9x - 18, x + 2$
e. $x^3 + 3x^2 - 9x - 27, x + 3$
f. $-x^3 + x^2 + 9x - 9, x - 1$
g. $-2x^3 + 9x^2 - x - 12, x - 4$
h. $3x^3 + 22x^2 + 37x + 10, x + 5$

Problem solving

8. When $x^4 + ax^3 - 4x^2 + b$ and $x^3 - ax^2 - 7x + b$ are each divided by $(x - 2)$, the remainders are 26 and 8 respectively. Find the values of a and b.

9. Both $(x - 1)$ and $(x - 2)$ are factors of $P(x) = x^4 + ax^3 - 7x^2 + bx - 30$. Find the values of a and b and the remaining two linear factors.

Reflection

How are the remainder and factor theorems related?

CHALLENGE 17.1

The remainder when $2x - 1$ is divided into $6x^3 - x^2 + 3x + k$ is the same as when it is divided into $4x^3 - 8x^2 - 5x + 2$. What is the value of k?

17.7 Factorising polynomials

17.7.1 Using long division

- Once one factor of a polynomial has been found (using the factor theorem as in the previous section), long division may be used to find other factors. In the case of a cubic polynomial, one — possibly two — other factors may be found.

WORKED EXAMPLE 9

Use long division to factorise the following.

a $x^3 - 5x^2 - 2x + 24$ **b** $x^3 - 19x + 30$ **c** $-2x^3 - 8x^2 + 6x + 4$

THINK	WRITE
a 1 Name the polynomial.	**a** $P(x) = x^3 - 5x^2 - 2x + 24$

2 Look for a value of x such that $P(x) = 0$. For cubics containing a single x^3, try a factor of the constant term (24 in this case). Try $P(1)$.	$P(1) = 1^3 - 5 \times 1^2 - 2 \times 1 + 24$ $= 1 - 5 - 2 + 24$ $= 18$ $\neq 0$
$P(1) \neq 0$, so $(x - 1)$ is not a factor. Try $P(2)$.	$P(2) = 2^3 - 5 \times 2^2 - 2 \times 2 + 24$ $= 8 - 20 - 4 + 24$ $\neq 0$
$P(2) \neq 0$, so $(x - 2)$ is not a factor. Try $P(-2)$.	$P(-2) = (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24$ $= -8 - 20 + 4 + 24$ $= -28 + 28$ $= 0$
$P(-2)$ does equal 0, so $(x + 2)$ is a factor.	$(x + 2)$ is a factor.
3 Divide $(x + 2)$ into $P(x)$ using long division to nd a quadratic factor.	$$\begin{array}{r} x^2 - 7x + 12 \\ x + 2\overline{)x^3 - 5x^2 - 2x + 24} \\ \underline{x^3 + 2x^2} \qquad\qquad \\ -7x^2 - 2x \qquad \\ \underline{-7x^2 - 14x} \qquad \\ 12x + 24 \\ \underline{12x + 24} \\ 0 \end{array}$$
4 Write $P(x)$ as a product of the two factors found so far.	$P(x) = (x + 2)(x^2 - 7x + 12)$
5 Factorise the quadratic factor if possible.	$P(x) = (x + 2)(x - 3)(x - 4)$
b 1 Name the polynomial. *Note*: There is no x^2 term, so include $0x^2$.	**b** $P(x) = x^3 - 19x + 30$ $P(x) = x^3 + 0x^2 - 19x + 30$
2 Look at the last term in $P(x)$, which is 30. This suggests it is worth trying $P(5)$ or $P(-5)$. Try $P(-5)$. $P(-5) = 0$ so $(x + 5)$ is a factor.	$P(-5) = (-5)^3 - 19 \times (-5) + 30$ $= -125 + 95 + 30$ $= 0$ So $(x + 5)$ is a factor.
3 Divide $(x + 5)$ into $P(x)$ using long division to nd a quadratic factor.	$$\begin{array}{r} x^2 - 5x + 6 \\ x + 5\overline{)x^3 + 0x^2 - 19x + 30} \\ \underline{x^3 + 5x^2} \qquad\qquad \\ -5x^2 - 19x \qquad \\ \underline{-5x^2 - 25x} \qquad \\ 6x + 30 \\ \underline{6x + 30} \\ 0 \end{array}$$
4 Write $P(x)$ as a product of the two factors found so far.	$P(x) = (x + 5)(x^2 - 5x + 6)$
5 Factorise the quadratic factor if possible.	$P(x) = (x + 5)(x - 2)(x - 3)$

c **1** Write the given polynomial.

c Let $P(x) = -2x^3 - 8x^2 + 6x + 4$

2 Take out a common factor of -2. (We could take out $+2$ as the common factor, but taking out -2 results in a positive leading term in the part still to be factorised.)

$= -2(x^3 + 4x^2 - 3x - 2)$

3 Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$. (We have already used P earlier.)

Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$.

4 Evaluate $Q(1)$. $Q(1) = 0$, so $(x - 1)$ is a factor.

$Q(1) = 1 + 4 - 3 - 2$
$= 0$
So $(x - 1)$ is a factor.

5 Divide $(x - 1)$ into $Q(x)$ using long division to find a quadratic factor.

$$\begin{array}{r} x^2 + 5x + 2 \\ x - 1\,\overline{)\,x^3 + 4x^2 - 3x - 2} \\ \underline{x^3 - x^2} \\ 5x^2 - 3x \\ \underline{5x^2 - 5x} \\ 2x - 2 \\ \underline{2x - 2} \\ 0 \end{array}$$

6 Write the original polynomial $P(x)$ as a product of the factors found so far.

$P(x) = -2(x - 1)(x^2 + 5x + 2)$

7 In this case, it is not possible to further factorise $P(x)$.

- *Note:* In these examples, $P(x)$ may have been factorised without long division by finding all three values of x that make $P(x) = 0$ (and hence three factors) and then checking that the three factors multiply to give $P(x)$.

17.7.2 Using short division

- The process of long division can be quite time (and space) consuming. An alternative is short division, which may take a little longer to understand, but is quicker once mastered.
- Consider $P(x) = x^3 + 2x^2 - 13x + 10$. Using the factor theorem, we can find that $(x - 1)$ is a factor of $P(x)$. So, $P(x) = (x - 1)(\ ?\ )$.

