The power theory\(^1\) of electric circuits in its present form is the result of research of a few generations of scientists and electrical engineers. The term power theory is often used in phrases such as Fryze’s power theory, the \(p-q\) power theory or the power theory based on the current’s physical components theory, etc. In this context it means the suggested interpretation of power phenomena taking place in electric circuits, the definition of quantities related to them and the mathematical relations. The term power theory of electric circuits can also be understood as the general state of knowledge about their properties. In this case it would be the collection of real statements and interpretations, definitions and formulas describing these properties. The power theory is developing for two main reasons. The first reason is cognitive. In relation to electric circuits the power theory is searching for an answer to the question: why does an electrical load usually require the apparent power of the supply source to be bigger than the active power? This question is strongly connected with the need for an interpretation of power phenomena in electric circuits. The second reason is practical. The power theory is trying to answer the question: how can the apparent power of the supply source be reduced without decreasing the active power of an electrical load? These two deceptively simple questions turn out to be exceptionally difficult. In spite

\(^1\)In this case the term power theory means the state of knowledge about the power properties of electric circuits. The power theory understood in that way is a collective result of the intellectual work of those who contribute to explaining the power properties of electric circuits [16],[24].

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of the fact that it has been over a hundred years since the first power theory questions were asked, some answers are still controversial [16],[24].

A3.1 CONVENTIONAL CONCEPTS OF THE POWER PROPERTIES OF ELECTRIC CIRCUITS

This annex presents a basic review of the simplest cases from the point of view of power properties (power phenomena) which are considered in single-phase and three-phase electric circuits. This will remind us about the way in which the commonly used quantities in electrotechnics were defined, such as the instantaneous power \( p(t) \), the active power \( P \), the reactive power \( Q \), the apparent power \( S \), the distortion (harmonic) power \( H \), the deformation power \( D \), power factors \( DPF \) and \( PF \), etc. This kind of description of power properties (power phenomena) of electric circuits is defined as conventional and is valid for the steady-state analysis.

A3.1.1 Single-Phase Circuits

A3.1.1.1 Sinusoidal Waveform of Supply Voltage and Linear Load

Figure A3.1 shows the sinusoidal waveforms of supply voltage \( u(t) \) and load current \( i(t) \) in a single-phase electric circuit with linear (resistance–inductive) load under steady-state operation.

![Waveforms of supply voltage, load current, and instantaneous power](image)

**Figure A3.1** Waveforms of: supply voltage \( u(t) \), load current \( i(t) \) and instantaneous power \( p(t) \) for an a.c. circuit with a linear (resistance–inductive) load under steady-state operation
If the linear load is supplied by sinusoidal voltage $u(t)$ with the angular frequency $\omega$:

$$u(t) = \sqrt{2}U \sin(\omega t)$$  \hspace{1cm} (A3.1)

then load current $i(t)$ can be expressed by the following relation (Figure A3.1):

$$i(t) = \sqrt{2}I \sin(\omega t - \varphi)$$  \hspace{1cm} (A3.2)

where $U$ is the r.m.s. value of supply voltage $u(t)$; $I$ is the r.m.s. value of load current $i(t)$; and $\varphi$ is the phase angle between voltage $u(t)$ and current $i(t)$ ($\varphi = \angle(U, I)$).

The instantaneous power $p(t)$ is defined as the rate of electric energy flow $w(t)$ from the supply source to the load and it is equal to the product of $u(t)$ and $i(t)$, which is

$$p(t) = \frac{dw(t)}{dt} = u(t)i(t) = 2UI \sin(\omega t) \sin(\omega t - \varphi)$$  \hspace{1cm} (A3.3)

Relation (A3.3) can be transformed into the following:

$$p(t) = UI \cos(\varphi) - UI \cos(2\omega t - \varphi)$$  \hspace{1cm} (A3.4)

$$p(t) = UI \cos(\varphi)(1 - \cos(2\omega t)) - UI \sin(\varphi) \sin(2\omega t).$$  \hspace{1cm} (A3.5)

Figure A3.1 shows the waveform of instantaneous power $p(t)$. It is a periodic waveform of period $T/2$, where $T$ is the common period of signals $u(t)$ and $i(t)$. In relations (A3.4) and (A3.5) are components with constant value for given $\varphi$, components with double angular frequency $2\omega t$, components which always take the value greater than or equal to 0 and components of which the average value always equals zero. On the basis of (A3.4) and (A3.5) the following quantities, commonly known and used in electrotechnics to describe the power properties of electric circuits, are defined:

$$P = UI \cos(\varphi) \quad \text{active power,}$$  \hspace{1cm} (A3.6)

$$Q = UI \sin(\varphi) \quad \text{reactive power,}$$  \hspace{1cm} (A3.7)

$$S = UI \quad \text{apparent power.}$$  \hspace{1cm} (A3.8)

Using (A3.4)–(A3.8) the instantaneous power $p(t)$ can be shown to be

$$p(t) = S \cos(\varphi) - S \cos(2\omega t - \varphi) = P - P_{osc}(t)$$  \hspace{1cm} (A3.9)

$$p(t) = S \cos(\varphi)(1 - \cos(2\omega t)) - S \sin(\varphi) \sin(2\omega t) = P(1 - \cos(2\omega t)) - Q \sin(2\omega t) = p_a(t) - p_b(t)$$  \hspace{1cm} (A3.10)

The waveforms of separated components of instantaneous power $p(t)$ are shown in Figure A3.2.