  Actually, we know more than this: as $P(x)$ begins with x^3 and ends with $+10$, we could write

  $$P(x) = (x - 1)(x^2 + ? - 10)$$

  The x^2 in the second pair of grouping symbols produces the desired x^3 (the leading term in $P(x)$) when the expressions are multiplied. The -10 in the second pair of grouping symbols produces $+10$ (the last term in $P(x)$) when the expressions are multiplied.
- Imagine expanding this version of $P(x)$. Multiplying x in the first pair of grouping symbols by x^2 in the second would produce x^3, which is what we want, but multiplying -1 in the first pair of grouping symbols by x^2 in the second gives $-1x^2$.

  Since $P(x) = x^3 + 2x^2 - 13x + 10$, we really need $+2x^2$, not $-1x^2$. That is, we need $+3x^2$ more. To get this, the? must be $3x$, because when x in the first pair of grouping symbols is multiplied by $3x$ in the second pair, $+3x^2$ results. That is, we have deduced $P(x) = (x - 1)(x^2 + 3x - 10)$.

  Factorising the expression in the second pair of grouping symbols gives

  $$P(x) = (x - 1)(x + 5)(x - 2).$$
- This procedure, which we will call *short division*, can be confusing at first, but with persistence it can be a quick and easy method for factorising polynomials.
- The following worked example is a repeat of a previous one, but explains the use of short, rather than long, division.

WORKED EXAMPLE 10 TI | CASIO

Use short division to factorise $x^3 - 5x^2 - 2x + 24$.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 - 5x^2 - 2x + 24$.
2 Look for a value of x such that $P(x) = 0$. Try $P(-2)$.	$P(-2) = (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24$ $= -8 - 20 + 4 + 24$ $= -28 + 28$ $= 0$
$P(-2)$ does equal 0, so $(x + 2)$ is a factor.	So $(x + 2)$ is a factor.
3 Look again at the original $P(x) = x^3 - 5x^2 - 2x + 24$. The first term in the grouping symbols must be x^2, and the last term must be 12.	$P(x) = (x + 2)(x^2 + ax + 12)$
4 Imagine the expansion of the expression in step 3. We have x^3 and $2x^2$, but require $-5x^2$. We need an extra $-7x^2$. We get this by inserting a $-7x$ term in the second pair of grouping symbols.	$P(x) = (x + 2)(x^2 - 7x + 12)$
5 Factorise the expression in the second pair of grouping symbols if possible.	$P(x) = (x + 2)(x - 3)(x - 4)$

learnON RESOURCES — ONLINE ONLY

Complete this digital doc: SkillSHEET: Factorising quadratic trinomials
Searchlight ID: doc-5368

Exercise 17.7 Factorising polynomials

assessON

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–d, 3a–d, 4a–c, 5, 6	Questions: 1d–f, 2e–h, 3e–h, 4d–g, 5, 6, 8	Questions: 1d–f, 2i–n, 3g–j, 4h–k, 5–9

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE9 Use long division to factorise each dividend.

a. $x + 1\overline{)x^3 + 10x^2 + 27x + 18}$
b. $x + 2\overline{)x^3 + 8x^2 + 17x + 10}$
c. $x + 9\overline{)x^3 + 12x^2 + 29x + 18}$
d. $x + 1\overline{)x^3 + 8x^2 + 19x + 12}$
e. $x + 3\overline{)x^3 + 14x^2 + 61x + 84}$
f. $x + 7\overline{)x^3 + 12x^2 + 41x + 42}$
g. $x + 2\overline{)x^3 + 4x^2 + 5x + 2}$
h. $x + 3\overline{)x^3 + 7x^2 + 16x + 12}$
i. $x + 5\overline{)x^3 + 14x^2 + 65x + 100}$
j. $x\overline{)x^3 + 13x^2 + 40x}$
k. $x\overline{)x^3 + 7x^2 + 12x}$
l. $x + 5\overline{)x^3 + 10x^2 + 25x}$
m. $x + 1\overline{)x^3 + 6x^2 + 5x}$
n. $x + 6\overline{)x^3 + 6x^2}$

2. **WE9, 10** Factorise the following as fully as possible.

a. $x^3 + x^2 - x - 1$
b. $x^3 - 2x^2 - x + 2$
c. $x^3 + 7x^2 + 11x + 5$
d. $x^3 + x^2 - 8x - 12$
e. $x^3 + 9x^2 + 24x + 16$
f. $x^3 - 5x^2 - 4x + 20$
g. $x^3 + 2x^2 - x - 2$
h. $x^3 - 7x - 6$
i. $x^3 + 3x^2 - 4$
j. $x^3 + x^2 + x + 6$
k. $x^3 + 8x^2 + 17x + 10$
l. $x^3 + x^2 - 9x - 9$
m. $x^3 - x^2 - 8x + 12$
n. $x^3 + 9x^2 - 12x - 160$

Understanding

3. Factorise as fully as possible.

a. $2x^3 + 5x^2 - x - 6$
b. $3x^3 + 14x^2 + 7x - 4$
c. $3x^3 + 2x^2 - 12x - 8$
d. $4x^3 + 35x^2 + 84x + 45$
e. $5x^3 + 9x^2 + 3x - 1$
f. $x^3 + x^2 + x + 1$
g. $4x^3 + 16x^2 + 21x + 9$
h. $6x^3 - 23x^2 + 26x - 8$
i. $10x^3 + 19x^2 - 94x - 40$
j. $7x^3 + 12x^2 - 60x + 16$

4. Factorise as fully as possible.

a. $3x^3 - x^2 - 10x$
b. $4x^3 + 2x^2 - 2x$
c. $3x^3 - 6x^2 - 24x$
d. $-2x^3 - 12x^2 - 18x$
e. $6x^3 - 6x^2$
f. $-x^3 - 7x^2 - 12x$
g. $-x^3 - 3x^2 + x + 3$
h. $-2x^3 + 10x^2 - 12x$
i. $-6x^3 - 5x^2 + 12x - 4$
j. $-5x^3 + 24x^2 - 36x + 16$
k. $-x^5 - x^4 + 21x^3 + 49x^2 - 8x - 60$

Reasoning

5. Factorise $x^4 - 9x^2 - 4x + 12$.
6. Factorise $-x^5 + 6x^4 + 11x^3 - 84x^2 - 28x + 240$.
7. Two of the factors of $x^3 + px^2 + qx + r$ are $(x + a)$ and $(x + b)$. Find the third factor.

Problem solving

8. $(x - 1)$ and $(x - 2)$ are known to be factors of $x^5 + ax^4 - 2x^3 + bx^2 + x - 2$. Find the values of a and b and hence fully factorise this fth-degree polynomial.
9. Factorise $x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$.

Re□ection

Explain the steps in factorising polynomials.

CHALLENGE 17.2

The polynomial $x^4 - 6x^3 + 13x^2 - 12x - 32$ has three factors, one of which is $x^2 - 3x + 8$. What are the other two factors?

17.8 Solving polynomial equations

- A polynomial equation of the form $P(x) = 0$ may be solved by factorising $P(x)$ and applying the Null Factor Law. The solutions are also called zeros. They are the x-intercepts on the graph of $P(x)$. If $P(x)$ is of degree n, expect n zeros.
- The Null Factor Law applies to polynomial equations just as it does for quadratics.
- If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions can be found as follows.
  Let each factor = 0:
  $$x - a = 0 \qquad x - b = 0 \qquad x - c = 0$$
  Solving each of these equations produces the solutions (roots)

$$x = a \qquad x = b \qquad x = c.$$

- If $P(x) = k(lx - a)(mx - b)(nx - c) = 0$, then the solutions can be found as follows. Let each factor $= 0$:

  $$lx - a = 0 \qquad mx - b = 0 \qquad nx - c = 0$$

  Solving each of these equations produces the solutions

  $$x = \frac{a}{l} \qquad x = \frac{b}{m} \qquad x = \frac{c}{n}.$$

  Note: The coef cient k used in this example does not produce a solution because $k \neq 0$.

WORKED EXAMPLE 11 — TI | CASIO

Solve:

a $x^3 = 9x$ **b** $-2x^3 + 4x^2 + 70x = 0$ **c** $2x^3 - 11x^2 + 18x - 9 = 0.$

THINK	WRITE
a 1 Write the equation.	**a** $x^3 = 9x$
2 Rearrange so all terms are on the left.	$x^3 - 9x = 0$
3 Take out a common factor of x.	$x(x^2 - 9) = 0$
4 Factorise the expression in the grouping symbols using the difference of squares rule.	$x(x + 3)(x - 3) = 0$
5 Use the Null Factor Law to solve.	$x = 0, x + 3 = 0$ or $x - 3 = 0$ $x = 0, x = -3$ or $x = 3$
b 1 Write the equation.	**b** $-2x^3 + 4x^2 + 70x = 0$
2 Take out a common factor of $-2x$.	$-2x(x^2 - 2x - 35) = 0$
3 Factorise the expression in the grouping symbols.	$-2x(x - 7)(x + 5) = 0$
4 Use the Null Factor Law to solve.	$-2x = 0, x - 7 = 0$ or $x + 5 = 0$ $x = 0, x = 7$ or $x = -5$
c 1 Name the polynomial.	**c** Let $P(x) = 2x^3 - 11x^2 + 18x - 9.$
2 Use the factor theorem to nd a factor (search for a value a such that $P(a) = 0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.	$P(1) = 2 - 11 + 18 - 9$ $= 0$ So $(x - 1)$ is a factor.
3 Use long or short division to nd another factor of $P(x)$.	$\begin{array}{r} 2x^2 - 9x + 9 \\ x - 1 \overline{) 2x^3 - 11x^2 + 18x - 9} \\ \underline{2x^3 - 2x^2} \qquad\qquad \\ -9x^2 + 18x \qquad \\ \underline{-9x^2 + 9x} \qquad \\ 9x - 9 \\ \underline{9x - 9} \\ 0 \end{array}$ $P(x) = (x - 1)(2x^2 + 9x - 9)$
4 Factorise the quadratic factor.	$P(x) = (x - 1)(2x - 3)(x - 3)$
5 Consider the factorised equation to solve.	For $(x - 1)(2x - 3)(x - 3) = 0$
6 Use the Null Factor Law to solve.	$x - 1 = 0,\ 2x - 3 = 0$ or $x - 3 = 0$ $x = 1, x = \frac{3}{2}$ or $x = 3$

learnON RESOURCES — ONLINE ONLY

- Complete this digital doc: SkillSHEET: Factorising difference of two squares expressions
  Searchlight ID: doc-5369
- Complete this digital doc: SkillSHEET: Solving quadratic equations
  Searchlight ID: doc-5370
- Complete this digital doc: WorkSHEET 17.2
  Searchlight ID: doc-14619

Exercise 17.8 Solving polynomial equations

assessON

Individual Pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3, 4, 5a–d, 6a–c, 7, 10	Questions: 1e–h, 2e–h, 3, 4, 5e–h, 6d–f, 8, 10	Questions: 1i–n, 2i–n, 3, 4, 5e–h, 6d–f, 7–11

To answer questions online and to receive **immediate feedback** and **sample responses** for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Fluency

1. WE11a, b Solve the following.

a. $x^3 - 4x = 0$ **b.** $x^3 - 16x = 0$ **c.** $2x^3 - 50x = 0$
d. $-3x^3 + 81 = 0$ **e.** $x^3 + 5x^2 = 0$ **f.** $x^3 - 2x^2 = 0$
g. $-4x^3 + 8x = 0$ **h.** $12x^3 + 3x^2 = 0$ **i.** $4x^2 - 20x^3 = 0$
j. $x^3 - 5x^2 + 6x = 0$ **k.** $x^3 - 8x^2 + 16x = 0$ **l.** $x^3 + 6x^2 = 7x$
m. $9x^2 = 20x + x^3$ **n.** $x^3 + 6x = 4x^2$