According to the definition, the active power $P$ equals the average value over one period $T$ of the instantaneous power $p(t)$ (see Equation (A3.11)). It is also equal to the d.c. component of the waveform of $p(t)$ (see Equations (A3.4), (A3.6) and (A3.9)):

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} p(t)dt = UI \cos(\varphi)$$  \hspace{1cm} (A3.11)
Figure A3.2 Waveforms of separated components of instantaneous power $p(t)$
In both theory and practice the active power is quite significant because it is reliable enough to define electric energy provided from the source to the load and transformed into other forms in it, such as heat energy, mechanical energy, light energy, etc., so it is an important parameter for the productive processes – its dimension is watt, W. The reactive power \( Q \) is an amplitude of the component \( p_b(t) \) from the relations in Equations (A3.5), (A3.7) and (A3.10) – its dimension is volt-ampere reactive, VAr. If the voltage of the supply source \( U \) is known, the reactive power \( Q \) and the active power \( P \) define the r.m.s. value of the load current \( I \) (see Equation (A3.12)). Compensation of reactive power to the value \( Q = 0 \) VAr reduces the r.m.s. value of load current \( I \) to its minimal value and eliminates the component \( p_b(t) \) (it eliminates the power oscillation):

\[
I = \sqrt{\left( \frac{P}{U} \right)^2 + \left( \frac{Q}{U} \right)^2}
\]

(A3.12)

Besides the active power \( P \) and the reactive power \( Q \) there is also the term of apparent power \( S \) – its dimension is volt-ampere, VA. It is the amplitude of the component \( p_{osc}(t) \) of the instantaneous power \( p(t) \) (see Equations (A3.4), (A3.8) and (A3.9)). It gives information on the biggest value of the active power that can be obtained from the supply source with the given supply voltage \( U \) and the load current \( I \). In the above case the power equation (A3.13) is fulfilled:

\[
S^2 = P^2 + Q^2
\]

(A3.13)

In Figure A3.3 a so-called power triangle is shown, whereas Equation (A3.14) shows the relation for the complex power \( \overline{S} \). The magnitude of the complex power \( \overline{S} \) is equal to the apparent power \( S \):

\[
\overline{S} = U \cdot I = U e^{j\alpha} (I e^{j\beta})^* = \overline{Se^{j(\alpha-\beta)}} = S e^{j\varphi} = S \cos(\varphi) + jS \sin(\varphi) = P + jQ
\]

(A3.14)

where a bar and a bar with an asterisk denote respectively a complex number and a conjugate complex number.

The parameter \( \cos(\varphi) \), called the power factor, is defined as

\[
\cos(\varphi) = \frac{P}{S} = DPF
\]

(A3.15)

Figure A3.3  Graphical representation of power components – power triangle
It gives information on the extent to which the apparent power \( S \) is used, e.g. \( \cos(\varphi) = 0.5 \) means that only half of the apparent power \( S \) is used as the active power \( P \). The parameter \( \cos(\varphi) \) is also called a displacement power factor (DPF) or the power factor in the first-harmonic domain.

**A3.1.1.2 Sinusoidal Waveform of the Supply Voltage and Non-linear Load**

The following consideration assumes that the voltage of the supply source is sinusoidal, while the current waveform will be non-sinusoidal (distorted) because of the non-linear character of the load (Figure A3.4). Therefore, the equation of load current is as follows:

\[
i(t) = \sum_{n=1}^{\infty} \sqrt{2} I_n \sin(n \omega t - \varphi_n)
\]

(A3.16)

where \( n \) is the harmonic order; \( I_n \) is the r.m.s. value of the \( n \)th harmonic of current \( i(t) \); and \( \varphi_n \) is the phase angle between the \( n \)th voltage \( u(t) \) and current \( i(t) \) harmonics, \( \varphi_n = \angle(U_n, I_n) \).

The r.m.s. value of the distorted load current \( i(t) \) can be expressed as

\[
I = \sqrt{I_1^2 + I_2^2 + I_3^2 + I_4^2 + \cdots} = \sqrt{\sum_{n=2}^{\infty} I_n^2} = \sqrt{\frac{1}{T} \int_{0}^{T} i^2(t) dt}
\]

(A3.17)

---

**Figure A3.4** Waveforms of: voltage \( u(t) \), current \( i(t) \) and instantaneous power \( p(t) \) for an a.c. circuit with a non-linear load under steady-state operation
In that case the equation of instantaneous power \( p(t) \) is as follows:

\[
p(t) = UI_{(1)} \cos(\varphi_{(1)}) (1 - \cos(2\omega t)) - UI_{(1)} \sin(\varphi_{(1)}) \sin(2\omega t)
\]

\[
+ \sum_{n=2}^{\infty} 2UI_{(n)} \sin(\omega t) \sin(n\omega t - \varphi_{(n)}) 
\]

(A3.18)

In relations (A3.17) and (A3.18) the component of the first harmonic (the fundamental frequency) – denoted by subscript \((1)\) – was separated. The definitions of active power \( P \) (Equation (A3.6)), reactive power \( Q \) (Equation (A3.7)) were specified in the first-harmonic domain and do not change, therefore:

\[
P_{(1)} = UI_{(1)} \cos(\varphi_{(1)}) = P \quad \text{active power} \quad \text{(A3.19)}
\]

\[
Q_{(1)} = UI_{(1)} \sin(\varphi_{(1)}) = Q \quad \text{reactive power} \quad \text{(A3.20)}
\]