2. WE11c Use the factor theorem to solve the following.

a. $x^3 - x^2 - 16x + 16 = 0$ **b.** $x^3 - 6x^2 - x + 30 = 0$
c. $x^3 - x^2 - 25x + 25 = 0$ **d.** $x^3 + 4x^2 - 4x - 16 = 0$
e. $x^3 - 4x^2 + x + 6 = 0$ **f.** $x^3 - 4x^2 - 7x + 10 = 0$
g. $x^3 + 6x^2 + 11x + 6 = 0$ **h.** $x^3 - 6x^2 - 15x + 100 = 0$
i. $x^3 - 3x^2 - 6x + 8 = 0$ **j.** $x^3 + 2x^2 - 29x + 42 = 0$
k. $2x^3 + 15x^2 + 19x + 6 = 0$ **l.** $-4x^3 + 16x^2 - 9x - 9 = 0$
m. $-2x^3 - 9x^2 - 7x + 6 = 0$ **n.** $2x^3 + 4x^2 - 2x - 4 = 0$

3. MC *Note:* There may be more than one correct answer.
Which of the following is a solution to $x^3 - 7x^2 + 2x + 40 = 0$?
A. 5 **B.** −4 **C.** −2 **D.** 1

4. MC A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solutions are there?
A. 0 **B.** 1 **C.** 2 **D.** 3

Understanding

5. Solve $P(x) = 0$.

a. $P(x) = x^3 + 4x^2 - 3x - 18$ **b.** $P(x) = 3x^3 - 13x^2 - 32x + 12$
c. $P(x) = -x^3 + 12x - 16$ **d.** $P(x) = 8x^3 - 4x^2 - 32x - 20$
e. $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ **f.** $P(x) = -72 - 42x + 19x^2 + 7x^3 - 2x^4$
g. $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ **h.** $P(x) = 4x^4 + 12x^3 - 24x^2 - 32x$

6. Solve each of the following equations.

a. $x^3 - 3x^2 - 6x + 8 = 0$

b. $x^3 + x^2 - 9x - 9 = 0$

c. $3x^3 + 3x^2 - 18x = 0$

d. $2x^4 + 10x^3 - 4x^2 - 48x = 0$

e. $2x^4 + x^3 - 14x^2 - 4x + 24 = 0$

f. $x^4 - 2x^2 + 1 = 0$

Reasoning

7. Solve for a if $x = 2$ is a solution of $ax^3 - 6x^2 + 3x - 4 = 0$.

8. Solve for p if $x = \frac{p}{2}$ is a solution of $x^3 - 5x^2 + 2x + 8 = 0$.

9. Show that it is possible for a cuboid of side lengths x cm, $(x - 1)$ cm and $(x + 2)$ cm to have a volume that is 4 cm^3 less than twice the volume of a cube of side length x cm. Comment on the shape of such a cuboid.

Problem solving

10. Solve for x.

$x^3 + 8 = x(5x - 2)$

11. Solve for z.

$z(z - 1)^3 = -2(z^3 - 5z^2 + z + 3)$

Reflection

Can you predict the number of solutions a polynomial might have?

17.9 Review

17.9.1 Review questions

Fluency

1. Which of the following is *not* a polynomial?

a. $x^3 - \frac{x^2}{3} + 7x - 1$

b. $a^4 + 4a^3 + 2a + 2$

c. $\sqrt{x^2 + 3x + 2}$

d. 5

2. Consider the polynomial $f(x) = -\frac{1}{7}x^4 + x^5 + 3$.

a. What is the degree of $f(x)$?

b. What is the coefficient of x^4?

c. What is the constant term?

d. What is the leading term?

3. The expansion of $(x + 5)(x + 1)(x - 6)$ is:

a. $x^3 - 30$

b. $x^3 + 12x^2 - 31x + 30$

c. $x^3 - 31x - 30$

d. $x^3 + 5x^2 - 36x - 30$

4. $x^3 + 5x^2 + 3x - 9$ is the expansion of:

a. $(x + 3)^3$

b. $x(x + 3)(x - 3)$

c. $(x - 1)(x + 3)^2$

d. $(x - 1)(x + 1)(x + 3)$

5. Expand each of the following.

a. $(x - 2)^2(x + 10)$

b. $(x + 6)(x - 1)(x + 5)$

c. $(x - 7)^3$

d. $(5 - 2x)(1 + x)(x + 2)$

6. Consider the following long division.

$$
\begin{array}{rl}
 & x^2 + x + 2 \\
x - 4\,\big) & \overline{x^3 + 5x^2 + 6x - 1} \\
 & \underline{x^3 + 4x^2} \\
 & \quad x^2 + 6x \\
 & \quad \underline{x^2 + 4x} \\
 & \qquad 2x - 1 \\
 & \qquad \underline{2x + 8} \\
 & \qquad\quad -9
\end{array}
$$

a. The quotient is:
A. -9 **B.** 9 **C.** $x + 4$ **D.** $x^2 + x + 2$
b. The remainder is:
A. -9 **B.** 2 **C.** 4 **D.** $2x - 1$

7. Find the quotient and remainder when the rst polynomial is divided by the second in each case.
a. $x^3 + 2x^2 - 16x - 3,\ x + 2$ **b.** $x^3 + 3x^2 - 13x - 7,\ x - 3$ **c.** $-x^3 + x^2 + 4x - 7,\ x + 1$

8. If $P(x) = x^3 - 3x^2 + 7x + 1$, then $P(-2)$ equals:
a. -34 **b.** -33 **c.** -9 **d.** 9

9. If $P(x) = -3x^3 + 2x^2 + x - 4$, nd:
a. $P(1)$ **b.** $P(-4)$ **c.** $P(2a)$

10. Without dividing, nd the remainder when $x^3 + 3x^2 - 16x + 5$ is divided by $x - 1$.

11. Show that $x + 3$ is a factor of $x^3 - 2x^2 - 29x - 42$.

12. Factorise $x^3 + 4x^2 - 100x - 400$.

13. Solve:
a. $(2x + 1)(x - 3)^2 = 0$ **b.** $x^3 - 9x^2 + 26x - 24 = 0$
c. $x^4 - 4x^3 - x^2 + 16x - 12 = 0$

Problem solving

14. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial where the coef cients are integers. Also let $P(w) = 0$ where w is an integer. Show that w is a factor of a_0.

15. Find the area of a square whose sides are $(2x - 3)$ cm. Expand and simplify your answer. If the area is 16 cm^2, nd x.