However, the equation of apparent power \( S \) changes, as follows:

\[
S = UI = U \sqrt{I_{(1)}^2 + \sum_{n=2}^{\infty} I_{(n)}^2} \quad \text{apparent power} \quad \text{(A3.21)}
\]

Therefore the power equation takes the form of

\[
S^2 = U^2 I^2 = U^2 I_{(1)}^2 + U^2 \sum_{n=2}^{\infty} I_{(n)}^2 = S_{(1)}^2 + H^2 = P_{(1)}^2 + Q_{(1)}^2 + H^2 = P_{(1)}^2 + D^2 
\]

(A3.22)

In this way in Equation (A3.22) the apparent power in the first-harmonic domain \( S_{(1)} \), the distortion (harmonic) power \( H \) and the deformation power \( D \) were separated. Figure A3.5 shows the so-called power tetrahedron.

Therefore on the basis of Equations (A3.17), (A3.21) and (A3.22) the following was obtained:

\[
S_{(1)} = UI_{(1)} \quad \text{apparent power in the first harmonic domain} \quad \text{(A3.23)}
\]

\[
H = U \sum_{n=2}^{\infty} I_{(n)} \quad \text{distortion (harmonic) power} \quad \text{(A3.24)}
\]

\[
D = \sqrt{Q_{(1)}^2 + H^2} \quad \text{deformation power} \quad \text{(A3.25)}
\]
In this case of non-linear load the power factor, denoted as \( PF = \cos(\psi) \), was defined as
\[
PF = \frac{P_{(1)}}{S} = \frac{P_{(1)}}{\sqrt{S_{(1)}^2 + H^2}} = \frac{P_{(1)}}{\sqrt{P_{(1)}^2 + Q_{(1)}^2 + H^2}}
\]
(A3.26)
\[
= \cos(\psi) = \cos(\varphi_{(1)}) \cos(\gamma)
\]
whereas the equation of the \( DPF \) (Equation (A3.15)) does not change and is associated only with the first harmonic domain:
\[
DPF = \cos(\varphi_{(1)}) = \frac{P_{(1)}}{S_{(1)}}
\]
(A3.27)
Between the factors \( PF \) and \( DPF \) the following relation exists:
\[
PF = \frac{P_{(1)}}{S} = \frac{UI_{(1)} \cos(\varphi_{(1)})}{UI} = \frac{I_{(1)}}{I} \cos(\varphi_{(1)}) = \frac{1}{\sqrt{1 + THD_i^2}} DPF
\]
(A3.28)
where
\[
THD_i = \sqrt{\sum_{n=2}^{n_{\text{min}}} \frac{I_{(n)}^2}{I_{(1)}^2}}
\]
is the total harmonic distortion of \( i(t) \).

**A3.1.2 Three-phase circuits**

Consider the three-phase, three-wire electric circuit shown in Figure A3.6.

Phase voltages and currents are expressed in the form of column vectors:
\[
u = u(t) = [u_R, u_S, u_T]^T \quad \text{and} \quad i = i(t) = [i_R, i_S, i_T]^T
\]
(A3.29)

![Figure A3.6 Phase voltages and currents at the cross-section R–S–T](image-url)
Figure A3.7 shows the sinusoidal symmetrical waveforms of supply voltage $u(t)$ and load current $i(t)$ in the three-phase electric circuit with linear load. Therefore the simplest case from the point of view of power phenomena which take place in three-phase electric circuits is considered.

Figure A3.7  Waveforms of: supply voltage $u(t)$, load current $i(t)$ and phase instantaneous powers $p_R(t)$, $p_S(t)$, $p_T(t)$ and instantaneous power $p_{3f}(t)$
Referring to the definitions which were given in Section A3.1.1.1, the equation of instantaneous power \( p_{3f}(t) \) of the three-phase circuit can take the form

\[
p_{3f}(t) = p_R(t) + p_S(t) + p_T(t) = u_R(t)i_R(t) + u_S(t)i_S(t) + u_T(t)i_T(t)
\]  
(A3.30)

In this case the waveform of instantaneous power \( p_{3f}(t) \) is constant, in contrast to the single-phase circuit (Figure A3.1, Figure A3.2):

\[
p_{3f}(t) = 3P = P_{3f}
\]  
(A3.31)

The relations of powers of the three-phase circuit take the form

\[
P_R = U_RI_R \cos(\varphi) = P_S = P_T = P \Rightarrow P_{3f} = 3P
\]  
(A3.32)

\[
Q_R = U_RI_R \sin(\varphi) = Q_S = Q_T = Q \Rightarrow P_{3f} = 3P
\]  
(A3.33)

\[
S_R = S_{R(1)} = U_RI_R = S_S = S_T = S \Rightarrow S_{3f} = S_{3f(1)} = 3S
\]  
(A3.34)

and power factors \( PF_{3f} \) and \( DPF_{3f} \) take the form

\[
PF_{3f} = DPF_{3f} = \frac{P_{3f}}{S_{3f}}
\]  
(A3.35)

In the above-mentioned case the interpretation, definitions and equations describing these properties for single-phase circuits (Section A3.1.1.1) with sinusoidal supply voltage and current were applied.