16. A window is in the shape of a semicircle above a rectangle. The height of the window is $(6x + 1)$ cm and its width is $(2x + 2)$ cm.
a. Find the total area of the window.
b. Expand and simplify your answer.
c. What is the perimeter of the window?

17. a. Find the volume of a cube of side $(x + 4)$ cm.
b. Find the surface area of the cube.
c. Find the value of x for which the volume and surface are numerically equal.
d. Find x if the numerical value of the volume is 5 less than the numerical value of the surface area.

18. Find the quotient and remainder when $mx^2 + nx + q$ is divided by $(x - p)$.

19. When $P(x)$ is divided by $(x - n)$, the quotient is $x^2 - 2x + n$ and the remainder is $(n + 1)$. Find $P(x)$.

learnON RESOURCES — ONLINE ONLY

Try out this interactivity: Word search
Searchlight ID: int-2874

Try out this interactivity: Crossword
Searchlight ID: int-2875

Try out this interactivity: Sudoku
Searchlight ID: int-3892

Complete this digital doc: Concept map
Searchlight ID: doc-14621

Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

2. Let us now look at the effect that the exponent of each factor has on the shape of the graph of the polynomial. Consider the following functions.

a. $y_1 = (x + 1)(x - 2)(x + 3)$
b. $y_2 = (x + 1)^2(x - 2)(x + 3)$
c. $y_3 = (x + 1)^2(x - 2)^2(x + 3)$
d. $y_4 = (x + 1)^2(x - 2)(x + 3)^3$
e. $y_5 = (x + 1)^3(x - 2)(x + 3)^4$
f. $y_6 = (x + 1)^5(x - 2)^3(x + 3)^2$

i. On a separate sheet of paper, draw a sketch of each of the polynomials, marking in the x-intercepts.
ii. Explain how the power of the factor affects the behaviour of the graph at the x-intercept.

3. Create and draw a sketch of polynomials with the following given characteristics. Complete your graphs on a separate sheet of paper.

a. A first-degree polynomial that:
i. crosses the x-axis
ii. does not cross the x-axis.

b. A second-degree polynomial that:
i. crosses the x-axis twice
ii. touches the x-axis at one and only one point.

c. A third-degree polynomial that crosses the x-axis:
i. three times
ii. twice
iii. once.

d. A fourth-degree polynomial that crosses the x-axis:
i. four times
ii. three times
iii. twice
iv. once.

4. Considering the powers of factors of polynomials, write a general statement outlining the conditions under which the graph of a polynomial will pass through the x-axis or just touch the x-axis.

learnon RESOURCES — ONLINE ONLY

Complete this digital doc: Code puzzle: Who were Australia's first three Prime Ministers?
Searchlight ID: doc-15949

Answers

Topic 17 Polynomials

Exercise 17.2 Polynomials

1. a. 3 **b.** 7 **c.** 2 **d.** 6 **e.** 8 **f.** 5 **g.** 5 **h.** 1 **i.** 6

2. a. x **b.** x **c.** x **d.** x **e.** y **f.** u **g.** e **h.** g **i.** f

3. a. Polynomial 1h **b.** Polynomial 1c
c. Polynomial 1a **d.** Polynomials 1a, 1d and 1e

4. a. N **b.** P **c.** P **d.** N **e.** N **f.** P **g.** P **h.** N **i.** N

5. a. 3 **b.** x **c.** 4 **d.** 5 **e.** $3x$ **f.** $-2x^3$

6. a. 7 **b.** w **c.** 7 **d.** 0 **e.** -9 **f.** $6w^7$

7. a. 4 **b.** 1 **c.** x^4 **d.** 1

8. a. 6 **b.** t **c.** 6
d, e. Check with your teacher.

9. a. 5 units to the right of the origin **b.** 4 units to the right of the origin
c. The body moves towards the origin, then away.

10. a. $4x^3 + 2x^2 - 10x + 18$ **b.** $3x^4 - 3x^3 - x^2 + 7x - 7$ **c.** $5x^3 - 4x^2 - 13x - 6$

11. $a = 4, b = -6$

12. $a = \pm 3, b = \pm 2$

Exercise 17.3 Adding, subtracting and multiplying polynomials

1. a. $x^4 + 2x^3 - x^2 - 10$ **b.** $x^6 + 2x^4 - 3x^3 + 9x^2 + 5$ **c.** $5x^3 - 5x^2 + 7x - 13$
d. $2x^4 + 3x^3 + 12x^2 - 4x + 14$ **e.** $x^5 + 13x^4 - 10$

2. a. $x^4 + 2x^2 + 2x + 4$ **b.** $x^6 - x^5 + x^3 + x^2 + 2$ **c.** $5x^7 - 4x^3 + 5x$
d. $10x^4 - 7x^2 + 20x + 5$ **e.** $2x^3 + 6x^2 - 10x + 15$

3. a. $x^3 + 7x^2 + 6x$ **b.** $x^3 - 7x^2 - 18x$ **c.** $x^3 + 8x^2 - 33x$ **d.** $2x^3 + 10x^2 + 12x$
e. $48x - 3x^3$ **f.** $5x^3 + 50x^2 + 80x$ **g.** $x^3 + 4x^2$ **h.** $2x^3 - 14x^2$
i. $-30x^3 - 270x^2$ **j.** $-7x^3 - 56x^2 - 112x$

4. a. $x^3 + 12x^2 + 41x + 42$ **b.** $x^3 - 3x^2 - 18x + 40$ **c.** $x^3 + 3x^2 - 36x + 32$ **d.** $x^3 - 6x^2 + 11x - 6$
e. $x^3 + 6x^2 - x - 6$ **f.** $x^3 + 5x^2 - 49x - 245$ **g.** $x^3 + 4x^2 - 137x - 660$ **h.** $x^3 + 3x^2 - 9x + 5$
i. $x^3 - 12x^2 + 21x + 98$ **j.** $x^3 + x^2 - x - 1$