However, in three-phase circuits a phenomenon occurs which is not present in single-phase circuits, namely the asymmetry of waveforms of supply voltages \( u(t) \) and/or load currents \( i(t) \). Considering only the asymmetry of load currents of resistive load causes the occurrence of a phase shift between the voltages and currents in three-phase, three-wire electric circuits (despite the lack of passive elements!). Therefore, the displacement power factor \( DPF_{3f} \) equals a value less than 1, \( DPF_{3f} < 1 \). Figure A3.8 shows the waveforms of voltages \( u(t) \) and currents \( i(t) \) in three-phase, three-wire electric circuits with balanced (Figure A3.8a) and unbalanced (Figure A3.8b) resistive load.

Note in Figure A3.8(b) that the instantaneous power \( p_{3f}(t) \) is no longer constant. In this situation the interpretation, definitions and formulas for single-phase circuits proposed in Section A3.1.1.1 do not apply.

Figure A3.9 shows the sinusoidal symmetrical waveforms of supply voltages \( u(t) \) and load current \( i(t) \) in a three-phase electric circuit with symmetrical non-linear load.

The presence of distorted phase load currents results in the fact that instantaneous power \( p_{3f}(t) \) is no longer constant. However, in this case it is possible to use the interpretation, definitions and formulas for single-phase circuits proposed in Section A3.1.1.2. Then the quantities in Equations (A3.19)–(A3.21) and (A3.23)–(A3.25) must be multiplied by 3, similar to Equations (A3.32)–(A3.34). This would not be possible if the waveforms of supply voltages \( u(t) \) and/or load currents \( i(t) \) were asymmetrical.

These presented considerations showed the lack of a conventional approach to the interpretation, definitions and formulas of power properties of electric circuits. This was
Figure A3.8  Waveforms of: supply voltage $u(t)$, load current $i(t)$ and instantaneous power $p_{3f}(t)$ in the three-phase, three-wire electric circuit with balanced (a) and unbalanced (b) resistive load.
Figure A3.9 Waveforms of: supply voltage $u(t)$, load current $i(t)$, phase instantaneous powers $p_R(t)$, $p_S(t)$, $p_T(t)$ and instantaneous power $p_{3f}(t)$ especially apparent in the case of three-phase circuits with asymmetrical waveforms of voltages and/or currents. Moreover, it should be emphasized that this kind of approach does not take into consideration the distorted supply voltage and is valid only for steady-state analysis.
A3.1.2.1 Apparent Power in Three-phase Circuits

It is worth noting that in electrotechnics three definitions are used to calculate the apparent power of three-phase electric circuits, namely

\[ S_A = U_R \cdot I_R + U_S \cdot I_S + U_T \cdot I_T \]  
\[ S_G = \sqrt{P_{3f}^2 + Q_{3f}^2} \]  
\[ S_B = \sqrt{U_R^2 + U_S^2 + U_T^2 \cdot I_R^2 + I_S^2 + I_T^2} \]

Buchholz’s apparent power \( \Rightarrow PF_B = \frac{P_\Sigma}{S_B} \) (A3.38)

where \( P_\Sigma = P_R + P_S + P_T \) total active power, \( Q_\Sigma = Q_R + Q_S + Q_T \) total reactive power.

The first two, the arithmetic apparent power \( S_A \) and the geometric apparent power \( S_G \), are most often used in three-phase electric circuits. Buchholz’s definition of apparent power \( S_B \) is rather unknown among the electrical engineering community. As long as the supply voltage is sinusoidal and symmetrical, and the load is balanced, these relations give the same correct result. If one of the above-mentioned conditions is not met the obtained results will differ. The consequence of this will be that we get different values of power factors for the same electric circuit. Therefore, the value of power factor depends on the choice of the apparent power definitions. For obvious reasons such a situation is neither desired nor admissible.

In [25] it has been proved that in such a case only Buchholz’s definition of apparent power \( S_B \) allows for the correct calculation of apparent power, and therefore of the power factor value. Thus, the arithmetic apparent power \( S_A \) and the geometric apparent power \( S_G \) do not properly characterize the power phenomena in the case of asymmetrical waveforms of supply voltage and/or load currents in unbalanced electric circuits. Moreover, it can be proven that Buchholz’s apparent power can be extended to circuits with non-sinusoidal voltages and currents [25].