5. a. $x^3 + 13x^2 + 26x - 112$ **b.** $3x^3 + 26x^2 + 51x - 20$ **c.** $4x^4 + 3x^3 - 37x^2 - 27x + 9$
d. $10x^3 - 49x^2 + 27x + 36$ **e.** $-6x^3 - 71x^2 - 198x + 35$ **f.** $21x^4 - 54x^3 - 144x^2 + 96x$
g. $54x^3 + 117x^2 - 72x$ **h.** $24x^3 - 148x^2 + 154x + 245$ **i.** $20x^4 - 39x^3 - 50x^2 + 123x - 54$
j. $4x^3 + 42x^2 + 146x + 168$

6. a. $x^3 + 6x^2 + 12x + 8$ **b.** $x^3 + 15x^2 + 75x + 125$
c. $x^3 - 3x^2 + 3x - 1$ **d.** $x^4 - 12x^3 + 54x^2 - 108x + 81$
e. $8x^3 - 72x^2 + 216x - 216$ **f.** $81x^4 + 432x^3 + 864x^2 + 768x + 256$

7. $(2a + 5b)x + (2b - 5c)$

8. $x^4 + (a - 4b)x^3 + (2a - 4ab + 3b^2)x^2 + (2a^2 - 2ab + 3ab^2)x - 2a^2b$

9. $a = 1, b = -12, c = 54, d = -108, e = 81$

10. $8x^3 - 45x^2 + 78x - 43$

11. $\frac{1}{8}(11x^3 - 105x^2 + 73x - 27)$

12. $a = 1, b = -2$ and $c = 1$

13. $a = 1, b = 4$ and $c = -1$

Exercise 17.4 Long division of polynomials

1. a. $x^2 + 2x$, 9 **b.** $x^2 + x + 3, -2$ **c.** $x^2 + 3x - 6, 19$ **d.** $x^2 - x + 5, -17$
e. $x^2 + 2x - 1, 6$ **f.** $x^2 + 4x - 6, 14$ **g.** $x^2 + 1, 2$ **h.** $x^2 + 5, -36$
i. $x^2 - x + 6, -11$ **j.** $x^2 + 4x - 17, 87$

2. **a.** $x^2 + 4x + 3, -3$ **b.** $x^2 + 4x + 13, 48$ **c.** $x^2 + 3x - 3, -11$ **d.** $x^2 - 3x + 7, 5$
e. $x^2 - 2x - 3, -17$ **f.** $x^2 - 6x + 3, -4$ **g.** $x^2 + 14x + 72, 359$ **h.** $x^2 + 8x + 27, 104$

3. **a.** $3x^2 - 7x + 20, -35$ **b.** $4x^2 - 8x + 18, -22$ **c.** $2x^2 - 3x + 3, 7$ **d.** $2x^2 - 9, 35$
e. $4x^2 + 2x - 3, -1$ **f.** $3x^2 + x - 1, -2$

4. **a.** $3x^2 - 2x + 1, 5$ **b.** $2x^2 + 5x - 6, -7$ **c.** $4x^2 - 7x - 2, -3$ **d.** $x^2 - 4x + 3, 8$
e. $x^2 + x - 6, -11$ **f.** $3x^2 + 2x + 1, 13$

5. **a.** $-x^2 - 5x - 2, -14$ **b.** $-3x^2 - 2x + 4, -3$ **c.** $-x^2 + 5x + 6, 9$ **d.** $-2x^2 + 7x - 1, 1$

6. **a.** $x^2 - x - 2, 3$ **b.** $x^2, -7$ **c.** $x^2 - x - 2, -8$ **d.** $-x^2 - x - 8, 0$
e. $5x - 2, 7$ **f.** $2x^2 - 2x + 10, -54$ **g.** $-2x^2 - 4x - 9, -16$ **h.** $-2x^2 + 4x - 1, 1$

7. **a.** $x^3 + 2x^2 + 5x - 2, -2$ **b.** $x^3 + 2x^2 - 9x - 18, 0$
c. $x^4 - 3x^3 + 6x^2 - 18x + 58, -171$ **d.** $2x^5 - 4x^4 + 7x^3 - 13x^2 + 32x - 69, 138$
e. $6x^3 + 17x^2 + 53x + 155, 465$ **f.** $x^3 - \frac{7}{3}x^2 + \frac{7}{9}x + 3\frac{20}{27}, -3\frac{20}{27}$

8. Quotient $= ax + (b + ad)$
Remainder $= Rc + d(b + ad)$

9. $q = \dfrac{2p}{c}$

10. $a = -3$

11. $a = 3, b = -5$

Exercise 17.5 Polynomial values

1. **a.** 10 **b.** 11 **c.** 18
d. 43 **e.** 3 **f.** −22
g. −77 **h.** $2a^3 - 3a^2 + 2a + 10$ **i.** $16b^3 - 12b^2 + 4b + 10$
j. $2x^3 + 9x^2 + 14x + 18$ **k.** $2x^3 - 21x^2 + 74x - 77$ **l.** $-128y^3 - 48y^2 - 8y + 10$

2. to **6.**

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a	4	15	0	−5	4	15	0	−5
b	10	28	−2	−8	10	28	−2	−8
c	3	11	−7	−21	3	11	−7	−21
d	−7	−19	5	−7	−7	−19	5	−7

7. **a.** $P(-8)$ **b.** $P(7)$ **c.** $P(a)$

8. **a.** −1 **b.** −6249 **c.** $2(a - 5)^5 + 1$ **d.** $-2(2a + 5)^5 + 1$

9. $b = 1, 4$

10. **a.** $-2a^3 - 3a^2 + a + 4$ **b.** $-2a^3 - 9a^2 - 11a - 1$

11. $c = 2$

12. $b = -2, c = 5$

Exercise 17.6 The remainder and factor theorems

1. **a.** −30 **b.** 0 **c.** 0
d. −24 **e.** −24 **f.** $k^3 + 3k^2 - 10k - 24$
g. $-n^3 + 3n^2 + 10n - 24$ **h.** $-27c^3 + 27c^2 + 30c - 24$

2. **a.** 58 **b.** −8 **c.** 11 **d.** −9 **e.** −202 **f.** 6 **g.** 158 **h.** −6 **i.** 35 **j.** 441

3. **a.** 6 **b.** 3 **c.** 1 **d.** −2 **e.** 2 **f.** 2 **g.** −5, 2 **h.** $a = -5, b = -3$