A3.2 MODERN (SELECTED) CONCEPTS OF THE POWER PROPERTIES OF ELECTRIC CIRCUITS

The first propositions of the power theory appeared in the 1920s and 1930s. Two basic trends of power theory development came into being at that time. The first makes use of Fourier series to describe the power properties of electric circuits. This trend treats the electrical waveforms as the sum of components with different frequencies and that is why the power properties of an electric circuit are defined in the frequency domain. Almost at the same time that the frequency trend came into being, a trend stressing the direct definition of power quantities in electric circuits, without using Fourier series, appeared. They are defined as the functional of the waveforms of current and voltage that is in the time domain. The problem of a lack of a universally accepted power theory describing the power properties of an electric circuit with non-sinusoidal waveforms of voltages and currents and valid for steady and transient states has taken on greater and greater significance since the discovery of the semiconductor and the development of power electronics. Besides the undeniable advantages of power converters, they are also a source of negative phenomena.
The growth in the number of power converters (non-linear devices) has caused a significant increase in the level of harmonics and has revealed their negative influence on the power system. For this reason, since the 1970s the interest in describing the power properties of such devices and the interest in methods of improving power factor have significantly increased [44]. There are many methods for the description of power phenomena occurring in electric circuits under non-sinusoidal conditions, which, de facto, are proposals of the power theory. The extension of the reactive power to non-sinusoidal and asymmetrical waveforms is now a subject of controversy. Many new power theories have been proposed and they are not accepted by all researchers around the world. The suggested definitions of this topic are still confusing and it is difficult to find a general and unified interpretation of power phenomena and the definitions of power properties under non-sinusoidal conditions, particularly when three-phase unbalanced circuits are analyzed. The most popular power theory has been proposed by Akagi and co-authors, which is known as the $p$–$q$ power theory [2],[3]. In many researchers’ opinion and also in the author’s opinion the most correct interpretation of power phenomena and description of power properties was presented by Czarnecki, which is known as the power theory based on the current’s physical components theory [19],[20]. Both theories are of particular importance for the development of power theory.

A3.2.1 The $p$–$q$ Power Theory Proposed by Akagi et al.

H. Akagi, A. Nabae and Y. Kanazawa proposed in 1983 the generalized theory of the instantaneous reactive power [2],[3], also known as the $p$–$q$ power theory. This theory has been developed in the time domain. It is valid for steady and transient states and for three-phase, three-wire or four-wire circuits. Its definitions are formulated on the basis of the transformation of a three-phase system in natural $R$–$S$–$T$ coordinates, to the orthogonal $\alpha$–$\beta$ (or $\alpha$–$\beta$–0) coordinates (the Clarke transformation). The transformation allows for the time-domain analysis of the power properties of three-phase circuits, as well as the physical interpretation of the defined quantities. But the suggested interpretation is still controversial. The instantaneous imaginary power proposed by Akagi and co-authors does not have a clear physical meaning.

The present analysis will be focused on three-wire circuits (Figure A3.6). Therefore, zero-sequence voltage or current components are not present. The $p$–$q$ power theory (in its original form) transforms instantaneous voltage $\mathbf{u}$ and current $\mathbf{i}$ (see Equation (A3.29)) measurements by the following matrix equations:

$$
\begin{bmatrix}
e_{\alpha} \\
e_{\beta}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix} \begin{bmatrix}
u_R \\
u_S \\
u_T
\end{bmatrix} \quad \text{and} \quad 
\begin{bmatrix}
i_{\alpha} \\
i_{\beta}
\end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{bmatrix} \begin{bmatrix}
i_R \\
i_S \\
i_T
\end{bmatrix}
$$

(A3.39)

where $\vec{e} = \vec{e}_a + \vec{e}_b + \vec{e}_c$, $\vec{i} = \vec{i}_a + \vec{i}_b + \vec{i}_c$, $|\vec{e}_a| = e_a$, $|\vec{e}_b| = e_b$, $|\vec{i}_a| = i_a$, $|\vec{i}_b| = i_b$ are space vectors of voltage and current in $\alpha$–$\beta$ coordinates and their amplitudes (the arrow denotes a space vector).
The instantaneous active power $p$ and the instantaneous imaginary power $\tilde{q}$ were defined as

\[
p = e_\alpha \cdot i_\alpha + e_\beta \cdot i_\beta \tag{A3.40}
\]

\[
\tilde{q} = e_\alpha \times i_\beta + e_\beta \times i_\alpha \tag{A3.41}
\]

where:

$p$ is the real power in the three-phase circuit equal to the conventional equation of the instantaneous power $p_3(t)$ (Equation (A3.30)). This power represents the total energy flow per unit time in the three-wire, three-phase circuit, in terms of $\alpha$–$\beta$ components (its dimension is $W$).

$q$ is the imaginary power (the imaginary axis vector is perpendicular to the real plane in $\alpha$–$\beta$ coordinates), has a non-traditional physical meaning and gives the measure of the quantity of current or power that flows in each phase without transporting energy at any instant. Akagi and co-authors introduced the instantaneous imaginary power space vector $\tilde{q}$ in order to define the instantaneous reactive power. This means that $\tilde{q}$ could not be dimensioned in $W$, VA or VAr, so its dimension is imaginary volt-amperes, IVA.

Therefore, on the basis of Equations (A3.40) and (A3.41) the following was obtained:

\[
\begin{bmatrix}
  p \\
  q
\end{bmatrix} =
\begin{bmatrix}
  e_\alpha & e_\beta \\
  -e_\beta & e_\alpha
\end{bmatrix}
\begin{bmatrix}
  i_\alpha \\
  i_\beta
\end{bmatrix} \quad (A3.42)
\]

where $q$ is the amplitude of space vector $\tilde{q}$ ($|\tilde{q}| = q$).

To calculate the currents $i_\alpha$, $i_\beta$ in $\alpha$–$\beta$ coordinates the expression (A3.42) is changed into the following:

\[
\begin{bmatrix}
  i_\alpha \\
  i_\beta
\end{bmatrix} =
\begin{bmatrix}
  e_\alpha & e_\beta \\
  -e_\beta & e_\alpha
\end{bmatrix}^{-1}
\begin{bmatrix}
  p \\
  q
\end{bmatrix} = \frac{1}{e_\alpha^2 + e_\beta^2}
\begin{bmatrix}
  e_\alpha & -e_\beta \\
  e_\beta & e_\alpha
\end{bmatrix}
\begin{bmatrix}
  p \\
  q
\end{bmatrix} \quad (A3.43)
\]

In general, when the load is non-linear and unbalanced the real and imaginary powers can be divided into average and oscillating components, as follows:

\[
p = \overline{p} + \tilde{p} = \overline{p} + \tilde{p}_h + \tilde{p}_{2f_1} \tag{A3.44}
\]

\[
q = \overline{q} + \tilde{q} = \overline{q} + \tilde{q}_h + \tilde{q}_{2f_1} \tag{A3.45}
\]

where $\overline{p}$, $\overline{q}$ are the average components, $\tilde{p}_h$, $\tilde{q}_h$ oscillating components ($h$ stands for harmonic) and $\tilde{p}_{2f_1}$, $\tilde{q}_{2f_1}$ oscillating components ($2f_1$ refers to double the fundamental component frequency). From these power components it is possible to calculate the current components in $\alpha$–$\beta$ coordinates. Then, by using the Clarke inverse transformation, it is possible to calculate the currents in the natural $R$–$S$–$T$ coordinates.

Finally, according to the $p$–$q$ power theory, the three-phase unbalanced non-linear load is expressed in four components

\[
i = i_p + i_\eta + i_h + i_{2f_1} \tag{A3.46}
\]
where \( i_p \) is associated with \( \bar{p} \), i.e. with the active power \( P \) defined in the traditional way, \( \bar{p} = P_R + P_S + P_T = P_S \); \( i_q \) is associated with \( \bar{q} \), i.e. with the reactive power \( Q \) (in the case of a symmetrical sinusoidal supply voltage and balanced linear load, \( \bar{q} \) is equal to the reactive power \( Q_{3f} \) defined in the fundamental harmonic domain, \( -\bar{q} = -q = Q_{3f} \) (Equation (A3.33)); \( i_h \) is associated with \( \tilde{p}_h \) and \( \tilde{q}_h \), i.e. with the presence of harmonics in voltage and current waveforms; and \( i_{2f/1} \) is associated with \( \tilde{p}_{2f/1} \) and \( \tilde{q}_{2f/1} \), i.e. the unbalance load currents. Figure A3.10 shows a graphical representation of defined power components in the three-phase circuit.

The \( p-q \) power theory turned out to be a very useful tool for building and developing the control algorithms of active power filters. This power theory has been the basis of many other proposals of power theory. Among them the most interesting is the power theory proposed by Peng [48], [50].

### A3.2.2 The Power Theory Based on the Current’s Physical Components Theory Proposed by Czarnecki

The power theory based on the current’s physical components theory proposed by L. S. Czarnecki has been developed in the frequency domain. This theory is a proposal for the physical interpretation of power phenomena occurring in electric circuits under unbalanced conditions and in the presence of non-sinusoidal waveforms. The complete Czarnecki theory for three-phase unbalanced circuits with periodic non-sinusoidal source voltage was presented in 1994 [20]. The theory deals comprehensively with all situations in circuits with periodic waveforms, from the interpretation in physical terms to methods for power factor improvement in single-phase and three-phase circuits. For the purpose of the present analysis the power theory for three-phase, three-wire electric circuits (Figure A3.6) has been applied under the assumption that the voltage unbalance at the cross-section \( R-S-T \) is negligibly small; it has therefore been neglected in the analysis.

![Figure A3.10](image-url)  
**Figure A3.10** The \( p-q \) power theory – a graphical representation of defined power components in a three-phase circuit
Phase voltages are expressed in terms of Fourier series and presented in the form of the vector $\mathbf{u}$ (see Equation (A3.29)):

$$
\mathbf{u} = \begin{bmatrix} u_R \\ u_S \\ u_T \end{bmatrix} = \sum_{n \in N} \mathbf{u}_n(n) = \sum_{n \in N} \begin{bmatrix} u_{(n)R} \\ u_{(n)S} \\ u_{(n)T} \end{bmatrix} = \sqrt{2}\text{Re} \sum_{n \in N} \begin{bmatrix} U_{(n)R} \\ U_{(n)S} \\ U_{(n)T} \end{bmatrix} e^{jn\omega_1 t}
$$

(A3.47)

where $\mathbf{U}_n(n) = [U_{(n)R}, U_{(n)S}, U_{(n)T}]^T$ is the vector of voltage complex r.m.s. values. $N$ denotes the set of voltage harmonics observed at the cross-section $R-S-T$. The supply source is assumed to have internal impedance, therefore the same harmonics occur in both the voltages and currents.

Phase currents are presented, analogously to the voltages, by means of the vector $\mathbf{i}$ (see Equation (A3.29)):

$$
\mathbf{i} = \begin{bmatrix} i_R \\ i_S \\ i_T \end{bmatrix} = \sum_{n \in N} \mathbf{i}_n(n) = \sum_{n \in N} \begin{bmatrix} i_{(n)R} \\ i_{(n)S} \\ i_{(n)T} \end{bmatrix} = \sqrt{2}\text{Re} \sum_{n \in N} \begin{bmatrix} I_{(n)R} \\ I_{(n)S} \\ I_{(n)T} \end{bmatrix} e^{jn\omega_1 t}
$$