4. **a.** D **b.** C, D **c.** D **d.** A, C, D

5. **a.** $(x - 1)$ **b.** $(x - 3)$ or $(x - 2)$ **c.** $(x - 3)$ or $(x + 2)$
d. $(x - 6)$ or $(x + 4)$ or $(x + 5)$

6. Show $P(-2) = 0, P(3) = 0$ and $P(-5) = 0$.

7. a. Show $P(1) = 0$ **b.** Show $P(7) = 0$ **c.** Show $P(2) = 0$ **d.** Show $P(-2) = 0$
e. Show $P(-3) = 0$ **f.** Show $P(1) = 0$ **g.** Show $P(4) = 0$ **h.** Show $P(-5) = 0$

8. $a = 3, b = 2$

9. $a = -5, b = 41, (x + 3)$ and $(x - 5)$

Challenge 17.1

$k = -4$

Exercise 17.7 Factorising polynomials

1. a. $(x + 1)(x + 3)(x + 6)$ **b.** $(x + 1)(x + 2)(x + 5)$ **c.** $(x + 1)(x + 2)(x + 9)$ **d.** $(x + 1)(x + 3)(x + 4)$
e. $(x + 3)(x + 4)(x + 7)$ **f.** $(x + 2)(x + 3)(x + 7)$ **g.** $(x + 1)^2(x + 2)$ **h.** $(x + 2)^2(x + 3)$
i. $(x + 4)(x + 5)^2$ **j.** $x(x + 5)(x + 8)$ **k.** $x(x + 3)(x + 4)$ **l.** $x(x + 5)^2$
m. $x(x + 1)(x + 5)$ **n.** $x^2(x + 6)$

2. a. $(x - 1)(x + 1)^2$ **b.** $(x - 2)(x - 1)(x + 1)$ **c.** $(x + 1)^2(x + 5)$ **d.** $(x - 3)(x + 2)^2$
e. $(x + 1)(x + 4)^2$ **f.** $(x - 5)(x - 2)(x + 2)$ **g.** $(x - 1)(x + 1)(x + 2)$ **h.** $(x - 3)(x + 1)(x + 2)$
i. $(x - 1)(x + 2)^2$ **j.** $(x + 2)(x^2 - x + 3)$ **k.** $(x + 1)(x + 2)(x + 5)$ **l.** $(x - 3)(x + 1)(x + 3)$
m. $(x - 2)^2(x + 3)$ **n.** $(x - 4)(x + 5)(x + 8)$

3. a. $(2x + 3)(x - 1)(x + 2)$ **b.** $(3x - 1)(x + 1)(x + 4)$ **c.** $(3x + 2)(x - 2)(x + 2)$ **d.** $(4x + 3)(x + 3)(x + 5)$
e. $(5x - 1)(x + 1)^2$ **f.** $(x + 1)(x^2 + 1)$ **g.** $(x + 1)(2x + 3)^2$ **h.** $(x - 2)(2x - 1)(3x - 4)$
i. $(x + 4)(2x - 5)(5x + 2)$ **j.** $(7x - 2)(x - 2)(x + 4)$

4. a. $x(x - 2)(3x + 5)$ **b.** $2x(x + 1)(2x - 1)$ **c.** $3x(x - 4)(x + 2)$ **d.** $-2x(x + 3)^2$
e. $6x^2(x - 1)$ **f.** $-x(x + 4)(x + 3)$ **g.** $-(x - 1)(x + 1)(x + 3)$ **h.** $-2x(x - 3)(x - 2)$
i. $-(x + 2)(2x - 1)(3x - 2)$ **j.** $-(x - 2)^2(5x - 4)$ **k.** $-(x - 1)(x + 3)(x - 5)(x + 2)^2$

5. $(x - 1)(x + 2)(x + 2)(x - 3)$

6. $-(x - 2)(x + 2)(x + 3)(x - 4)(x - 5)$

7. $(x - p + (a + b))$

8. $a = -2, b = 4, (x - 1)^2\ (x + 1)^2\ (x - 2)$

9. $x(x - 1)(x + 1)(x - 2)(x - 3)$

Challenge 17.2

The other two factors are $(x - 4)$ and $(x + 1)$.

Exercise 17.8 Solving polynomial equations

1. a. $-2, 0, 2$ **b.** $-4, 0, 4$ **c.** $-5, 0, 5$ **d.** 3 **e.** $-5, 0$ **f.** $0, 2$
g. $-\sqrt{2}, 0, \sqrt{2}$ **h.** $-\frac{1}{4}, 0$ **i.** $0, \frac{1}{5}$ **j.** $0, 2, 3$ **k.** $0, 4$ **l.** $-7, 0, 1$
m. $0, 4, 5$ **n.** 0

2. a. $-4, 1, 4$ **b.** $-2, 3, 5$ **c.** $-5, 1, 5$ **d.** $-4, -2, 2$ **e.** $-1, 2, 3$ **f.** $-2, 1, 5$
g. $-3, -2, -1$ **h.** $-4, 5$ **i.** $-2, 1, 4$ **j.** $-7, 2, 3$ **k.** $-6, -\frac{1}{2}, -1$ **l.** $-\frac{1}{2}, \frac{3}{2}, 3$
m. $-3, -2, \frac{1}{2}$ **n.** $-2, -1, 1$

3. A, C

4. B

5. a. $-3, 2$ **b.** $-2, \frac{1}{3}, 6$ **c.** $-4, 2$ **d.** $-1, \frac{5}{2}$ **e.** $-4, -2, 1, 3$ **f.** $-2, -\frac{3}{2}, 3, 4$
g. $-3, -2, 1, 2$ **h.** $-4, -1, 0, 2$

6. a. $-2, 1, 4$ **b.** $-3, -1, 3$ **c.** $-3, 0, 2$ **d.** $-4, -3, 0, 2$ **e.** $-2, \frac{3}{2}, 2$ **f.** $-1, 1$

7. 3.75

8. $-2, 4, 8$

9. Proof — check with your teacher.

10. $x = -1, 4$ and 2

11. $z = -1, 1, -2$ and 3

17.9 Review

1. C

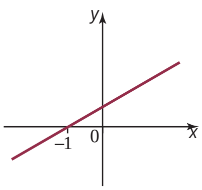
2. a. 5 **b.** $-\frac{1}{7}$ **c.** 3 **d.** x^5