(A3.48)

where $\mathbf{I}_n(n) = [I_{(n)R}, I_{(n)S}, I_{(n)T}]^T$ is the vector of current complex r.m.s. values. The complex power $S_n$ of the $n$th-order harmonic is

$$
S_{(n)} = \mathbf{U}_n^T(n)\mathbf{I}_n^*(n) = P_{(n)} + jQ_{(n)}
$$

(A3.49)

where $P_{(n)}$ and $Q_{(n)}$ are the $n$th-order harmonic active and reactive power of a load. If the load is passive, linear and time invariant, then the harmonic active power $P_{(n)}$ is transmitted from the source to the load, i.e. $P_{(n)} \geq 0$. If any of the above conditions is not satisfied the current harmonics can be generated in the load. The energy at the generated harmonic frequencies can be transmitted from the load to the source, i.e. $P_{(n)} \leq 0$. Hence the set of all orders of harmonics $N$ has been divided into two subsets $N_A$ and $N_B$:

$$
\text{if } P_{(n)} \geq 0 \text{ then } n \in N_A \quad \text{if } P_{(n)} < 0 \text{ then } n \in N_B
$$

(A3.50)

Thus the voltages $\mathbf{u}$, currents $\mathbf{i}$ and active power $P$ observed at the cross-section $R-S-T$ (Figure A3.6) have been decomposed into the following components:

$$
i = \sum_{n \in N} \mathbf{i}_n(n) = \sum_{n \in N_A} \mathbf{i}_n(n) + \sum_{n \in N_B} \mathbf{i}_n(n) = \mathbf{i}_A + \mathbf{i}_B
$$

(A3.51)

$$
\mathbf{u} = \sum_{n \in N} \mathbf{u}_n(n) = \sum_{n \in N_A} \mathbf{u}_n(n) - \sum_{n \in N_B} \mathbf{u}_n(n) = \mathbf{u}_A - \mathbf{u}_B
$$

(A3.52)

$$
P = \sum_{n \in N} P_{(n)} = \sum_{n \in N_A} P_{(n)} - \sum_{n \in N_B} P_{(n)} = P_A - P_B
$$

(A3.53)
According to the current’s physical components theory, the current of the three-phase, unbalanced, non-linear load has been decomposed into five components:

\[ i = i_a + i_s + i_r + i_u + i_B \]  \hspace{1cm} (A3.54)

The components are defined by the following relations:

\[ i_a = \sqrt{2} \text{Re} \sum_{n \in N} G_e U_{A(n)} e^{jna_1 t} \] \hspace{1cm} (A3.55) \quad \text{active current}

\[ i_s = \sqrt{2} \text{Re} \sum_{n \in N_A} (G_{e(n)} - G_e) U_{A(n)} e^{jna_1 t} \] \hspace{1cm} (A3.56) \quad \text{scattered current}

\[ i_r = \sqrt{2} \text{Re} \sum_{n \in N_A} jB_{e(n)} U_{A(n)} e^{jna_1 t} \] \hspace{1cm} (A3.57) \quad \text{reactive current}

\[ i_u = \sum_{n \in N_A} i_{u(n)} = \sqrt{2} \text{Re} \sum_{n \in N_A} A_{e(n)} U_{A(n)} e^{jna_1 t} \] \hspace{1cm} (A3.58) \quad \text{unbalanced current}

\[ i_B = \sum_{n \in N_B} i_{B(n)} \] \hspace{1cm} (A3.59) \quad \text{load-generated current}

where \( G_e \) is the equivalent conductance; \( Y_{e(n)} = G_{e(n)} + jB_{e(n)} \) the equivalent admittance and \( A_{e(n)} \) the unbalanced admittance. Appropriate relationships, which enable the determination of \( G_e, Y_{e(n)}, A_{e(n)}, \) and powers \( P_A, D_s, Q_r, Q_u, S_B, S_F, S \) are given in [19],[20].

Each component is associated with a different power phenomenon and is orthogonal with respect to the others. Thus the current decomposition (A3.54) reveals five different physical phenomena, which determine the load current value, namely:

- active energy transmission to the load, \( i_a \);
- change of the load conductance \( G_{e(n)} \) with frequency, \( i_s \);
- reciprocating flow of energy, \( i_r \);
- load current unbalance, \( i_u \);
- active energy transmission back to the source, \( i_B \).

If harmonic currents are generated by the supply source, due to the distorted voltage, then the current components \( i_a, i_s, i_r, i_u, i_B \) may also contain, besides the fundamental, also high-order harmonics of orders already present in the voltage.\(^2\) Thus, each of the components \( i_a, i_s, i_r, i_u, i_B \) can be additionally decomposed into the fundamental and high-order harmonics, e.g.

\[ i_a = i_{a(1)} + i_{a(h)} \] \hspace{1cm} (A3.60)

where the subscript (h) means harmonics without the first harmonic. The subscript (1) means the first harmonic. The equivalent electric circuits for harmonics of order \( n \in N_A \) and \( n \in N_B \) are shown in Figure A3.11 and Figure A3.12, respectively.

\(^2\)This is decided by the sign of the active power calculated for the given harmonic [19],[20].
Due to the orthogonality of the current’s physical components, their r.m.s. values satisfy the relation

$$\|\mathbf{i}\|^2 = \|\mathbf{i}_a\|^2 + \|\mathbf{i}_b\|^2 + \|\mathbf{i}_c\|^2 + \|\mathbf{i}_d\|^2$$  \hspace{1cm} (A3.61)

where $\|\mathbf{i}\|$ is the r.m.s. value of $\mathbf{i}$ (see Equation (A3.29)) and

$$\|\mathbf{i}\| = \sqrt{\sum_{n \in M} \mathbf{i}_{(n)}^T \cdot \mathbf{i}_{(n)}} = \sqrt{\sum_{n \in M} \mathbf{i}_{(n)}^T \cdot \mathbf{i}_{(n)}} = \sqrt{\sum_{n \in M} (I_{(n)R}^2 + I_{(n)S}^2 + I_{(n)T}^2)} = \sqrt{\|i_R\|^2 + \|i_S\|^2 + \|i_T\|^2}$$

where $M$ denotes respectively the set $N_A$ or $N_B$.

Multiplying Equation (A3.61) by the square of the supply voltage r.m.s. value $\|\mathbf{u}\|$, we obtain the equation of powers for an unbalanced three-phase circuit with non-sinusoidal waveforms:

$$S^2 = P_A^2 + D_s^2 + Q_r^2 + Q_s^2 + S_B^2 + S_F^2$$  \hspace{1cm} (A3.62)
where

\[ S = \|\mathbf{u}\| \cdot \|\mathbf{i}\| = \sqrt{S_A^2 + S_B^2 + S_F^2} \]  

apparent power \hfill (A3.63)

\[ S_A = \|\mathbf{u}_A\| \cdot \|\mathbf{i}_A\| = \sqrt{P_A^2 + D_s^2 + Q_r^2 + Q_u^2} \]  
supply-generated apparent power \hfill (A3.64)

\[ S_B = \|\mathbf{u}_B\| \cdot \|\mathbf{i}_B\| \]  
load-generated apparent power \hfill (A3.65)

\[ S_F = \sqrt{\|\mathbf{u}_A\|^2 \cdot \|\mathbf{i}_B\|^2 + \|\mathbf{u}_B\|^2 \cdot \|\mathbf{i}_A\|^2} \]  
forced apparent power \hfill (A3.66)

\[ P_A = \sum_{n \in N_A} P_n \]  
active power for harmonics \hfill (A3.67)

of order \( n \in N_A \) \hfill (A3.53)

\[ D_s = \|\mathbf{u}_A\| \cdot \|\mathbf{i}_s\| \]  
scattered power \hfill (A3.68)

\[ Q_r = \|\mathbf{u}_A\| \cdot \|\mathbf{i}_r\| \]  
reactive power \hfill (A3.69)

\[ Q_u = \|\mathbf{u}_A\| \cdot \|\mathbf{i}_u\| \]  
unbalanced power \hfill (A3.70)

The last component of the apparent power, \( S_F \), occurs even in an ideal circuit, when the supply voltage source is connected with a current source of harmonic orders different than the supply voltage harmonics. In the case of unidirectional flow of energy from the supply source to the load, some power components are equal to zero \( (S_B = S_F = 0) \), and then \( S_A = S \) and for all orders of harmonics \( P_n \geq 0 \) (the subset \( N_B \) is empty).

The power factor \( PF_{ CPC} \) has been determined by means of the components of the apparent power \( S \), and also by means of the current’s physical components:

\[
PF_{ CPC} = \frac{P}{S} = \frac{P_A - P_B}{\sqrt{P_A^2 + D_s^2 + Q_r^2 + Q_u^2 + S_B^2 + S_F^2}} = \frac{\|\mathbf{i}_s\|}{\|\mathbf{i}\|} = \frac{\|\mathbf{i}_s\|}{\sqrt{\|\mathbf{i}_s\|^2 + \|\mathbf{i}_r\|^2 + \|\mathbf{i}_u\|^2 + \|\mathbf{i}_B\|^2 + \|\mathbf{i}_A\|^2}} \hfill (A3.71)
\]

In this way the decomposition \( (A3.54) \) reveals the influence on the source power factor \( PF_{ CPC} \) of all physical phenomena in three-phase unbalanced circuits with periodic distorted waveforms. This theory provides the basis for developing control algorithms, among others, for reactive power compensators, balancing compensators, hybrid filters and active power filters.

**A3.2.3 Other Proposed Power Theories**

As was mentioned above, many proposals of power theory and many suggested definitions of power properties (power phenomena) exist in the scientific literature. Table A3.1 shows a list of proposals of power theories. This list is limited due to the large number of attempts to define them. Due to the fact that the list is a summary of the different concepts, only their general characteristics are presented: the domain in which the power theory was defined, the date of its publication, the type of electric circuits (single-, three-, multi-phase), and the kind of quantities that were used in the definitions (average, r.m.s, instantaneous) [54].
<table>
<thead>
<tr>
<th>Author(s) of The Power Theory</th>
<th>Date of publication</th>
<th>Number of phases of the circuit</th>
<th>Value of the quantities</th>
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<td></td>
<td></td>
<td>Single</td>
<td>Three</td>
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<td>Fryze (Fryze’s power theory)</td>
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<td>– orthogonal currents</td>
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<td></td>
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<tr>
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<tr>
<td>– the first harmonic of voltage and current</td>
<td></td>
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<tr>
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<td>1983</td>
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<td></td>
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<td>Ferrero, Superti-Furga</td>
<td>1991</td>
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<tr>
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**Defined in time domain**
Table A3.1  (Continued)

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<th>Number of phases of the circuit</th>
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<tr>
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BIBLIOGRAPHY