3. C
4. C
5. **a.** $x^3 + 6x^2 - 36x + 40$ **b.** $x^3 + 10x^2 + 19x - 30$ **c.** $x^3 - 21x^2 + 147x - 343$ **d.** $-2x^3 - x^2 + 11x + 10$
6. **a.** D **b.** A
7. **a.** $x^2 - 16,\ 29$ **b.** $x^2 + 6x + 5,\ 8$ **c.** $-x^2 + 2x + 2,\ -9$
8. B
9. **a.** -4 **b.** 216 **c.** $-24a^3 + 8a^2 + 2a - 4$
10. -7
11. Show $P(-3) = 0$.
12. $(x - 10)(x + 4)(x + 10)$
13. **a.** $-\frac{1}{2},\ 3$ **b.** 2, 3, 4 **c.** $-2,\ 1,\ 2,\ 3$
14. Teacher to check.
    For example, given $P(x) = x^3 - x^2 - 34x - 56$ and $P(7) = 0$ □ $(x - 7)$ is a factor and 7 is a factor of 56.
15. $4x^2 - 12x + 9;\ x = -\frac{1}{2},\ \frac{7}{2}$
16. **a, b** Area $= (\frac{1}{2}\pi + 10)x^2 + (\pi + 10)x + \frac{\pi}{2}$
    c Perimeter $= (12 + \pi)x + (2 + \pi)$
17. **a.** $(x + 4)^3$
    b. $6(x + 4)^2$
    c. $x = 2$
    d. $-3,\ \dfrac{-3 + 3\sqrt{5}}{2},\ \dfrac{-3 - 3\sqrt{5}}{2}$
18. $mx + (n + mp);\ q + p(n + mp)$
19. $x^3 - (2 + n)x^2 + 3nx - (n^2 - n - 1)$

Investigation — Rich task

1. **a. i.** 1

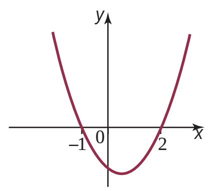
ii.



iii. The graph is linear and crosses the x-axis once (at $x = -1$).

b. i. 2

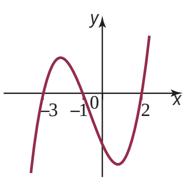
ii.



iii. The graph is quadratic and crosses the x-axis twice (at $x = -1$ and $x = 2$).

c. i. 3

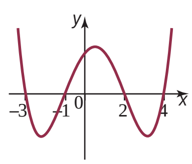
ii.



iii. The graph is a curve and crosses the x-axis 3 times (at $x = -1$, $x = 2$ and $x = -3$).

d. i. 4

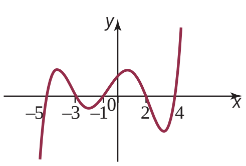
ii.



iii. The graph is a curve and crosses the x-axis 4 times (at $x = -1$, $x = 2$, $x = -3$ and $x = 4$).

e. i. 5

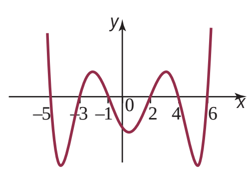
ii.



iii. The graph is a curve and crosses the x-axis 5 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$ and $x = -5$).

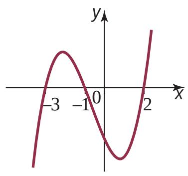
f. i. 6

ii.



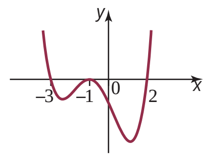
iii. The graph is a curve and crosses the x-axis 6 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$, $x = -5$ and $x = 6$).

2. a. i.



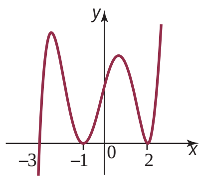
ii. Each factor is raised to the power 1. The polynomial is of degree 3 and the graph crosses the *x*-axis in 3 places (-3, -1 and 2).

b. i.



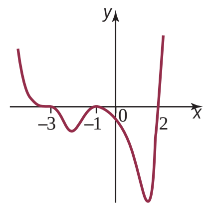
ii. The factor $(x + 1)$ is raised to the power 2 while the other two factors are raised to the power 1. The power 2 causes the curve not to cross the *x*-axis at $x = -1$ but to be curved back on itself.

c. i.



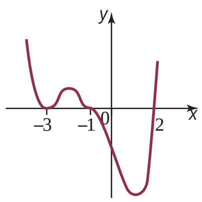
ii. The power 2 on the two factors $(x + 1)$ and $(x - 2)$ causes the curve to be directed back on itself and not to cross the *x*-axis at those two points ($x = -1$ and $x = 2$).

d. i.



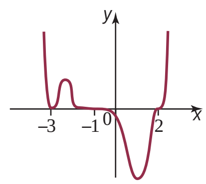
ii. The power 3 on the factor $(x + 3)$ causes the curve to run along the axis at that point then to cross the axis (at $x = -3$).

e. i.



ii. The power 3 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis. The power 4 on the factor $(x + 3)$ causes the curve to be directed back on itself without crossing the axis at $x = -3$.

f. i.



ii. The power 5 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis.

3. Answers will vary. Teacher to check. Possible answers could be as follows.

a. **i.** $y = 3x + 2$ **ii.** $y = 4$

b. **i.** $y = (x + 1)(x + 2)$ **ii.** $y = (x + 1)^2$

c. **i.** $y = (x + 1)(x + 2)(x + 3)$ **ii.** Not possible **iii.** $y = (x + 1)^2(x + 2)$

d. **i.** $y = (x + 1)(x + 2)(x + 3)(x + 4)$ **ii.** Not possible

iii. $y = (x + 1)^2(x + 2)(x + 3)$, $y = (x + 1)^3(x + 2)$ **iv.** Not possible

4. If the power of the factor of a polynomial is an odd integer, the curve will pass through the *x*-axis. If the power is 1, the curve passes straight through. If the power is 3, 5 . . ., the curve will run along the *x*-axis before passing through it. On the other hand, an even power of a factor causes the curve to just touch the *x*-axis then move back on the same side of the *x*-axis.